Fundamental Journal of Mathematics and Mathematical Science
Vol. 3, Issue 1, 2015, Pages 83-90
This paper is available online at http://www.frdint.com/
Published online July 31, 2015

WEAKLY *P* PROPERTIES

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Abstract

For the topological property *P* of Hausdorff or Urysohn, there is a corresponding topological property called weakly *P* characterized by *P* and T_0-identification spaces: A space (X, T) is weakly *P* iff its T_0-identification space has property *P*. Thus the question of whether other topological properties and T_0-identification spaces could be used in the same manner as Hausdorff and Urysohn to define weakly *P* properties arose leading to the work in this paper. Within this paper, well-defined weakly *P* properties behaving in the same manner as for Hausdorff and Urysohn are defined and investigated, examples are given, and for weakly *P* properties, a never before imagined relationship between *P* and weakly *P* is given.

1. Introduction

In 1975, N. Levine submitted a paper, published in 1978 [6], in which he used nets to define and investigate a separation axiom weaker than T_2 that he named weakly Hausdorff. Within a 1975 paper [5], W. Dunham characterized weakly

Keywords and phrases: topological properties, T_0-identification spaces, weakly *P*.
2010 Mathematics Subject Classification: 54A05, 54B15, 54D10.
Received June 29, 2015; Accepted July 14, 2015

Hausdorff using T_0 -identification spaces.

T_0 -identification spaces were introduced in 1936 by M. Stone [7].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N [7].

Within the 1936 paper [7], it was shown that metrizable and pseudometrizable behave in the same manner as Hausdorff and weakly Hausdorff.

Theorem 1.1. *A space (X, T) is weakly Hausdorff iff its T_0 -identification space $(X_0, Q(X, T))$ is Hausdorff* [5].

In 1988 [2], the Urysohn separation axiom was generalized to weakly Urysohn and, as in the case of weakly Hausdorff, was characterized using T_0 -identification spaces.

Theorem 1.2. *A space (X, T) is weakly Urysohn iff its T_0 -identification space $(X_0, Q(X, T))$ is Urysohn* [2].

Definition 1.2. A space (X, T) is weakly Urysohn iff for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist open sets U and V such that $Cl(\{x\}) \subseteq U$, $Cl(\{y\}) \subseteq V$, and $Cl(U) \cap Cl(V) = \phi$ [2].

Thus the question of whether, as in the case of Hausdorff and Urysohn, T_0 -identification spaces could be used to uniquely define other weakly P properties behaving in the same manner as weakly Hausdorff and weakly Urysonh arose leading to the results below. If the answer to the question seems simple, work on the problem before reading further.

2. Generalized Weakly *P* Properties

Within the 1975 paper [5], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 2.1. A space (X, T) is R_1 iff for $x, y \in X$, such that

$Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

In the 1961 paper [1], A. Davis was interested in separation axioms R_i; $i = 0, 1$, respectively, which together with T_i, are equivalent to T_{i+1}, which led to the introduction of the R_1 and R_0 separation axioms.

Definition 2.2. A space (X, T) is R_0 iff for each $O \in T$ and for each $x \in O$, $Cl(\{x\}) \subseteq O$ [1].

The R_1 and R_0 separation axioms did exactly as intended: A space is R_i and T_i iff it is T_{i+1}, $i = 0, 1$, respectively [1]. Thus to move from weakly Hausdorff, which is R_1, to T_2, the T_0 separation axiom is required. The exact thing is true moving from weakly Urysohn to Urysohn [2]. Thus to generalize weakly Hausdorff and weakly Urysohn to weakly P, in the same manner as above, T_0 would be needed, which led to the definitions below.

Definition 2.3. Let P and S be topological properties. Then a space (X, T) satisfies the property P implies S iff (X, T) is a P space that is also S.

Theorem 2.1. *Let P and S be topological properties such that P implies S exists. Then P implies S is a topological property.*

The straightforward proof is omitted.

As stated above, for weakly P to exist, P must be P implies T_0, which will be denoted by Po.

Definition 2.4. Let P be a topological property for which Po exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

Thus for a topological property P to have the weakly Po property, it must be true that P implies T_0 exists raising the question of whether P could be T_0.

Definition 2.5. A space (X, T) is "not-T_0" iff there exist distinct elements $x, y \in X$ such that each open set containing one of x and y contains both x and y.

Theorem 2.2. "*Not-*T_0" *is a topological property.*

The proof is straightforward and omitted.

Theorem 2.3. *Weakly "not-*T_0*" is undefined.*

Proof. Since "not-T_0" does not imply T_0, weakly "not-T_0" does not exist.

Theorem 2.4. T_0 *is not a weakly Po property.*

Proof. Suppose weakly T_0 exists. Let (X, T) be T_0. Since for each space, its T_0-identification space is T_0 [7], then $(X_0, Q(X, T))$ is T_0. Let (Y, S) be "not-T_0". Then $(Y_0, Q(Y, S))$ is T_0. Thus T_0 implies weakly T_0 and "not-T_0 implies weakly T_0," which is a contradiction.

