Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 6, Issue 1, 2016, Pages 33-43
This paper is available online at http://www.frdint.com/
Published online July 22, 2016

WEAKLY $P1$, WEAKLY Po AND T_0-IDENTIFICATION P PROPERTIES

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Abstract

In 1975, T_0-identification spaces were used to further characterize weakly Hausdorff spaces raising the question of whether the process used to characterize weakly Hausdorff could be generalized to include additional properties. The consideration of that question led to the introduction and investigation of weakly Po properties. As in the 1975 characterization of weakly Hausdorff, the T_0 separation axioms have a major role in the definition and properties of weakly Po properties. Thus the question of what would happen if T_0 in the definition of weakly Po was replaced by T_1 arose leading to the work in this paper.

1. Introduction and Preliminaries

In 1975 [7], T_0-identification spaces were used to further characterize weakly

Keywords and phrases: weakly $P1$ properties, weakly Po properties, T_0-identification P properties.

2010 Mathematics Subject Classification: 54A05, 54B15, 54D10.

Received May 31, 2016; Accepted June 13, 2016

Hausdorff spaces.

T_0 -identification spaces were introduced in 1936 [8].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, T))$ is the T_0 -identification space of (X, T).

Within the 1936 paper [8], T_0 -identification spaces were used to further characterize pseudometrizable spaces.

Theorem 1.1. *A space* (X, T) *is pseudometrizable iff* $(X_0, Q(X, Q(X, T))$ *is metrizable* [8].

Theorem 1.2. *A space* (X, T) *is weakly Hausdorff iff* $(X_0, Q(X, T))$ *is Hausdorff* [7].

In the 1975 paper [7], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.2. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

Within the 1961 paper [1], A. Davis was interested in separation axioms R_i, which together with T_i, are equivalent to T_{i+1}; $i = 0, 1$, respectively, leading to the definition of R_1 and the rediscovery of the R_0 separation axiom.

Definition 1.3. A space (X, T) is R_0 iff for each $O \in T$ and each $x \in O$, $Cl(\{x\}) \subseteq O$ [1].

The separation axioms R_i; $i = 0, 1$, satisfied Davis' expectations [1].

Within a recent paper [2], weakly Hausdorff was generalized to weakly Po properties.

Definition 1.4. Let P be a topological property for which $P\text{o} = (P \text{ and } T_0)$ exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

As a result of the role of T_0 in the weakly Po property process within the introductory paper [2], it was proven that for a topological property P for which weakly Po exists, a space is weakly Po iff its T_1-identification space has property Po.

Even though weakly Po properties were undefined at the time, since (pseudometrizable)o equals metrizable, metrizable was the first known weakly Po property and weakly (pseudometrizable)o = weakly (metrizable)= pseudometrizable. Within the paper [2], it was established that both T_2 and T_1 are weakly Po properties, with weakly (R_1)o = weakly $T_2 = R_1$ and weakly (R_0)o = weakly $T_1 = R_0$.

In the introductory weakly Po property paper [2], it was shown that both T_0 and "not-T_0" are not weakly Po properties, where "not-T_0" is the negation of T_0. Also, within the paper [2], it was shown that a space is weakly Po iff its T_0-identification space is weakly Po. The combination of this result with the fact that other topological properties are simultaneously shared by a space and its T_0-identification space led to the introduction and investigation of T_0-identification P properties [3].

Definition 1.5. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property S [3].

Within the paper [3], it was proven that property Q is a T_0-identification P property iff Qo exists and $Q =$ weakly Qo, which is combined with results above to give a new characterization of T_0-identification P properties.

Corollary 1.1. *The* $\{Q \mid Q$ *is a* T_0*-identification* P *property*$\}$ = $\{$*weakly* $\{Q\text{o} \mid Q\text{o}$ *is a weakly* $P\text{o}$ *property*$\}$.

Corollary 1.2. *Let* Q *be a* T_0*-identification* P *property. Since weakly* $Q\text{o}$ *exists*, $Q\text{o}$ *is a weakly* $P\text{o}$ *property*.

