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WEAKLY Po AND PRODUCT PROPERTIES REVISITED

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Abstract

Within this paper, each of weakly Po and product properties continue to be investigated and many new characterizations of product properties are given.

1. Introduction and Preliminaries

Topological product properties were introduced in 1930 [11].

Definition 1.1. Let P be a topological property. Then P is a product property iff a product space, with the Tychonoff topology, has property P iff each factor space has property P [11].

The 1930 definition began the search within topology for product properties and not product properties and great progress has been made. However, there continued

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to be unanswered, very natural questions concerning product properties: (1) If *P* and *Q* are product properties, is (*P* and *Q*) a product property?, (2) If *P* is a topological property and each of *Q* and (*P* and *Q*) are product properties, must *P* be a product property? and (3) Is there some unknown topological property that is a product property by the 1930 definition that totally destroys the intent of the 1930 defined product properties?; just to give a few such questions. The answer to question (1) seems obvious, but, to prove the statement, one would have to prove first that (*P* and *Q*) exists. With the continual increase and diversity in 1930 defined product properties, nobody found tools needed to resolve the above question. However, that difficulty was overcome with the discovery of a never before imagined topological property within a 2016 paper [2] that, by the 1930 definition, is a product property, but created an unexpected reality for product properties as defined in 1930 that destroyed the 1930 intent of product properties. The never before imagined topological property was discovered in the continued study of T_0-identification spaces introduced in 1936 [10] and through the many years seemed far removed from product properties.

Definition 1.2. Let (X, T) be a space, let *R* be the equivalence relation on *X* defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of *R* equivalence classes of *X*, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map *N*. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T) [10].

Within the 1936 paper [10], T_0-identification spaces were used to jointly characterize pseudometrizable and metrizable.

Theorem 1.1. *A space is pseudometrizable iff its T_0-identification space is metrizable. T_0-identification spaces were cleverly created to add T_0 to the externally generated, strongly (X, T) related T_0-identification space of (X, T), making T_0-identification spaces a strong, useful topological tool* [10].

Similarly, in 1975 [9], T_0-identification spaces were used to jointly characterize the R_1 separation axiom and Hausdorff.

Definition 1.3. A space (X, T) is R_1 iff for $x, y \in X$ such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$ [1].

In a 2015 paper [3], the T_0-identification space property for each of pseudometrizable and R_1 was generalized to weakly Po.

Definition 1.4. Let P be a topological property for which $P\text{o} = (P$ and $T_0)$ exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

Within the 2015 paper [3], the search for a topological property that is not weakly Po led to the need and use of T_0 and "not-T_0". Thus both T_0 and "not-T_0" proved to be useful topological properties motivating the further investigation of T_0 and the addition of "not-*P*", where *P* is a topological property for which "not-*P*" exists, to the study of topology [3]. The addition of the many new properties provided tools not before used in the study of topology and, in a short time period, has revealed a mathematically fertile, never before imagined territory long overlooked within topology that has already changed the study of topology.

In the paper [2], the use of "not-T_0" and "not-*P*", where "not-*P*" exists, not only provided needed tools to prove the existence of the never before imagined least of all topological properties *L*, but, also, provided the needed tools for a quick, easily understood proof of the existence of *L*.

Theorem 1.2. *L*, *the least of all topological properties*, *is given by* $L = (T_0$ *or* "*not-*T_0") $= (P$ *or* "*not-P*"), *where P is a topological property for which* "*not-P*" *exists* [2].

Within the paper [2], it was shown that every space has property *L*. Thus each product space and each of its factor spaces simultaneously share property *L*, regardless of how diverse or even if factor spaces have properties that are known not product properties, and, by the 1930 definition, L is a product property, a reality far different than the intent of product properties in 1930.

The topological property L is unique in several special ways. As given above, (1) it is the least of all topological properties and (2) its existence created a disconnect in the study of product properties. In addition, it is the only topological property whose negation does not exist [4], which proved to be useful knowledge in the continued expansion of topology.

As given above, the existence of L created a disconnect in the study of product properties and, if possible, needed fixing. A quick, easy fix to restoring continuity in the study of product properties was the removal of L as a product property, which has proven to be a productive fix.

Definition 1.4. Let P be a topological property. Then P is a product property iff $P \neq L$ and a product space with the Tychonoff topology has property P iff each factor space has property P [2].

Within this paper, Definition 1.4 will be used as the definition of product properties. About the question: If P is a topological property and both Q and (P and Q) are product properties, must P be a product property? was asked, which is easily resolved using L and Definition 1.4. L is not a product property and for each product property Q, (L and Q) $= Q$ is a product property. In the paper [5], it is shown that, once again, L is unique in that it is the only topological property P for which both Q and (P and Q) are product properties and P is not a product property.

