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WEAKLY *P*2 AND RELATED PROPERTIES

CHARLES DORSETT

Department of Mathematics
Texas A&M University-Commerce
Commerce, Texas 75429
USA
e-mail: charles.dorsett@tamuc.edu

Abstract

In 1975, T_0-identification spaces were used to further characterize weakly Hausdorff spaces raising the question of whether the process used to characterize weakly Hausdorff could be generalized to include additional properties. The consideration of that question led to the introduction and investigation of weakly Po properties. As in the 1975 characterization of weakly Hausdorff, the Po separation axioms has a major role in the definition and properties of weakly Po properties. Thus the question of what would happen if T_0 in the definition of weakly Po was replaced by T_1 or T_2 arose leading to the definition and investigation of weakly $P1$ properties. Within this paper, the investigation continues with the definition and investigation of weakly $P2$ properties.

1. Introduction

In 1975 [8], T_0-identification spaces were used to further characterize weakly Hausdorff spaces.

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T_0-identification spaces were introduced in 1936 [9].

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Within the 1936 paper [9], T_0-identification spaces were used to further characterize pseudometrizable spaces.

Theorem 1.1. *A space (X, T) is pseudometrizable iff $X_0, Q(X, Q(X, T))$ is metrizable* [9].

Theorem 1.2. *A space (X, T) is weakly Hausdorff iff $(X_0, Q(X, T))$ is Hausdorff* [8].

In the 1975 paper [8], it was proven that weakly Hausdorff is equivalent to the R_1 separation axiom, which was introduced in 1961 [1].

Definition 1.2. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in V$ and $y \in V$ [1].

Within the 1961 paper [1], A. Davis was interested in separation axioms R_i, which together with T_i, are equivalent to T_{i+1}; $i = 0, 1$, respectively, leading to the definition of R_1 and the rediscovery of the R_0 separation axiom, which is weaker than R_1.

Definition 1.3. A space (X, T) is R_0 iff for each $O \in T$ and each $x \in O$, $Cl(\{x\}) \subseteq O$ [1].

The separation axioms R_i; $i = 0, 1$ satisfied Davis' expectations [1].

Within a recent paper [2], weakly Hausdorff was generalized to weakly *Po* properties.

Definition 1.4. Let *P* and *S* be topological properties. Then a space has property

P implies S iff the space is a P space that satisfies S [2].

For convenience, for a topological property P, P implies T_0 is denoted by Po.

Definition 1.5. Let P be a topological property for which Po exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

As a result of the role of T_0 in the weakly Po property process, within the introductory paper [2], it was proven that for a topological property P for which weakly Po exists, a space is weakly Po iff its T_0-identification space has property Po.

Even though weakly Po properties were undefined at the time, since (pseudometrizable)*o* equals metrizable, metrizable was the first known weakly Po property and weakly (metrizable) = pseudometrizable. Within the paper [2], it was established that both T_2 and T_1 are weakly Po properties, with weakly $T_2 = R_1$ and weakly $T_1 = R_0$.

In the introductory weakly Po property paper [2], it was shown that both T_0 and "not-T_0" are not weakly Po properties, where "not-T_0" is the negation of T_0. Also, within the paper [2], it was shown that a space is weakly Po iff its T_0-identification space is weakly Po. The combination of this result with the fact that other topological properties are simultaneously shared by a space and its T_0-identification space led to the introduction and investigation of T_0-identification P properties, which generalize weakly Po properties [3].

Definition 1.6. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously share property S [3].

Within the paper [4], it was proven that both R_0 and R_1 are T_0-identification P properties.

As in the case of weakly Po properties, both T_0 and "not-T_0" fail to be T_0-identification P properties [3] and weakly $P1$ properties [5]. Within the paper [5], the knowledge and insights obtained from the investigations of weakly Po and T_0-

identification P properties were used to define and investigate weakly $P1$ properties and to further investigate weakly Po and T_0-identification P properties.

For convenience of notation, let $P1$ denote P implies T_1.

Definition 1.7. Let P be a topological property for which $P1$ exists. Then (X, T) is weakly $P1$ iff $(X_0, Q(X, T))$ is $P1$. A topological property $P1$ for which weakly $P1$ exists is called a weakly $P1$ property.