Theorem 2.5. *Let P be a topological property. Then* (X, T) *is weakly Po iff* $(X_0, Q(X, T))$ *has property Po.*

Proof. Suppose (X, T) is weakly *Po*. Then $(X_0, Q(X, T))$ has property *P* and since $(X_0, Q(X, T))$ is also T_0, then $(X_0, Q(X, T))$ has property *Po*.

The converse is immediate since *Po* implies *P*.

Theorem 2.6. *Let P be a topological property for which weakly Po is defined. Then weakly Po is unique.*

Proof. Let *S* and *W* be weakly *Po* properties. Let (X, T) be a space with property *S*. Then $(X_0, Q(X, T))$ has property *Po*, which implies (X, T) has property *W*. Thus *S* implies *W*. Similarly *W* implies *S* and *S* and *W* are equivalent.

Theorem 2.7. *Let* $f : (X, T) \to (Y, S)$ *be a homeomorphism. For each* $x \in X$, *let* C_x *be the element in* X_0 *containing x and for each* $y \in Y$, *let* K_y *be the element in* Y_0 *containing y and let* $f_0 : (X_0, Q(X, T)) \to (Y_0, Q(Y, S))$ *defined by* $f_0(C_x) = K_{f(x)}$. *Then* $f_0 : (X_0, Q(X, T)) \to (Y_0, Q(Y, S))$ *is a homeomorphism.*

Proof. Since $f : (X, T) \to (Y, S)$ is continuous and onto, $f_0 : (X_0, Q(X, T)) \to (Y_0, Q(Y, S))$ is continuous and onto [3]. Since *f* is open and onto, f_0 is open [3].

Since f is a homeomorphism, for each $u, v \in X$, $Cl_T(\{u\}) = Cl_T(\{v\})$ iff

$Cl_S(\{f(u)\}) = Cl_S(\{f(v)\})$. Let $C_u, C_v \in X_0$ such that $f_0(C_u) = f_0(C_v)$. Then $K_{f(u)} = K_{f(v)}$ and $Cl_S(\{f(u)\}) = Cl_S(\{f(v)\})$, which implies $C_u = C_v$. Thus f_0 is one-to-one and a homeomorphism.

Theorem 2.8. *Let P be a topological property such that weakly Po is defined, let* (X, T) *be weakly Po, and let* $f : (X, T) \to (Y, S)$ *be a homeomorphism. Then* (Y, S) *is weakly Po.*

Proof. Since (X, T) be weakly *Po*, $(X_0, Q(X, T))$ has property *P*. Since $f_0 : (X_0, Q(X, T)) \to (Y_0, Q(Y, S))$, as defined above, is a homeomorphism, then $(Y_0, Q(Y, S))$ has property *P*, which implies (Y, S) is weakly *Po*.

Thus weakly *Po* is a unique, topological property.

Investigations of T_0-identification spaces revealed that the natural map $N : (X, T) \to (X_0, Q(X, T))$ has strong properties: onto, continuous, open, closed, for each $O \in T$, $N^{-1}(N(O)) = O$, and for each closed set *C* in X, $N^{-1}(N(C)) = C$ [3], which, as in the case of weakly Hausdorff and weakly Urysohn, provide a convenient, strong path connecting weakly *Po* spaces and *P* spaces. Below another strong, useful property for T_0-identification spaces is given.

Theorem 2.9. *Let* (X, T) *be a space. Then* $(X_0, Q(X, T))$ *and* $((X_0)_0, Q(X_0, Q(X, T)))$ *are homeomorphic.*

Proof. Since $(X_0, Q(X, T))$ is T_0, then the natural map $N_0 : (X_0, Q(X, T)) \to ((X_0)_0, Q(X_0, Q(X, T)))$ is one-to-one and thus a homeomorphism [4].

Theorem 2.10. *Let P be a topological property that is neither* T_0 *nor "not-*T_0*". Then P implies* T_0 *and P implies "not-*T_0*" exist.*

The straightforward proof is omitted.

Theorem 2.11. *Let P be a topological property such that weakly Po exists. Then weakly Po is neither* T_0 *nor "not-*T_0*" and (weakly Po)o = (weakly Po) implies* T_0 *exists.*

Proof. Suppose weakly *Po* is T_0 or "not-T_0". Let (X, T) be weakly *Po*. Then $(X_0, Q(X, T))$ is T_0, but not *P*, which is a contradiction. Thus, by Theorem 2.10, (weakly *Po*)*o* exists.

Theorem 2.12. *Let P be a topological property. Then weakly Po exists iff weakly* ((*weakly Po*)*o*) = *weakly Po exists.*

Proof. Suppose weakly *Po* exists. Let (X, T) be a space. Then (X, T) is weakly *Po* iff $(X_0, Q(X, T))$ has property *P*. Since $(X_0, Q(X, T))$ and $((X_0)_0, Q(X_0, Q(X, T)))$ are homeomorphic, then $((X_0)_0, Q(X_0, Q(X, T)))$ has property *P* and $(X_0, Q(X, T))$ is weakly *Po*, which implies (X, T) is weakly ((weakly *Po*)*o*). Thus weakly ((weakly *Po*)*o*) = weakly *Po*.

