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WEAKLY Po, NON-WEAKLY Po, AND WEAKLY Po PROPERTIES

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Abstract

Within this paper each of T_0-identification P properties, weakly Po, and weakly Po properties are further investigated and characterized, and infinitely many non-weakly Po topological properties are given.

1. Introduction and Preliminaries

T_0-identification spaces were introduced in 1936 and used to jointly characterize pseudometrizable and metrizable [9].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \rightarrow X_0$ be the natural map, and let $Q(X, T)$ be the decomposition

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topology on X_0 determined by (X, T) and the map *N*. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T) [9].

Theorem 1.1. *A space is pseudometrizable iff its T_0-identification space is metrizable. T_0-identification spaces were cleverly created to add T_0 to the externally generated, strongly (X, T) related T_0-identification space of (X, T), making T_0-identification spaces a strong, useful topological tool* [9].

Similarly, in 1975 [8] T_0-identification spaces were used to jointly characterize the R_1 separation axiom and Hausdorff.

Definition 1.2. A space (X, T) is R_1 iff for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets *U* and *V* such that $x \in U$ and $y \in V$ [1].

In a 2015 paper [2], the T_0-identification space property for each of pseudometrizable and R_1 was generalized to weakly *P*o.

Definition 1.3. Let *P* be a topological property for which *P*o = (*P* and T_0) exists. Then (X, T) is weakly *P*o iff $(X_0, Q(X, T))$ has property *P*. A topological property *P*o for which weakly *P*o exists is called a weakly *P*o property.

Within the 2015 paper [2], it was shown that for a topological property *P* for which weakly *P*o exists, weakly *P*o is a unique topological property. Also, in that paper [2], it was shown that if *P* is a topological property for which *P*o exists and $\mathcal{S}$ is the collection of all topological properties *Q* for which *Q*o exists and *Q*o implies *P*o, then weakly *P*o exists iff weakly *P*o is the least element of $\mathcal{S}$.

Also, in the 2015 paper [2], the search for a topological property that is not weakly *P*o led to the need and use of T_0 and "not-T_0". Thus both T_0 and "not-T_0" proved to be useful topological properties motivating the further investigation of T_0 and the addition of "not-*P*", where *P* is a topological property for which "not-*P*" exists, to the study of topology [2]. The addition of the many new properties provided tools not before used in the study of topology and, in a short time period, has revealed a mathematically fertile, never before imagined territory long overlooked within topology that has already changed the study of topology.

In the paper [3], the use of "not-T_0" and "not-P", where "not-P" exists, not only provided needed tools to prove the existence of the never before imagined least of all topological properties *L*, but, also, provided the needed tools for a quick, easily understood proof of the existence of *L*.

Theorem 1.2. *L*, *the least of all topological properties*, *is given by* $L = (T_0$ *or* "*not*-T_0"$) = (P$ *or* "*not*-P"$)$, *where P is a topological property for which* "*not*-P" *exists* [3].

As is often the case, the existence of something not imagined can create problems as was the case of *L*. Within the paper [3], it was shown that every space has property *L*. Thus each product space and each of its factor spaces simultaneously share property *L*, regardless of how diverse or even if factor spaces have properties that are known not product properties, and, by the 1930 definition [10], *L* is a product property, a reality far different than the intent of product properties defined in 1930. Within the paper [3], the discontinuity in the study of product properties was removed by the removal of *L* as a product property. Thus, the continued study of T_0-identification spaces has already been a productive study revealing new, basic, fundamental topological properties that require corrections in classical topology.

In the 2015 paper [2], it was shown that a space is weakly *P*o iff its T_0-identification space is weakly *P*o, motivating the introduction of T_0-identification *P* properties.

Definition 1.4. A topological property *Q* is a T_0-identification *P* property iff *Q* is simultaneously shared by both a space and its T_0-identification space [4].

Within that paper [4], it was shown that for a T_0-identification *P* property *Q*, *Q* = weakly *Q*o, which for a while clouded the obvious: A topological property *Q* is weakly *P*o iff Q is a T_0-identification P property.

Initially, to search for a weakly *P*o property or equivalently, a T_0-identification P property, a classical topological property *Q*o = (Q and T_0) was chosen and a topological property *W* was sought such that if a space (X, T) has property *W*, then $(X_0, Q(X, T))$ has property *Q*o, which then implies (X, T) has property *W*, all with no certainty that such a *W* exists. Not knowing which topological properties are weakly *P*o properties in the initial investigations made the requirement that weakly *P*o exist necessary. Had that search process continued, the study of weakly *P*o spaces and properties would continue to be uncertain, tedious, and never ending, greatly hindering the exploration of the newly revealed mathematical territory. Thus to make the search process more certain, the question of precisely which topological properties are weakly *P*o properties arose leading to successes in a 2017 paper [5].

