Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 5, Issue 1, 2016, Pages 11-20
This paper is available online at http://www.frdint.com/
Published online February 29, 2016

WEAKLY *P* CORRECTIONS AND NEW, FUNDAMENTAL TOPOLOGICAL PROPERTIES AND FACTS

CHARLES DORSETT

Department of Mathematics
Texas A&M University-Commerce
Commerce, Texas 75429
USA
e-mail: charles.dorsett@tamuc.edu

Abstract

In 1975, T_0-identification spaces were used to further characterize weakly Hausdorff spaces raising the question of whether the process used in the characterization of weakly Hausdorff could be generalized and used to characterize other topological properties. In response, in 2015, weakly Po properties were introduced and investigated. Within the 2015 investigation, it was erroneously stated that for a topological property *P*, weakly Po exists iff weakly Po is the least element of $\mathcal{S} = \{S \mid S$ is a topological property, (S and T_0) exists, and (S and T_0) implies (P and T_0)$\}$. In this paper, an example is given showing the statement is incorrect, necessary corrections are given, and additional insights for weakly Po properties and foundation topology are given.

1. Introduction and Preliminaries

Within a 1975 paper [4], weakly Hausdorff was characterized using T_0-

Keywords and phrases: topological properties, T_0-identification spaces, weakly *P*.

2010 Mathematics Subject Classification: 54A05, 54B15, 54D10.

Submitted by T. M. G. Ahsanullah

Received December 24, 2015; Accepted January 27, 2016

identification spaces, which were introduced in 1936 [5].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T) [5].

Theorem 1.1. *A space* (X, T) *is weakly Hausdorff iff its* T_0*-identification space is Hausdorff* [4].

In 2015 [2], the question of whether T_0-identification spaces could be used to uniquely define other weakly P properties behaving in the same manner as weakly Hausdorff led to the introduction and investigation of weakly P properties.

Definition 1.2. Let P and S be topological properties. Then a space (X, T) has property P implies S iff (X, T) is a P space that also has property S [2].

For convenience, a topological property P for which P implies T_0 is denoted by Po.

Definition 1.3. Let P be a topological property for which Po exists. Then (X, T) is weakly Po iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

In the 2015 paper [2], it was proven that for a topological property P for which weakly Po exists, weakly Po is a unique, topological property.

Within the 1975 paper [4], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.4. A space (X, T) is R_1 iff for $x, y \in X$, such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

In the 1961 paper [1], separation axioms R_i; $i = 0, 1$, respectively, which together with T_i, are equivalent to T_{i+1} were sought, leading to the introduction of the R_1 and rediscovery of the R_0 separation axioms.

Definition 1.5. A space (X, T) is R_0 iff for each $O \in T$ and for each $x \in O$, $Cl(\{x\}) \subseteq O$ [1].

The R_1 and R_0 separation axioms did exactly as intended: A space is R_i and T_i iff it is T_{i+1}, $i = 0, 1$, respectively [1]. Thus to move from weakly Hausdorff, which is R_1, to T_2, the T_0 separation axiom is required.

Since for each space (X, T), $(X_0, Q(X, T))$ is T_0 [5], then, as given in the 2015 paper [2], a space is weakly Po iff its T_0-identification space has property Po.

In the 2015 paper [2], the following statement was given: "Let *P* be a topological property and let $\mathcal{S} = \{S \mid S$ is a topological property, So exists, and So implies $Po\}$. Then weakly Po exists iff weakly Po is the least element of $\mathcal{S}$." Fortunately, continued work on the weakly Po process has shown the statement to be false and, at the same time, revealed new fundamental topological properties and facts. Below an example is given showing the statement is false, new insights into weakly Po properties and foundation topology obtained from the continued study are given, and needed corrections are made.

2. Example

As given above, weakly $T_2 = R_1$. Let $W = (R_1$ or "not-T_0"), where "not-T_0" is the negation of T_0. Since each of R_1 and "not-T_0" is topological property, then *W* is a topological property and since (*W* and T_0) = ((R_1 or "not-T_0") and T_0) = (R_1 and T_0) = T_2 [1], then Wo exists and Wo implies T_2. Thus $W \in \mathcal{S} = \{S \mid S$ is a topological property, So exists, and So implies $T_2\}$. Since for propositions *P* and *Q*, *P* implies (*P* or *Q*), then R_1 implies *W*. Let *X* be a set with three or more elements, let *x* and *y* be distinct elements of *X*, and let $T = \{\phi, X, \{x, y\}\}$. Then (X, T) is "not-T_0", but not R_1, and *W* is weaker than R_1. Thus R_1 = weakly T_2 is not the least element of $\mathcal{S}$.