As given above, product properties have been long studied, but, not until a recent paper [4] was it known whether for arbitrary product properties P and Q, (P and Q) is a product property. Within that paper [4], properties of L and use of "not-P" were used to show that for product properties P and Q, (P and Q) is a product property, "not-P" is not a product property, and if P and W are product properties and (P and "not-W") exists, then (P and "not-W") is not a product property.

With the addition of the topological property weakly Po and the continual search for product properties, a natural question to ask is whether there is any connection between product properties and weakly Po? Below this question is resolved and never before known characterizations of product properties are given.

2. Additional Characterizations of Product Properties

Combining the results above gives the following useful characterization of product properties.

Corollary 2.1. *Let P be a topological property. Then P is a product property iff* $P \neq L$ *and for Q a product property*, (*P and Q*) *is a product property*.

Theorem 2.1. *Let P be a topological property and let W be any one of* T_0, T_1, T_2, *Urysohn*, T_3, *and* $T_{3\frac{1}{2}}$. *Then the following are equivalent*: (a) *P is a product property*, (b) $P \neq L$ *and* (*P and* T_0) *is a product property*, (c) $P \neq L$ *and* (*P and W*) *is a product property*, (d) $P \neq L$ *and for each product property Q*, (*P and Q*) *is a product property*, *and* (e) $P \neq L$ *and for some product property S*, (*P and S*) *is a product property*.

Proof. (a) implies (b): Since each of P and T_0 are product properties, then, by the results above, $P \neq L$ and (P and T_0) is a product property.

(b) implies (c): Since P is a topological property different from L and each of T_0 and (P and T_0) are product properties, then P is a product property. Since W is a product property, then (P and W) is a product property.

(c) implies (d): Let Q be a product property. Since $P \neq L$ and each of W and (P and W) are product properties, then P is a product property. Thus, both P and Q are product properties and (P and Q) is a product property.

(d) implies (e): Let S be a product property for which (P and S) is a product property. Then $P \neq L$ and each of S and (P and S) are product properties, which implies P is a product property.

(e) implies (a): Since $P \neq L$ and for some product property S, (P and S) are product properties, then P is a product property.

Thus never before known characterizations of product properties are now known.

3. Weakly Po and Product Properties

In the 2015 paper [3], it was shown that a space is weakly Po iff its T_0-identification space is weakly Po, motivating the introduction of T_0-identification P properties.

Definition 3.1. A topological property Q is a T_0-identification P property iff Q is simultaneously shared by both a space and its T_0-identification space [6].

Within that paper [6], it was shown that for a T_0-identification P property Q, $Q =$ weakly Qo, which for a while clouded the obvious: A topological property Q is weakly Po iff Q is a T_0-identification P property.

Initially, to search for a weakly Po property or equivalently, a T_0-identification P property, a classical topological property $Q\text{o} = (Q \text{ and } T_0)$ was chosen and a topological property W was sought such that if a space (X, T) has property W, then $(X_0, Q(X, T))$ has property Qo, which then implies (X, T) has property W, with no certainty that such a W exists. Not knowing which topological properties are weakly Po properties in the initial investigations made the requirement that weakly Po exist necessary. Had that search process continued, the study of weakly Po spaces and properties would continue to be uncertain, tedious, and never ending greatly hindering the exploration of the newly revealed mathematical territory. Thus to make the search process more certain, the question of precisely which topological properties are weakly Po properties arose leading to an answer in a 2017 paper [7].

Answer: $\{Q\text{o} \mid Q$ is a T_0-identification P property $\} = \{Q\text{o} \mid Q\text{o}$ is a weakly

Po property } = {Qo| Q is a topological property and Qo exists} [7].

Thus, the uncertainty of the starting place in the search for a weakly Po property was replaced with certainty.

By the Answer above, for a topological property Q for which Qo exists, Qo is a weakly Po property, raising the question: For what topological property W is W = weakly Qo?

Within the 2017 paper [7], for a topological property Q for which Qo exists, a property QNO was defined.

Definition 3.2. Let Q be a topological property such that Qo exists. A space (X, T) has property QNO iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo.

In that paper [7], it was shown that for a topological property Q for which Qo exists, QNO exists and is a topological property, and a space has property (Qo or QNO) iff its T_0-identification space has property (Qo or QNO). Thus for a topological property Q for which Qo exists, (Qo or QNO) is a T_0-identification P property and (Qo or QNO) = weakly (Qo or QNO)o. Since QNO is "not-T_0", then (Qo and QNO)o = Qo and (Qo or QNO) = weakly (Qo or QNO)o = weakly Qo.