Answer. {*Q*o| Q is a T_0-identification P property} = {Q*o* | *Q*o is a weakly *P*o property} = {*Q*o| Q is a topological property and *Q*o exists} [5].

Within the 2017 paper [5], for a topological property *W* for which *W*o exists, a property *WNO* was defined. Let *W* be a topological property such that *W*o exists. A space (X, T) has property *WNO* iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property *W*o. In that paper [5], it was shown that for a topological property *W* for which *W*o exists, *WNO* exists and is a topological property, and a space has property (*W*o or *WNO*) iff its T_0-identification space has property (*W*o or *WNO*). Thus for a topological property *W* for which *W*o exists, (*W*o or *WNO*) is a T_0-identification P property and (*W*o or *WNO*) = weakly (*W*o or *WNO*)o. Since *WNO* is "not-T_0", then (*W*o or *WNO*)o = *W*o and (*W*o or *WNO*) = weakly (*W*o or *WNO*)o = weakly *W*o. Hence, substantial progress was made in the 2017 paper [5] concerning T_0-identification P properties or equivalently, topological properties that are weakly *P*o.

In the paper [6], the results above were used to completely characterize weakly *P*o.

For a topological property Q, the following are equivalent: (a) Q is a T_0-identification P property, (b) Q is weakly Po, (c) both Qo and (Q and "not-T_0") exists, and (Q and "not-T_0") = QNO, and (d) Q = (Qo or QNO).

Thus, the uncertainty in the initial search for topological properties that are weakly Po or equivalently, T_0-identification P properties, or weakly Po properties was replaced with certainty and great progress has been made in the study of topological properties that are weakly Po. With the connection of weakly Po to other topological properties, as given above, knowledge of topological properties that are non-weakly Po could be useful and enlightening in the continued expansion of topology. Below additional properties of weakly Po and weakly Po properties are given and examples of topological properties that are non-weakly Po are given.

2. Weakly Po and Non-weakly Po

Corollary 2.1. *Let P be a topological property for which Po exists. Then $\mathcal{S}$, the collection of topological properties Q such that Qo exists and Qo implies Po, has a least element and the least element of $\mathcal{S}$ is weakly Po.*

Theorem 2.1. *Let Q be a topological property that is weakly Po. Then both Qo and (Q and "not-T_0") exist.*

Proof. Since Q is weakly Po, then Q is not T_0 and Q is not ("not-T_0"). Since Q is not ("not-T_0"), then (Q and T_0) = Qo exists and since Q is not T_0, then (Q and "not-T_0") exists.

Corollary 2.2. *Let Q be a topological property. If one of Qo and (Q and "not -T_0") does not exist, then Q is a non-T_0-identification P property that is non-weakly Po.*

It would be nice and simple if weakly Qo = (Qo or (Q and "not-T_0")) and Q is a non-T_0-identification P property that is non-weakly Po iff one of Qo and (Q and "not-T_0") does not exist, but such is not the case as established by an example later in this paper.

Theorem 2.2. *Let Q be a topological property. Then the following are equivalent*: (*a*) *Q is weakly P*o, (*b*) *Q is a* T_0*-identification P property, and* (*c*) *Q*o *exists and QNO has property Q*.

Proof. By the results above (a) and (b) are equivalent.

(b) implies (c): By the results above *Q*o exists and $QNO = (Q$ and "not-T_0"$)$. Thus *QNO* implies *Q* and *QNO* has property *Q*.

(c) implies (a): Since *QNO* has property *Q*, which exists, then *QNO* exists and *QNO* is "not-T_0". Thus *QNO* has properties *Q* and "not-T_0" and *QNO* has property $(Q$ and "not-T_0"$)$ and $Q = (Q\text{o}$ or $(Q$ and "not-T_0"$)) = (Q\text{o}$ or $QNO)$. Hence, by the results above, (c) implies (a).

As above for weakly *P*o, for each weakly *P*o property $Q = Q\text{o}$, there are two obvious non-T_0-identification *P* properties that are non-weakly *P*o: *Q*o and *QNO*. Are there additional non-T_0-identification *P* properties that are non-weakly *P*o associated with a topological property *Q* for which *Q*o exists?

In the paper [5], for a topological property *Q* for which *Q*o exists and each natural number $n \geq 2$, property $Q(1, n)$ was defined. A space (X, T) is $Q(1, n)$ iff there exist *n* distinct elements $a_1, \cdots, a_n$ all of whose closures are equal, and for all other $x, y \in X$, $Cl(\{x\}) = Cl(\{y\})$ iff $x = y$, and the T_0-identification space of (X, T) is *Q*o. Within that paper [5] it was shown that $Q(1, n)$ is a topological property. Clearly $Q(1, n)$ is stronger than *QNO* and $(Q\text{o}$ or $Q(1, n)) \neq (Q\text{o}$ or $QNO)$ and, thus, each of $(Q\text{o}$ or $Q(1, n))$ and $Q(1, n)$ are non-T_0-identification *P* properties that are non-weakly *P*o. Since for distinct natural numbers *m* and *n* greater than or equal to 2, $Q(1, n)$ and $Q(1, m)$ are distinct topological properties [5], then there are infinitely many distinct non-T_0-identification *P* properties that are non-weakly *P*o for a given *Q*. Below, the example indicated above is given.