3. Additional Insights

With the focus on least topological properties, the question of whether there is a weakest topological property arose leading to the work below.

Theorem 3.1. $L = (T_0$ *or* "*not*-T_0") *is the least topological property*, Lo *is not a weakly* Po *property*, *and* "*not*-L" *does not exist*.

Proof. Since both T_0 and "not-T_0" are topological properties, then L is a topological property. Let P be a topological property. Then P is T_0 or P is "not-T_0" or P is ((P and T_0) or (P and "not-T_0")), each of which implies L. Thus P implies L. Hence L is the least topological property. Suppose weakly Lo exists. Let (X, T) be weakly Lo. Then $(X_0, Q(X, T))$ has property L, which in $(X_0, Q(X, T))$ is equivalent to Lo, but $Lo = (T_0$ or "not-T_0")$o = (T_0$ and $T_0) = T_0 \neq L$, which is a contradiction. Thus Lo is not a weakly Po property. Since "not-L" = not-(T_0 or "not-T_0") = T_0 and "not-T_0", which is a contradiction, then "not-L" does not exist.

Theorem 3.2. *Let* L *be as in Theorem* 3.1 *and let* P *be a topological property*. *Then the following are equivalent*: (a) *the negation of* P; "*not*-P", *exists*, (b) "*not*-P" *is a topological property*, P *is stronger than* L, *and* $P \neq$ "*not*-P", (c) $P \neq L$ *and* $P \neq$ "*not*-P", (d) P *is stronger than* L, *and* (e) "*not*-P" *is a topological property stronger than* L.

Proof. (a) implies (b): By Theorem 3.1, P implies L. If $P = L$, then "not-P" does not exist and thus $P \neq L$ and P is stronger than L. If $P =$ "not-P", then P implies (P and not-P"), which is a contradiction. Thus $P \neq$ "not-P". Let (X, T) be a space with property "not-P" and let (Y, S) be a homeomorphic image of (X, T). If (Y, S) has property P, then (X, T) has property P, which is a contradiction. Thus (Y, S) has property "not-P" and "not-P" is a topological property.

Clearly (b) implies (c).

(c) implies (d): Since P implies L and $P \neq L$, then P is stronger than L.

(d) implies (e): Since P is stronger than L, there exists a space (X, T) that is L and not P. Thus (X, T) is "not-P" and "not-P" exists. Then, by the argument above, "not-*P*" is a topological property. Then "not-*P*" is a topological property and "not-("not-*P*")" $= P$ exists and by the arguments above "not-*P*" is stronger than *L*.

(e) implies (a): Since "not-*P*" is stronger than *L*, then "not-*P*" exists.

As proven next, there are many ways to define *L*.

Theorem 3.3. *Let P be a topological property not L. Then* $L = (P$ *or* "*not-P*").

Proof. By Theorem 3.2, "not-*P*" exists and is a topological property. By Theorem 3.1, each of *P* and "not-*P*" implies *L* and, thus, (*P* and "not-*P*") implies *L*. Since T_0 is *P* or T_0 is "not-*P*" or $T_0 = ((T_0$ and *P*) or (T_0 and "not-*P*")), each of which implies (*P* or "not-*P*"), then T_0 implies (*P* or "not-*P*"). In a similar manner, "not-T_0" implies (*P* or "not-*P*") and *L* implies (*P* or "not-*P*"). Hence $L = (P$ or "not-*P*").

Of the many ways to represent *L*, the definition given above is, perhaps, the most basic and easily used, particularly in this paper.

Corollary 3.1. $\mathcal{D} = \{L\} \cup \{\{P, \text{“}not\text{-}P\text{”}\} \mid P$ *is a topological property not* $L\}$ *is a decomposition of the set of all topological properties*.