In this paper, the investigation continues with the introduction and investigation of weakly $P2$ properties.

2. Weakly $P2$ Properties

For convenience of notation, let $P2$ denote P implies T_2.

Definition 2.1. Let P be a topological property for which $P2$ exists. Then (X, T) is weakly $P2$ iff $(X_0, Q(X, T))$ is $P2$. A topological property $P2$ for which weakly $P2$ exists is called a weakly $P2$ property.

Note that the definition of weakly $P2$ is totally consistent with the definitions of weakly Po and weakly $P1$ properties.

Theorem 2.1. *Let P be a topological property for which $P1$ exists. Then* $(P2)1 = (P2)o = P2 = P1$ *and* $R_1 = Po$ *and* R_1.

Proof. Since $P2$ implies each of T_0 and T_1, we have $(P2)1$ and $(P2)o$ exist, and $(P2)1 = ((P$ and $T_2)$ and $T_1) = P$ and $(T_2$ and $T_1) = P$ and $T_2 = P2$, $(P2)o = (P$ and $T_2)$ and $T_o = P$ and $(T_2$ and $T_0) = P$ and $T_2 = P2$, $P2 = P$ and $T_2 = P$ and $(T_1$ and $R_1) = (P$ and $T_1)$ and $R_1 = P1$ and $R_1 = (P$ and $(T_0$ and $R_0))$ and $R_1 = (P$ and $T_0)$ and $(R_0$ and $R_1)) = Po$ and R_1.

Theorem 2.2. *Let Q be a topological property for which $Q2$ exists. Then the following are equivalent*: (a) $Q2$ *is a weakly* $P2$ *property*, (b) $Q2$ *is a weakly* $P1$ *property*, (c) $Q2$ *is a weakly* Po *property*, (d) *weakly* $Q2 =$ (*weakly* $Q1$) *and* R_1, *and* (e) *weakly* $Q2 =$ (*weakly* Qo) *and* R_1.

Proof. (a) implies (b): Since $(Q2)1 = Q2$ and $Q2$ is a weakly $P2$ property, weakly $(Q2)1 =$ weakly $Q2$ exists and $Q2$ is a weakly $P1$ property.

(b) implies (c): Since $(Q2)o = (Q2)1 = Q2$ and $Q2$ is a weakly $P1$ property, weakly $Q2 =$ weakly $(Q2)o$ exists and $Q2$ is a weakly Po property.

(c) implies (d): Since $Q2$ is a weakly Po property, then weakly $Q2 =$ weakly $(Q2)o$ exists and $Q2$ is a weakly $Q2$ property. Let (X, T) be a space. Then (X, T) is weakly $Q2$ iff $(X_0, Q(X, T))$ is $Q2 = Q1$ and R_1 iff $(X_0, Q(X, T))$ is $Q1$ and $(X_0, Q(X, T))$ is R_1 iff (X, T) is (weakly $Q1$) and (X, T) is R_1. Thus weakly $Q2 =$ (weakly $Q1$) and R_1.

(d) implies (e): Since weakly $Q1 =$ (weakly Qo) and R_0, then weakly $Q2 =$ (weakly $Q1$) and R_1 = ((weakly Qo) and R_0) and R_1 = (weakly Qo) and (R_0 and R_1) = (weakly Qo) and R_1.

(e) implies (a): Since weakly $Q2$ exists, $Q2$ is a weakly $P2$ property.

Corollary 2.1. *Let $Q2$ be a weakly $Q2$ property. Since weakly $Q2$ is a weakly Po property, weakly $Q2$ is neither T_0 nor "not-T_0" and both ((weakly $Q2$) and T_0) and ((weakly $Q2$) and "not-T_0") exist.*

Corollary 2.2. *Let $Q2$ be a weakly $P2$ property. Then $Q2$ is a weakly $P1$ property and Qo is a weakly Po property.*

Theorem 2.3. *Let $Q2$ be a weakly $P2$ property. Then weakly $Q2$ is a topological property.*

Proof. Since weakly $Q2 =$ (weakly Qo) and R_1, weakly Qo is a topological property [2], and R_1 is a topological property, then weakly $Q2$ is a topological property.