Thus the question above has been resolved and the answer will be used below to further characterize product properties.

Theorem 3.1. *Let Q be a topological property for which Qo exists. Then the following are equivalent*: (a) *Qo is a product property*, (b) *(Qo or QNO) $\neq L$ and (Qo or QNO)o is a product property*, (c) *(Qo or QNO)o $\neq T_0$ and (Qo or QNO) is a product property*, (d) *(Qo or QNO) is a product space*, *and* (e) *(Qo or QNO) is a T_0-identification P product property.*

Proof. (a) implies (b): Since $(Q\text{o} \text{ or } QNO)\text{o} = Q\text{o}$, then $(Q\text{o} \text{ or } QNO)$ is a topological property and each of T_0 and $Q\text{o} = (Q\text{o} \text{ or } QNO)\text{o} = ((Q\text{o} \text{ or } QNO)$ and $T_0)$ are product properties, which implies $(Q\text{o} \text{ or } QNO)$ is a product property and, thus $(Q\text{o} \text{ or } QNO) \neq L$.

(b) implies (c): Since $(Q\text{o} \text{ or } QNO)\text{o}$ is a product property, then, by the argument above, $(Q\text{o} \text{ or } QNO)$ is a product property and $(Q\text{o} \text{ or } QNO)\text{o} \neq L$. If $(Q \text{ or } QNO)\text{o} = T_0$, then $(Q \text{ or } QNO)$ is a weakly $P\text{o}$ property and $(Q\text{o}$ or $QNO) =$ weakly $(Q\text{o} \text{ or } QNO)\text{o} =$ weakly T_0, but, since $L =$ weakly $L\text{o} =$ weakly T_0 [8] and weakly $P\text{o}$ is a unique topological property [3], then $(Q\text{o}$ or $QNO) = L$, which is a contradiction. Thus $(Q\text{o} \text{ or } QNO)\text{o}$ is not equal to T_0.

Clearly (c) implies (d).

(d) implies (e): Since $(Q\text{o} \text{ or } QNO)$ is a T_0-identification P property, then $(Q\text{o} \text{ or } QNO)$ is a T_0-identification P product property.

(e) implies (a): Since each of $(Q\text{o} \text{ or } QNO)$ and T_0 are product properties, then $((Q\text{o} \text{ or } QNO)$ and $T_0) = Q\text{o}$ is a product property.

From above, for each topological property Q for which $Q\text{o}$ exists, $(Q\text{o}$ or $QNO) =$ weakly $Q\text{o}$. Could there be another topological property M such that $M =$ weakly $Q\text{o}$?

Theorem 3.2. *Let M be a topological property such that* $M =$ *weakly* $Q\text{o}$ *for some topological property Q for which* $Q\text{o}$ *exists. Then* $M = (Q\text{o}$ *or* $QNO)$.

Proof. Since $M =$ weakly $Q\text{o}$, then a space (X, T) has property M iff $(X_0, Q(X, T))$ has property $Q\text{o}$ iff (X, T) has property $(Q\text{o} \text{ or } QNO)$. Thus, $M = (Q\text{o} \text{ or } QNO)$.

Combining the results above, gives the following result.

Corollary 3.1. *Let Q be a topological property for which weakly Qo exists and Q is a product property. Then* $(Q$o *or* $QNO)$ *is the only topological property M for which M is a product property that is weakly* Qo.

Theorem 3.3. *Let Q be a product property. Then weakly* $Q\text{o} \neq L$ *and* $Q\text{o} \neq T_0$.

Proof. By the results above, Q is a product property iff $Q\text{o} = (Q\text{o}$ or $QNO)\text{o}$ is a weakly Po product property and $Q\text{o} = (Q\text{o}$ or $QNO)\text{o} \neq T_0$. Since $L =$ weakly $L\text{o} =$ weakly T_0 [8] and weakly Po is a unique topological property, and for each topological property Q that is a product property, $(Q$ or $QNO)$ is weakly Po and a product property and $(Q$ or $QNO)\text{o} = Q\text{o} \neq T_0$, then weakly $Q\text{o} = (Q\text{o}$ or $QNO) \neq L$.

Thus, within the study of product properties, L and T_0 have unique roles. L is weakly Po and not a product property and $L\text{o} = T_0$ is a product property, but $T_0 \neq$ (weakly Qo$)\text{o} = Q\text{o}$ [3] for each product property Q.

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