If desired, $\mathcal{D}$, given above, could be used to give the decomposition equivalence relation on the set of all topological properties.

Since "not-T_0" does not imply T_0, then, as stated in the 2015 paper [2], "not-T_0" is not a weakly *Po* property. A detailed proof that T_0 is not a weakly *Po* property is given next.

Theorem 3.4. T_0 *is not a weakly Po property*.

Proof. Suppose weakly T_0 exists. Since for each space (X, T), $(X_0, Q(X, T))$ is T_0 [5], then if (X, T) is T_0, $(X_0, Q(X, T))$ is T_0, which implies (X, T) is weakly T_0. Thus T_0 implies weakly T_0. In a similar manner, "not-T_0" implies weakly T_0. Since ((T_0 implies weakly T_0) and ("not-T_0" implies weakly T_0)) is equivalent to ((T_0 or "not-T_0") implies weakly T_0), then *L* implies weakly T_0. Since weakly T_0 implies *L*, then weakly $T_0 = L$, but, then *Lo* is a weakly *Po* property, which is a contradiction. Thus T_0 is not a weakly *Po* property.

Within the 2015 paper [2], the definitions of weakly Po and T_0-identification spaces were combined with the facts that for each space (X, T), $(X_0, Q(X, T))$ is T_0 and the natural map $N : (X, T) \to (X_0, Q(X, T))$ is a homeomorphism iff (X, T) is T_0 [3] to prove that for a weakly Po property Q, weakly Qo is a weakly Po property and (weakly Qo)$o = Qo$, which is used below.

Theorem 3.5. *Let P be a topological property for which weakly Po exists. Then* (*weakly* Po) $\neq L$, "*not*-(*weakly* Po)" *exists and is a topological property*, *both* (*P and* T_0) *and* (*P and* "*not*-T_0") *exist*, ("*not-P*")o = (("*not*-(Po)")o), *weakly* (("*not-P*")o) *exists*, *weakly* (("*not-P*")o) = *weakly*(("*not*-(Po)")o) = "*not*-(*weakly* Po)" $\neq$ *weakly* Po, *and* ("*not-P*")$o \neq Po$.

Proof. Since weakly Po exists, weakly Po is a topological property and Po = (weakly Po)o. If weakly $Po = L$, then Po = (weakly Po)$o = Lo = T_0$ is a weakly Po property, which is a contradiction. Hence weakly Po is stronger than L and "not-(weakly Po)" exists and is a topological property. Since weakly Po exists, then (P and T_0) exists. If (P and "not-T_0") does not exist, then P is T_0 and $Po = T_0$, which is a contradiction. Thus both (P and T_0) = Po and (P and "not-T_0") exist. If ("not-P" and T_0) does not exist, then ("not-P") is "not-T_0") and (P and T_0) = Po is T_0, which is a contradiction. Thus ("not-P" and T_0) exists. Also, (("not-(Po)")o) = ((not-(P and T_0)") and T_0) = (("not-P") or ("not-T_0")) and T_0) = (("not-P") and T_0) = ("not-P")o. Since a space is weakly Po iff its T_0-identification space has property P, then a space has property "not-(weakly Po)" iff its T_0-identification space has property "not-P". Thus a space has property "not-(weakly Po)" iff its T_0-identification space has property ("not-P")o, which implies weakly (("not-P")o) = "not-(weakly Po)" exists and weakly ("not-P")$o \neq$ weakly Po. Also, weakly (("not-(Po)")o) = weakly (("not-P")o). If ("not-P")$o = Po$, then weakly Po = weakly (("not-P")o) = "not-(weakly Po)", which is a contradiction.

Theorem 3.6. *Let Q be a topological property. Then Qo is a weakly Po property iff* ("*not-Q*")o *is a weakly Po property.*

Proof. By Theorem 3.5, if Qo is a weakly Po property, then (“not-P” $)o$ is a weakly Po property. Thus consider the case that (“not-Q” $)o$ is a weakly Po property. Then weakly ((“not-Q” $)o) \neq L$ and “not-(weakly (“not-Q” $)o)$” exists. Thus, by the results above, weakly ((“not-(“not-Q”)”)o) exists and weakly ((“not-(“not-Q”)”)o) $=$ weakly Qo exists. Hence Qo is a weakly Po property.