Within the paper [4], it was shown that compact is a T_0-identification P property. Since (compact)o exists, (compact)o is a weakly Po property, since (compact)1 exists, (compact)1 is a weakly $P1$ property, and since (compact)2 exists, (compact)2 is a weakly $P2$ property. Thus, the converse of Corollary 2.2 is not true. Also, the example shows that weakly Po, weakly $P1$, and weakly $P2$ can all be different raising the question of when all three are equal.

Theorem 2.4. *Let* $Q2$ *be a weakly* $P2$ *property. Then the least topological property P for which* T_0*-identification* P = *weakly* Po = *weakly* $P1$ = *weakly* $P2$ *is* R_1.

Proof. Since R_1 is a T_0-identification property and weakly $(R_1)o$ = weakly $(R_1)1$ = weakly $(R_1)2 = R_1$, R_1 satisfies the required property. Let $Q2$ be a weakly $P2$ property satisfying the requirements. Then weakly $Q2$ = (weakly Qo) and R_1, which implies R_1. Thus R_1 is the least topological property satisfying the required properties.

A natural question to pose at this point is "If $Q2$ and $W2$ are weakly $P2$ properties and weakly $Q2$ = weakly $W2$, must $Q2 = W2$?", which is resolved below.

Theorem 2.5. *Let* $Q2$ *be a weakly* $P2$ *property and let* (X, T) *be a space. Then the following are equivalent*: (a) $(X_0, Q(X, T))$ *has property* $Q2$, (b) $(X_0, Q(X, T))$ *is weakly* $Q2$, *and* (c) $(X_0, Q(X, T))$ *is* (*weakly* $Q2$)*o*.

Proof. (a) implies (b): Since $(X_0, Q(X, T))$ is homeomorphic to $((X_0)_0, Q(X_0, Q(X_0, Q(X, T))))$ [2], then $((X_0)_0, Q(X_0, Q(X_0, Q(X, T))))$ has property $Q2$, which implies $(X_0, Q(X, T))$ is weakly $Q2$.

(b) implies (c): Since $(X_0, Q(X, T))$ is T_0 [9], $(X_0, Q(X, T))$ is (weakly $Q2$)*o*.

(c) implies (a): Since $(X_0, Q(X, T))$ is (weakly $Q2$)*o*, $(X_0, Q(X, T))$ is weakly $Q2$. Then $((X_0)_0, Q(X_0, Q(X_0, Q(X, T))))$ has property $Q2$, which, by the homeomorphic given above, implies $(X_0, Q(X, T))$ has property $Q2$.

Corollary 2.3. *Let* $Q2$ *be a weakly* $P2$ *property and let* (X, T) *be a space. Then* (X, T) *is weakly* $Q2$ *iff* $(X_0, Q(X, T))$ *is weakly* $Q2$.

Corollary 2.4. *Let* $Q2$ *be a weakly* $P2$ *property. Then weakly* $Q2$ *is a* T_0*-identification P property.*

Theorem 2.6. *Let* $Q2$ *be a weakly* $P2$ *property. Then* $Q2$ = (*weakly* $Q2$)*o*.

Proof. Let (X, T) be a space. Suppose (X, T) has property $Q2$. Then (X, T) is T_0 and (X, T) and $(X_0, Q(X, T))$ are homeomorphic [6]. Thus $(X_0, Q(X, T))$ is $Q2$, which implies $(X_0, Q(X, T))$ is (weakly $Q2)o$. Since each of (weakly $Q2$) and T_0 are topological properties, then, because of the homeomorphism, (X, T) is (weakly $Q2)o$. Thus $Q2$ implies (weakly $Q2)o$.

Suppose (X, T) has property (weakly $Q2)o$. Then (X, T) is T_0 and (X, T) and $(X_0, Q(X, T))$ are homeomorphic [6]. Thus $(X_0, Q(X, T))$ has property (weakly $Q2)o$, which implies $(X_0, Q(X, T))$ has property $Q2$ and (X, T) has property $Q2$. Thus (weakly $Q2)o$ implies $Q2$.

Therefore $Q2 =$ (weakly $Q2)o$.