Conversely, if weakly ((weakly *Po*)*o*) = weakly *Po*, then weakly *Po* exists.

Mathematical induction can be used to put any finite number of weakly to the let of ((weakly *Po*)*o*) in Theorem 2.12 and, thus for a topological property *P* for which weakly *Po* exists, only one topological property is generated by the use of weakly.

Corollary 2.1. *Let W be a topological property for which weakly Wo exists. Then weakly Wo is a weakly Po property.*

Theorem 2.13. *Let P be a topological property. Then weakly Po exists iff Po* = (*weakly Po*)*o*, *i.e.*, *a space* (X, T) *is Po iff it is weakly Po and* T_0.

Proof. Suppose weakly *Po* exists. Let (X, T) have property *Po*. Since (X, T) is homeomorphic to $(X_0, Q(X, T))$ which is homeomorphic to $((X_0)_0,$ $Q(X_0, Q(X, T)))$, then $((X_0)_0, Q(X_0, Q(X, T)))$ has property *Po*, which is equivalent to $(X_0, Q(X, T))$ is (weakly *Po*)*o*. Since (X, T) with the property (weakly *Po*)*o* is homeoporphic to $(X_0, Q(X, T))$ with property (weakly *Po*)*o*, then (X, T) with property *Po* is homeomorphic to (X, T) with the property (weakly *Po*)*o* and *Po* = (weakly *Po*)*o*.

Conversely, if *Po* = weakly *Po* implies T_0, then weakly *Po* exists.

Definition 2.6. Let $\mathcal{S}$ be a nonempty collection of topological properties. Then $\mathcal{S}$ has a least element *L* iff $L \in \mathcal{S}$ and for each $S \in \mathcal{S}$, S implies *L*.

Theorem 2.14. *Let P be a topological property and let* $\mathcal{S}$ *be a collection of topological properties S such that So exists and So implies Po. Then weakly Po exists iff weakly Po is the least element of* $\mathcal{S}$.

Proof. Suppose weakly Po exists. Since $Po = Po$ implies Po and Po implies Po, then $Po \in \mathcal{S}$ and $\mathcal{S} \neq \phi$. Let $S \in \mathcal{S}$. Let (X, T) has property So. Then $(X_0, Q(X, T))$ has property So, which implies $(X_0, Q(X, T))$ has property Po and (X, T) is weakly Po. By the results above weakly $Po \in \mathcal{S}$ and weakly Po is the least element of $\mathcal{S}$.

Conversely, if weakly Po is the least element of $\mathcal{S}$, then weakly Po exists.

Corollary 2.2. *Let P be a topological property. Then weakly Po exists iff weakly Po is the least element of all topological properties such that (S implies T_0) implies Po, i.e., weakly Po is the least element of all topological properties S for which S and T_0 implies Po.*

3. Examples

As established above, $R_0 =$ weakly $T_1 =$ weakly ((weakly $T_1)o) =$ weakly $(R_0)o$ and R_0 is the least of the topological properties S such that (S implies T_0) implies T_1. Also, from above $R_1 =$ weakly Hausdorff $=$ weakly ((weakly Hausdorff)o) $=$ weakly $(R_1)o$ and R_1 is the least of the topological properties S such that (S implies T_0) implies Hausdorff; weakly ((weakly Urysohn)o) $=$ weakly Urysohn and weakly Urysohn is the least of the topological properties S such that (S implies T_0) implies Urysohn; and pseudometrizable $=$ weakly metrizable $=$ weakly ((weakly metrizable)o) $=$ weakly (pseudometrizable)o and pseudometrizable is the least of the topological properties S such that (S implies T_0) implies metrizable.

Within the paper [3], it was proven that a space (X, T) is compact iff $(X_0, Q(X, T))$ is compact. Thus compact $=$ weakly (compact)o and compact is the least of the topological properties S such that (S implies T_0) implies compact and T_0. A search of the literature will give many more weakly Po properties.

References

[1] A. Davis, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly 68 (1961), 886-893.

[2] C. Dorsett, Generalized Urysohn spaces, Revista Columbiana de Mathematicas 22 (4) (1988), 149-160.

[3] C. Dorsett, T_0-identification spaces and R_1 spaces, Kyungpook Math. J. 18(2) (1978), 167-174.

[4] C. Dorsett, New characterizations of separation axioms, Bull. Cal. Math. Soc. 99(1) (2007), 37-44.

[5] W. Dunham, Weakly Hausdorff spaces, Kyungpook Math. J. 15(1) (1975), 41-50.

[6] N. Levine, Spaces in which the closure of a compact set is compact, Kyungpook Math. J. 18(1) (1978), 93-98.

[7] M. Stone, Application of Boolean algebras to topology, Mat. Sb. 1 (1936), 765-771.