As in the case of weakly $P\text{o}$ properties, both T_0 and "not-T_0" fail to be T_0-identification P properties [3]. The knowledge and insights obtained from the investigations of weakly $P\text{o}$ and T_0-identification P properties are used to define and investigate weakly $P1$ and to further investigate weakly $P\text{o}$ and T_0-identification P properties.

2. Weakly $P1$

Definition 2.1. Let P be a topological property for which $P1 = (P$ and $T_1)$ exists. Then (X, T) is weakly $P1$ iff $(X_0, Q(X, T))$ is $P1$. A topological property $P1$ for which weakly $P1$ exists is called a weakly $P1$ property.

Theorem 2.1. *Let* Q *be a topological property for which* $Q1$ *is a weakly* $P1$ *property. Then* $Q\text{o}$ *is a weakly* $P\text{o}$ *property.*

Proof. Since $Q1$ implies $Q\text{o}$, then $Q\text{o}$ exists. Let (X, T) be weakly $Q1$. Then $(X_0, Q(X, T))$ is $Q1$, which implies $(X_0, Q(X, T))$ has property Q and (X, T) is weakly $Q\text{o}$. Hence weakly $Q\text{o}$ exists and $Q\text{o}$ is a weakly $P\text{o}$ property.

Within the paper [4], it was shown that compact is a T_0-identification P property. Since (compact)o exists, then (compact)o is a weakly $P\text{o}$ property. Since (compact)1 exists, then (compact)1 is a weakly $P1$ property. Thus, the converse of Theorem 2.1 is not true.

Theorem 2.2. *Let* P *be a topological property for which* $P1$ *exists. Then* $(P\text{o})1 = (P1)\text{o} = P1 = (P\text{o}$ *and* $R_0)$.

Proof. Since $P1$ implies T_0, then $(P1)\text{o}$ exists, $(P1)\text{o} = ((P$ and $T_1)$ and $T_0) = (P$ and $(T_1$ and $T_0)) = (P$ and $T_1) = P1$, $(P\text{o})1 = ((P$ and $T_0)$ and

T_1) = (P and (T_0 and T_1)) = (P and T_1) = $P1$, and $P1$ = (P and T_1) = (P and (T_0 and R_0)) = ((P and T_0) and R_0) = (Po and R_0).

Theorem 2.3. *Let Q be a topological property for which $Q1$ exists. Then the following are equivalent*: (a) *$Q1$ is a weakly $P1$ property*, (b) *$Q1$ is a weakly Po property, and* (c) *weakly $Q1$ = ((weakly Qo) and R_0).*

Proof. (a) implies (b): Since ($Q1$)o = $Q1$ and $Q1$ is a weakly $P1$ property, then weakly ($Q1$)o = weakly $Q1$ exists and $Q1$ is a weakly Po property.

(b) implies (c): Since ($Q1$)o = $Q1$ and ($Q1$)o is a weakly Po property, then weakly $Q1$ = weakly ($Q1$)o exists. Thus $Q1$ is a weakly $P1$ property, which implies Qo is a weakly Po property. Then a space (X, T) is weakly $Q1$ iff (X, T) is weakly ($Q1$)o iff $(X_0, Q(X, T))$ is ($Q1$)o = ((Qo) and R_0)o = ((Qo) and T_1) iff (((X_0, $Q(X, T)$) is Q)o and ((X_0, $Q(X, T)$) is T_1)) iff ((X, T) is weakly Q)o and ((X, T) is R_0)) iff (X, T) is ((weakly Q)o and R_0). Hence weakly $Q1$ = ((weakly Q)o and R_0).

(c) implies (a): Since weakly $Q1$ exists, then $Q1$ is a weakly $P1$ property.

Theorem 2.4. *Neither T_0 nor "not-T_0" are weakly $P1$ properties.*

Proof. Since neither T_0 or "not-T_0" are weakly Po properties, then, by Theorem 2.1, neither T_0 nor "not-T_0" are weakly $P1$ properties.