The next result resolves the questions about what happens if the weakly $P2$ property process is repeated.

Theorem 2.7. *Let $Q2$ be a weakly $Q2$ property. Then weakly (weakly $Q2$) = weakly $Q2$.*

Proof. Let (X, T) be a space. Then (X, T) is weakly $Q2$ iff $(X_0, Q(X, T))$ is $Q2$ iff $(X_0, Q(X, T))$ is weakly $Q2$ iff (X, T) is weakly (weakly Q2). Thus weakly (weakly $Q2$) = weakly $Q2$.

If $Q2$ and $W2$ are weakly $P2$ properties and weakly $Q2 =$ weakly $W2$, must $Q2 = W2$?

Theorem 2.8. *Let $Q2$ and $W2$ be weakly $P2$ properties. Then $Q2 = W2$ iff weakly $Q2 =$ weakly $W2$.*

Proof. Clearly, if $Q2 = W2$, then weakly $Q2 =$ weakly $W2$. Thus, consider the case that weakly $Q2 =$ weakly $W2$. Then $Q2 =$ (weakly $Q2)o =$ (weakly $W2)o$ $=$ W2.

Within the paper [7], it was proven that for a topological property P for which weakly Po exists, weakly Po is strictly weaker than Po and thus Po is not a T_0-identification P property. Must a similar statement be true for weakly $P2$ properties?

Theorem 2.9. *Let $Q2$ be a weakly $P2$ property. Then weakly $Q2$ is strictly weaker than $Q2$ and $Q2$ is not a T_0-identification P property.*

Proof. Since weakly $Q2 =$ weakly $Q2)o$, weakly $Q2 =$ weakly $(Q2)o$ is strictly weaker than $(Q2)o = Q2$ and $Q2$ is not a T_0-identification *P* property.

Theorem 2.10. *Let $Q2$ be a weakly $P2$ property and let $\mathcal{S} = \{S \mid S$ is a topological property, So exists, and So implies $Q2\}$. Then $\mathcal{S} = \phi$ and weakly $Q2$ is the least element of $\mathcal{S}$.*

Proof. Since weakly $Q2$ is a topological property and (weakly $Q2)o = Q2$, weakly $Q2 \in \mathcal{S}$. Let $S \in \mathcal{S}$. Then (S and weakly $Q2$) is a topological property, (S and weakly $Q2)o$ implies So, and So implies $Q2$, which implies (S and weakly $Q2) \in \mathcal{S}$. Thus for each $S \in \mathcal{S}$, (S and weakly $Q2) \in \mathcal{S}$. Since for each $S \in \mathcal{S}$, (S and weakly $Q2)$ implies weakly $Q2$, then for each $S \in \mathcal{S}$, S implies weakly $Q1$. Hence weakly $Q2$ is the least element of $\mathcal{S}$.

Theorem 2.11. *Of all the topological properties S such that So implies T_2, R_1 is the least such topological property.*

Proof. Since weakly $T_2 = R_1$, R_1 is the least such topological property.

Theorem 2.12. *Let $Q2$ be a weakly $P2$ property. Then weakly $Q2 =$* ((weakly *$Q2)$ and $T_0)$ or ((weakly $Q2)$ and "not-T_0"), where both ((weakly $Q2$ and $T_0)$ and ((weakly $Q2)$ and "not-T_0") exist and neither are weakly $P2$ properties.*

Proof. By Corollary 2.1, both ((weakly $Q2)$ and $T_0)$ and ((weakly $Q2)$ and "not-T_0") exist. Thus weakly $Q2 =$ ((weakly $Q2)$ and $T_0)$ or ((weakly $Q2)$ and "not-T_0"), where both ((weakly $Q2)$ and $T_0)$ and ((weakly $Q2)$ and "not-T_0") exist. Since ((weakly $Q2)$ and "not-T_0") does not imply T_2, ((weakly $Q2)$ and "not-T_0") is not a weakly $P2$ property. Since ((weakly $Q2)$ and $T_0) =$ (weakly $Q2)o = Q2$ and weakly $Q2$ is strictly weaker than $Q2$, ((weakly $Q2)$ and $T_0)$ is not a weakly $P2$ property.