Corollary 3.2. *$\mathcal{D}o = \{\{Qo,$ “(not-(Q)” $o\} \mid Qo$ is a weakly Po property $\}$ is a decomposition of the set of all weakly Po properties.*

If desired, $\mathcal{D}o$, given above, could be used to give the decomposition equivalence relation on the set of all weakly Po properties.

Theorem 3.7. *Let Q be a topological property for which weakly Qo exists. Then weakly $Qo = ((\text{weakly } Qo)$ and $T_0))$ or ((weakly $Qo)$ and “not-T_0 ”) $= Qo$ or ((weakly $Qo)$ and “not-T_0 ”), where ((weakly $Qo)$ and “not-T_0 ”) exists and neither Qo nor ((weakly $Qo)$ and “not-T_0 ”) is weakly Po property.*

Proof. From above, ((weakly $Qo)$ and $T_0) = ($ weakly $Qo)o = Qo$, which exists. Since weakly Qo exists, then weakly $Qo \neq T_0$ and ((weakly $Qo)$ and “not-T_0 ”) exists. Thus weakly $Qo = Qo$ or ((weakly $Qo)$ and “not-T_0 ”), where neither Qo nor ((weakly $Qo)$ and “not-T_0 ”) is weakly Po property.

Corollary 3.3. *Let Q be a topological property for which weakly Qo exists. Then weakly ((“not-Q”$)o)$ exists, (weakly ((“not-Q”$)o))o =$ (“not-Q”$)o$, and weakly ((“not-Q”$)o) =$ (“not-Q”$)o$ or (weakly ((“not-Q”$)o)$ and “not-T_0 ”), where (weakly ((“not-Q”$)o)$ and “not-T_0 ”) exists and neither (“not-Q”$)o$ nor (weakly ((“not-Q”$)o)$ and “not-T_0 ”) is weakly Po property.*

Corollary 3.4. *If Q is a topological property that implies T_0,* weakly *Qo does not exist.*

Thus weakly Po does not exist for each of $T_0, T_1, T_2, T_3, T_{3(1/2)}, T_4, \ldots,$ and metrizable.

Corollary 3.5. *If Q is a topological property for which weakly Qo exists, then Q does not imply Qo.*

Returning to the question concerning $\mathcal{S}$, as given in Theorem 3.1, which led to the discoveries above and additional discoveries below, "*Is there a least topological property P such that* $Po \in \mathcal{S}$ *and each element of* $\mathcal{S}$ *implies P*?

4. Correction and Additional Insights

Theorem 4.1. *Let Q be a topological property for which weakly* Qo *exists and let* $\mathcal{S} = \{So \mid S$ *is a topological property*, So *exists*, *and* So *implies* $Qo\}$. *Then* (*weakly* Qo)$o \in \mathcal{S}$, *for each weakly* Po *property W such that* Wo *implies* Qo, (*weakly* Wo)$o \in \mathcal{S}$, *each element of* $\mathcal{S}$ *implies weakly* Qo, *and there exists a topological property* $Q_{\min}$ *weaker than weakly* Qo *such that* $(Q_{\min})o \in \mathcal{S}$.

Proof. *Since* Qo is a topological property and (weakly Qo)$o = Qo$, which implies Qo, then (weakly Qo)$o \in \mathcal{S}$. Thus $\mathcal{S} \neq \phi$. Let *W* be a weakly Po property such that Wo implies Qo. Let (X, T) be a space with property *W*. Then $(X_0, Q(X, T))$ has property *W*, which is equivalent to Wo in $(X_0, Q(X, T))$, which implies $(X_0, Q(X, T))$ has property Qo and (weakly Wo)$o = Wo \in \mathcal{S}$. Let $So \in \mathcal{S}$. Let (Y, Z) be a space with property So. Since So is a topological property and So is T_0, then the natural map $N : (Y, Z) \to (Y_0, Q(Y, Z))$ is a homeomorphism. Thus $(Y_0, Q(Y, Z))$ has So, which implies $(Y_0, Q(Y, Z))$ is Qo and (Y, Z) is weakly Qo. Hence, each element of $\mathcal{S}$ implies weakly Qo.