Natural questions to pose at this point are (1) "Is weakly $P1$ a T_0-identification P property?", (2) "What happens if the weakly $P1$ process is repeated?", and (3) "If $Q1$ and $W1$ are weakly $P1$ properties and weakly $Q1$ = weakly $W1$, must $Q1 = W1$?", which are resolved below.

Theorem 2.5. *Let $Q1$ be a weakly $P1$ property and let (X, T) be a space. Then the following are equivalent*: (a) *$(X_0, Q(X, T))$ has property $Q1$*, (b) *$(X_0, Q(X, T))$ is weakly $Q1$, and* (c) *$(X_0, Q(X, T))$ is (weakly $Q1$)o.*

Proof. (a) implies (b): Since $(X_0, Q(X, T))$ is homeomorphic to $((X_0)_0, Q(X_0, Q(X_0, Q(X, T))))$ [2], then $((X_0)_0, Q(X_0, Q(X_0, Q(X, T))))$ has property $Q1$, which implies $(X_0, Q(X, T))$ is weakly $Q1$.

(b) implies (c): Since $(X_0, Q(X, T))$ is T_0 [8], then $(X_0, Q(X, T))$ is (weakly $Q1$)o.

(c) implies (a): Since $(X_0, Q(X, T))$ is (weakly $Q1$)o, then $(X_0, Q(X, T))$ is weakly $Q1$. Then $((X_0)_0, Q(X_0, Q(X_0, Q(X, T))))$ has property $Q1$, which, by the homeomorphic given above, implies $(X_0, Q(X, T))$ has property $Q1$.

Corollary 2.1. *Let $Q1$ be a weakly $P1$ property and let (X, T) be a space. Then (X, T) is weakly $Q1$ iff $(X_0, Q(X, T))$ is weakly $Q1$.*

Corollary 2.2. *Let $Q1$ be a weakly $Q1$ property. Then weakly $Q1$ is a T_0-identification P property.*

Theorem 2.6. *Let $Q1$ be a weakly $P1$ property. Then $Q1 =$ (weakly $Q1$)o.*

Proof. *Let* (X, T) be a space. Suppose (X, T) has property $Q1$. Then (X, T) is T_0 and (X, T) and $(X_0, Q(X, T))$ are homeomorphic [5]. Thus $(X_0, Q(X, T))$ is $Q1$. which implies $(X_0, Q(X, T))$ is (weakly $Q1$)o. Since each of (weakly $Q1$) and T_0 are topological properties, then, because of the homeomorphism, (X, T) is (weakly $Q1$)o. Thus $Q1$ implies (weakly $Q1$)o.

Suppose (X, T) has property (weakly $Q1$)o. Then (X, T) is T_0 and, as above, (X, T) and $(X_0, Q(X, T))$ are homeomorphic. Thus $(X_0, Q(X, T))$ has property (weakly $Q1$)o. which implies $(X_0, Q(X, T))$ has property $Q1$ and (X, T) has property $Q1$. Thus (weakly $Q1$)o implies $Q1$. Therefore $Q1 =$ (weakly $Q1$)o.

The next two results resolve the questions about what happens if the weakly $P1$ property process is repeated.

Theorem 2.7. *Let $Q1$ be a weakly $P1$ property. Then* (*weakly* $Q1$)1 *exists and* *equals* $Q1$.

Proof. Since (weakly $Q1$)1 = ((weakly $Q1$) and (T_0 and T_1))) = (((weakly $Q1$) and T_0) and T_1) = (((weakly $Q1$)o) and T_1) = (($Q1$) and T_1) = $Q1$, which exists, then (weakly $Q1$)1 exists and equals $Q1$.

Corollary 2.3. *Let* $Q1$ *be a weakly* $P1$ *property. Then* (*weakly* $Q1$)1 = $Q1$ = (*weakly* $Q1$)o.

Theorem 2.8. *Let* $Q1$ *be a weakly* $P1$ *property. Then* (*weakly* $Q1$)1 *is a weakly* $P1$ *property and weakly* (*weakly* $Q1$)1 = *weakly* $Q1$.