Corollary 2.5. *Each weakly $P2$ property can be decomposed into two distinct topological properties, neither of which are weakly $P2$ properties.*

When investigating topological properties, questions concerning product spaces and subspaces naturally arise. Below known properties of weakly Po product spaces and weakly Po subspaces are used to answer questions concerning product spaces and subspaces of weakly $P1$ and weakly $P2$ properties.

3. Product Spaces and Subspaces of Weakly $P1$ and Weakly $P2$ Properties

In this section, a topological property P for which the product of a collection of spaces, with the Tychonoff topology, has property P iff each factor space has property P is called a product property.

Theorem 3.1. *Let* $\mathcal{P} = \{Z \mid Z$ *is a topological and product property for which weakly* $P1$ *exists*$\}$. *Let* $P \in \mathcal{P}$, *let* (X_α, T_α) *be a space for each* $\alpha \in A$, $X = \prod_{\alpha \in A} X_\alpha$, *and let W be the Tychonoff topology on X. Then* (X_α, T_α) *is weakly* $P1$ iff (X, W) *is weakly* $P1$.

Proof. Suppose (X_α, T_α) is weakly $P1$ for each $\alpha \in A$ Since weakly $P1 =$ (weakly Po) and R_0 [5], then weakly Po exists and (X_α, T_α) is (weakly Po) and R_0 for each $\alpha \in A$ and (X, W) is weakly Po [5] and R_0, which implies (X, W) is weakly $P1$.

Conversely, suppose (X, W) is weakly $P1$. Then (X, W) is (weakly Po) and R_0, which implies (X_α, T_α) is (weakly Po) [5] and R_0 for each $\alpha \in A$ and thus weakly $P1$.

Theorem 3.2. *Let* $\mathcal{P}$ *be as in Theorem* 3.1 *and let* $P \in \mathcal{P}$. *Then* (*weakly* $P1$) *and* $P1$ *are in* $\mathcal{P}$.

Proof. Since weakly $P1$ is a topological property, weakly $P1$ is a product property, and weakly (weakly $P1$) $=$ weakly $P1$ [5], then weakly $P1 \in \mathcal{P}$.

Since $P1 =$ (weakly Po) and T_0 [5], both of which are topological and product properties, then $P1 \in \mathcal{P}$.

Using the results above in this paper and arguments similar to those of Theorem 3.1 and Theorem 3.2, (weakly $P1$) and $P1$ in Theorems 3.1 and 3.2 can be replaced by weakly $P2$ and $P2$, respectively.

A topological property *P* for which a space has property *P* iff each subspace has property *P* is called a subspace property.

Theorem 3.3. *Let* $\mathcal{S} = \{Z \mid Z$ *is a topological, subspace property and weakly* $P1$ *exists* $\}$. *Let* $S \in \mathcal{S}$. *Then weakly* $S1$ *is a subspace property.*

Proof. Suppose (X, T) is weakly $S1$. Then weakly $S1 =$ (weakly So) and R_0, where weakly So is a subspace property [5] and R_0 is a subspace property [5], which implies each subspace of (X, T) is (weakly So) and R_0 = weakly $S1$.

Conversely, suppose each subspace of (X, T) is weakly $S1$. Then each subspace of (X, T) is (weakly So) and R_0, which implies (X, T) is (weakly So) and R_0 = weakly $S1$.

Theorem 3.4. *Let* $\mathcal{S}$ *be as in Theorem* 3.3 *and let* $S \in \mathcal{S}$. *Then* (*weakly* $S1$) *and* $S1$ *are in* $\mathcal{S}$.

Proof. Since (weakly $S1$) is a topological, subspace property and weakly (weakly $S1$) = weakly $S1$, then (weakly $S1$) $\in \mathcal{S}$.

Since $S1 =$ (weakly $S1$) and T_0, where both (weakly $S1$) and T_0 are topological, subspace properties, then $S1$ is a topological, subspace property.

Using the results above and arguments similar to those of Theorems 3.3 and 3.4, (weakly $S1$) and $S1$ in Theorems 3.3 and 3.4 can be replaced by (weakly $S2$) and $S2$, respectively.

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