Let $Q_{\min}$ = ((weakly Qo) or "not-T_0"). Then $Q_{\min}$ is a topological property, (weakly Qo) implies $Q_{\min}$, and since weakly Qo is "not-T_0", then $Q_{\min}$ is weaker than weakly Qo. Since each element in $\mathcal{S}$ implies weakly Qo, and weakly Qo implies $Q_{\min}$, then each element of $\mathcal{S}$ implies $Q_{\min}$.

Theorem 4.2. *Let Q and* $\mathcal{S}$ *be as in Theorem* 4.1. *Then* $Q_{\min}$, *as given in Theorem* 4.1, *is the least topological property P weaker than weakly* Qo *such that* $Po \in \mathcal{S}$.

Proof. Suppose there exists a topological property *W* weaker than $Q_{\min}$ that is weaker than weakly Qo, implied by each element of $\mathcal{S}$ with $Wo \in \mathcal{S}$. Then *W* is (“not-$Q_{\min}$”) = ((“(not-(weakly Qo))” and T_0))) = (“not-Q”)o, which is a contradiction. Thus $Q_{\min}$ is the least topological property weaker than weakly Qo implied by each element of $\mathcal{S}$ with $Q_{\min} \in \mathcal{S}$.

Theorem 4.3. *Let Q be a weakly Po property and let $Q_{(\min,\max)}$ = ((weakly Qo) and (“not-T_0”)). Then $(Q_{(\min,\max)})o = Qo$ and $Q_{(\min,\max)}$ is the least topological property weaker than Qo and stronger than weakly Qo.*

Proof. By Theorem 3.7, $Q_{(\min,\max)}$ exists and is stronger than weakly Qo. Then $Q_{(\min,\max)}$ is the weakest topological property stronger than weakly Qo, for suppose not. Let W be a topological property weaker than $Q_{(\min,\max)}$ and stronger than weakly Qo. Then W is (((weakly Qo) and (“not-((weakly Qo) and “not-T_0”))) = ((weakly Qo) *and* T_0) = Qo, which is a contradiction. Thus $Q_{(\min,\max)}$ is the least topological property stronger than weakly Qo. Since $Q_{(\min,\max)}$ implies weakly Qo and (weakly Qo and T_0) = (weakly Qo)o = Qo, then (($Q_{(\min,\max)}$) and T_0) is Qo. Since Qo implies weakly Qo and Qo is not $Q_{(\min,\max)}$, then $Q_{(\min,\max)}$ is weaker than Qo.

Corollary 4.1. *Let Q be a topological property for which weakly Qo* exists and let $(n\text{-}\mathcal{S} = \{So \mid S$ is a topological property, So exists, and So implies (“not-Q”)$o\}$. *Then (“not-Q”)$_{\min}$ = ((weakly (“not-Q”)o) or “not-T_0”) is the least topological property P such that $Po \in (n\text{-}\mathcal{S}$ and is implied by each element of $(n\text{-}\mathcal{S})$ and (“not-Q”)$_{(\min,\max)}$ = ((weakly (“not-Q”)o) and “not-T_0”) is the least topological property W weaker than (“not-W”)o and stronger than weakly (“not-W”)o for which $Wo \in (n\text{-}\mathcal{S})$.*

As given above, weakly $T_2 = R_1$. If desired, the results above could be applied to T_2 and (“not-R_1)o and used to further characterize T_2 and to characterize (“not-R_1”)o.

Thus, as established above, “not-topological properties” have an important role in the study of topology. Thanks to the use of “not- topological properties”, it is now known there is a least topological property and unanswered questions now have answers. Continued study of “not-topological properties” will continue to reveal other important roles of “not-topological properties” within topology and other mathematical studies.

References

[1] A. Davis, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly 68 (1961), 886-893.

[2] C. Dorsett, Weakly *P* properties, Fundamental J. Math. Math. Sci. 3(1) (2015), 83-90.

[3] C. Dorsett, New characterizations of separation axioms, Bull. Cal. Math. Soc. 99(1) (2007), 37-44.

[4] W. Dunham, Weakly Hausdorff spaces, Kyungpook Math. J. 15(1) (1975), 41-50.

[5] M. Stone, Application of Boolean algebras to topology, Mat. Sb. 1 (1936), 765-771.