Proof. Since (weakly $Q1$)1 = $Q1$, then (weakly $Q1$)1 is a weakly $P1$ property and since (weakly $Q1$)1 = $Q1$, then weakly (weakly $Q1$)1 = weakly $Q1$.

If $Q1$ and $W1$ are weakly $P1$ properties and weakly $Q1$ = weakly $W1$, must $Q1 = W1$?

Theorem 2.9. *Let* $Q1$ *and* $W1$ *be weakly* $P1$ *properties. Then* $Q1 = W1$ *iff weakly* $Q1$ = *weakly* $W1$.

Proof. Clearly, if $Q1 = W1$, then weakly $Q1$ = weakly $W1$. Thus, consider the case that weakly $Q1$ = weakly $W1$. Then $Q1$ = (weakly $Q1$)o = (weakly Qo = $W1$.

Theorem 2.9 raised corresponding questions for weakly Po and T_0-identification P properties, which are resolved below.

Theorem 2.10. *Let* Qo *and* Wo *be weakly* Po *properties. Then* Qo = Wo *iff weakly* Qo = *weakly* Wo.

Proof. Clearly, if Qo = Wo, then weakly Qo = weakly Wo. Thus consider the case that weakly Qo = weakly Wo. Since weakly Qo and weakly Wo exist, then Po = (((weakly Po) and T_0)) and Qo = ((weakly Qo) and T_0) [2] and since weakly Qo = weakly Wo, then Qo = Wo.

Theorem 2.11. *Let* Q *and* W *be* T_0*-identification* P *properties. Then the* *following are equivalent*: (a) $Q = W$, (b) *weakly* Qo = *weakly* Wo, *and* (c) Qo = Wo.

Proof. (a) implies (b): As given above, since Q and W are T_0-identification P properties, then weakly Qo and weakly Wo exist, and Q = weakly Qo and W = weakly Wo. Thus weakly Qo = weakly Wo.

(b) implies (c): Since Qo = (weakly Qo)o and Wo = (weakly Wo)o, then Qo = Wo.

(c) implies (a): Since Qo = Wo, then Q = weakly Qo = weakly Wo = W.

Within the paper [6], it was proven that for a topological property P for which weakly Po exists, weakly Po is strictly weaker than Po and thus Po is not a T_0-identification P property. Must a similar statement be true for weakly P1 properties?. Within the paper [3], product spaces and subspaces of weakly Qo spaces were investigated raising questions about product spaces and subspaces of weakly P1 spaces. These questions are addressed and resolved in the section below.

3. Weakly P1 Properties, Product Spaces, and Subspaces

Theorem 3.1. *Let Q be a topological property for which weakly* Q1 *exists. Then weakly* Q1 *is neither* T_0 *nor* "*not*-T_0".

Proof. If weakly Q1 = "not-T_0", then Q1 = (weakly Q1)o = ("not-T_0" and T_0), which is a contradiction. If weakly $Q1 = T_0$, then Q1 = (weakly Q1)o = T_0, which is a contradiction. Thus weakly Q1 is neither T_0 nor "not-T_0".

Theorem 3.2. *Let Q be a topological property for which weakly* Q1 *exists. Then both* ((*weakly* Q1) *and* T_0) *and* ((*weakly* Q1) *and* "*not*-T_0") *exist*, *weakly* Q1 = Q1 or ((*weakly* Q1) *and* "*not* T_0"), *and weakly* Q1 *is strictly weaker than* Q1.

Proof. Since weakly Q1 is not T_0, then ((weakly Q1) and "not-T_0") exists. Since weakly Q1 is not "not-T_0", then ((weakly Q1) and T_0) exists. Thus weakly

$Q1 = (((\text{weakly } Q1) \text{ and } T_0)$ or ((weakly $Q1$) and "not-T_0")) = ($Q1$ or ((weakly $Q1$) and "not-T_0")), both of which exist and are distinct. Since $Q1$ implies ($Q1$ or ((weakly $Q1$) and "not-T_0")) = weakly $Q1$, then $Q1$ is strictly stronger than weakly $Q1$.