Example 2.1. Let Q be a topological property such that Qo exists. Then $W = (Q\text{o}$ or $Q(1, 2))$ is non-T_0-identification P property that is a non-weakly Po for which both Wo and $(W$ and "not-T_0") exist.

Below, for a topological property Q for which Qo exists, infinitely many more non-T_0-identification P properties that are non-weakly Po are given.

Theorem 2.3. *Let Q be a topological property for which Qo exists and let $n_i; i = 1, \cdots, p$, be distinct natural numbers, where p is a natural number greater than or equal to 2. Then each of $(Q\text{o or } ((Q(1, n_1) \text{ or} \cdots \text{or } (Q(1, n_p)))$ and $((Q(1, n_1) \text{ or} \cdots \text{or } (Q(1, n_p))$ are non-T_0-identification P properties that are non-weakly Po.*

The proof is straightforward using the logic and information given above and is omitted.

If the space (X, T) above has 2 or more elements, then in the same manner for natural numbers m and n, each greater than one, $m \leq n$, $Q(2, m, n)$ could be defines and used to obtain additional distinct non-T_0-identification P properties that are non-weakly Po and, depending on the size of X, in a similar manner, additional non-T_0-identification P properties that are non-weakly Po can be obtained. Thus for a topological property Q for which Qo exists, there exists exactly one topological property W that is a T_0-identification P property that is weakly Po and W = weakly Qo and infinitely many topological properties that are non-T_0- identification P properties that are non-weakly Po.

3. Weakly *P*o Properties and their Matching Property that is Weakly *P*o

By the Answer above, if Q be a topological property such that $Q = Q$o, then $Q = Q$o is a weakly Po property that is a non-T_0-identification P property and non-weakly *P*o. Thus, in this case, the question of what topological property *W* is *W* = weakly *Q*o arises.

Theorem 3.1. *Let Q be a topological property for which Q* = *Q*o. *Then W = (Q*o *or QNO) is a topological property that is a* T_0*-identification P property that is weakly P*o *with* (*Q*o *or QNO*) = *weakly* (*Q*o *or QNO*)o = *weakly Q*o *and W*o = *Q*o.

Proof. Since *Q* is a topological property for which *Q*o exists, then, by the results above, *QNO* exists and *W* = (*Q*o or *QNO*) is a T_0-identification *P* property that is weakly *P*o with *W* = (*Q*o or *QNO*) = weakly (*Q*o or *QNO*)o = weakly *W*o = weakly *Q*o. Since weakly *W*o = weakly *Q*o, then *W*o = (weakly *W*o)o = (weakly *Q*o) o = *Q*o [2].

Thus, for the weakly *P*o property *Q* = *Q*o, weakly *Q*o has been completely determined with the use of *Q*o and *QNO* as given above. However, for a specific topological property *Q* = *Q*o, *QNO* is known to exist, but with little insight into its exact description, possibly creating a problem. Within the literature, there are many known topological properties that are weakly *P*o that could resolve the problem. As an alternative, within the paper [7], the T_0-identification space and weakly *P*o processes were internalized and can be, and has been, used to gain insight into the needed property, making the problem manageable.

References

[1] A. Davis, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly 68 (1961), 886-893.

[2] C. Dorsett, Weakly *P* properties, Fundamental J. Math. Math. Sci. 3(1) (2015), 83-90.

[3] C. Dorsett, Pluses and needed changes in topology resulting from additional properties, Far East J. Math. Sci. 101(4) (2017), 803-811.

[4] C. Dorsett, T_0-identification *P* and weakly *P* properties, Pioneer J. Math. Math. Sci. 15(1) (2015), 1-8.

[5] C. Dorsett, Complete characterization of weakly *P*o and related spaces and properties,

J. Math. Sci.: Adv. Appl. 45 (2017), 97-109.

[6] C. Dorsett, The Complete characterization of weakly *P*o and T_0-identification *P* properties with a correction, accepted by Journal of Mathematical Sciences: Advances and Applications.

[7] C. Dorsett, Additional properties for weakly *P*o and related properties with an application, J. Math. Sci.: Adv. Appl. 47 (2017), 53-64.

[8] W. Dunham, Weakly hausdorff spaces, Kyungpook Math. J. 15(1) (1975), 41-50.

[9] M. Stone, Application of Boolean algebras to topology, Mat. Sb. 1 (1936), 765-771.

[10] A. Tychonoff, Uber die Topogishe Erweiterung von Raumen, Math. Ann. 103 (1930), 544-561.