Theorem 3.3. *Let W be a topological property for which $W1$ exists. Then for each topological property Q such that weakly $Q1$ exists, weakly $Q1 \neq W1$.*

Proof. Suppose there exists a topological property Q such that weakly $Q1$ exists and weakly $Q1 = W1$. Then weakly $W1 =$ weakly (weakly $Q1$) $=$ weakly $Q1$. Thus $W1$ is a weakly $P1$ property and since weakly $W1 =$ weakly $Q1$, then $W1 = Q1$, but then $W1 = (W1$ or ((weakly $W1$) and "not-T_0")), which is a contradiction.

Theorem 3.4. *Let W be a topological property for which ((weakly $W1$) and "not-T_0") exists. Then for each topological property Q for which weakly $Q1$ exists, weakly $Q1 \neq$ ((weakly $W1$) and "not-T_0").*

Proof. Suppose there exists a topological property Q such that weakly $Q1$ exists and weakly $Q1 =$ ((weakly $W1$) and "not-T_0"). *Then* $Q1 =$ (weakly $Q1$)o $=$ (((weakly $Q1$) and "not-T_0") and T_0) does not exist, which is a contradiction.

Corollary 3.1. *Let Q be a topological property for which weakly $Q1$ exists. Then weakly $Q1 = (Q1$ or ((weakly $Q1$ and "not-T_0"))), neither of which are T_0-identification P or weakly $P1$ properties.*

In the following theorem, a topological property P for which the product space of a collection of topological spaces, with the Tychonoff topology, has property P iff each factor space has property P is called a product property.

Theorem 3.5. *Let $\mathcal{P} = \{Z \mid Z$ is a product property for which weakly $Z1$ exists$\}$, let $P \in \mathcal{P}$, let (X_α, T_α) be a space for each $\alpha \in A$, let $X = \prod_{\alpha \in A}$, and let W be the Tychonoff topology on X. Then (X_α, T_α) is weakly $P1$ for each $\alpha \in A$ iff (X, W) is weakly $P1$, and weakly $P1$ is a product property.*

Proof. Suppose (X_α, T_α) is weakly $P1$ for each $\alpha \in A$. Then (X_α, T_α) is ((weakly Po) and R_0) for each $\alpha \in A$, Since (X_α, T_α) is weakly Po for all $\alpha \in A$, then (X, W) is weakly Po [3] and since R_0 is a product property and (X_α, T_α) is R_0 for each $\alpha \in A$, then (X, W) is R_0. Thus (X, W) is ((weakly Po) and R_0) and (X, W) is weakly $P1$.

Conversely, suppose (X, W) is weakly $P1$. Then (X, W) is ((weakly Po) and R_0). Since (X, W) is weakly Po, then each factor space is weakly Po [3] and since (X, W) is R_0, then each factor space is R_0. Thus each factor space is ((weakly Po) and R_0) and each factor space is weakly $P1$.

Below, a topological property P for which a space has property P iff each subspace of the space has property P is called a subspace property.

Theorem 3.6. *Let* $\mathcal{S} = \{Z \mid Z$ *is a subspace property for which weakly* $Z1$ *exists*$\}$, *and let* $S \in \mathcal{S}$. *Then weakly* $S1$ *is a subspace property.*

Proof. *Let* (X, T) be weakly $S1$. Then (X, T) is ((weakly So) and R_0). Since weakly So is a subspace property, then every subspace of (X, T) is weakly So. Since R_0 is a subspace property, then every subspace of (X, T) is R_0. Thus every subspace of (X, T) is ((weakly So) and R_0) and every subspace of (X, T) is weakly $S1$.

Conversely, let (X, T) be a space for which every subspace of (X, T) is weakly $S1$. Then every subspace of (X, T) is weakly So, which implies (X, T) is weakly So and every subspace of (X, T) is R_0, which implies (X, T) is R_0. Thus (X, T) is weakly $S1$.

Therefore, weakly $S1$ is a subspace property.

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