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UNIQUE, FOUNDATIONAL PROPERTIES OF REGULAR, T_3, WEAKLY URYSOHN AND URYSOHN

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Abstract

In this paper additional unique, foundational roles for each of regular, T_3, weakly Urysohn, and Urysohn in the study of topology are established.

1. Introduction and Preliminaries

The regular separation axiom was introduced in 1921 [9].

Definition 1.1. A space (X, T) is regular iff for each closed set C and each $x \notin C$, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$. A regular T_1 space is denoted by T_3.

The further investigation of T_3 revealed that the requirement of T_1 in the definition of T_3 can be replaced by the weaker requirement of T_0 [1]. Thus a space

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is T_3 iff it is (regular and T_0) giving each of regular and T_0 an important role in defining T_3, but, at the same time raising additional questions about their roles: (1) Are there topological properties other than regular and T_3, which together with T_0, equals T_3 ? (2) If there are others, what are they? (3) If there are others, is there a least one? (4) If there are others, is there a strongest one? Regular and T_3 are long-defined, long-investigated, and long-used topological properties, but, as given above, there continues to be unanswered questions concerning each of them.

The Urysohn separation axiom was introduced in 1925 [8].

Definition 1.2. A space (X, T) is Urysohn iff for distinct elements *x* and *y* in *X*, there exist open sets *U* and *V* such that $x \in U$, $y \in V$, and $Cl(U) \cap Cl(V) = \phi$.

Unlike T_3, there was no topological property given in the 1925 paper [8] that played the role for Urysohn as regular did for T_3. Not until 1988 [2] was such a property defined.

Definition 1.3. A space (X, T) is weakly Urysohn iff for elements *x* and *y* in *X* such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist open sets *U* and *V* such that $x \in U$, $y \in V$, and $Cl(U) \cap Cl(V) = \phi$.

Within a 2015 paper [3], it was shown that a space is Urysohn iff it is (weakly Urysohn and T_0). Thus weakly Urysohn plays the same role for Urysohn as regular does for T_3 and the same questions for Urysohn and weakly Urysohn, as given above for T_3 and regular, arise.

The continued investigation of regular and T_3 revealed that T_3 is a weakly Po property with regular = weakly (regular)o = weakly T_3 [1]. Likewise, the continued investigation of weakly Urysohn and Urysohn revealed that Urysohn is a weakly Po property with (weakly Urysohn) = weakly (weakly Urysohn)o = weakly Urysohn [3].

Weakly Po spaces and properties were introduced in 2015 [4].

Definition 1.4. Let *P* be a topological property for which $P\text{o} = (P \text{ and } T_0)$

exists. Then a space is weakly Po iff its T_0-identification space has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

T_0-identification spaces were introduced in 1936 [7].

Definition 1.5. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the natural map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

In the paper [3], it was believed that (weakly Urysohn) was the least topological property, which together with T_0, equals Urysohn. However, further investigations of weakly Po spaces and properties determined that not to be true. Below recent discoveries for weakly Po spaces and properties are applied to correct the mistake above and to resolve the questions above about regular, T_3, (weakly Urysohn), and Urysohn.

2. Applications of Weakly Po Spaces and Properties for Regular and T_3

Within the paper [5], it was proven that for a weakly Po property Qo, the least topological property, which together with T_0, equals Qo is ((weakly Qo) or "not-T_0"). Applying this result to the weakly Po property T_3 gives the following result.

Corollary 2.1. *(Regular or "not-T_0") is the least topological property, which together with T_0, equals T_3.*

The continued investigation of weakly Po spaces and properties revealed that for a weakly Po property Qo, Qo is the only topological property stronger than weakly Qo, which together with T_0, equals Qo; the only topological property weaker than ((weakly Qo) and "not-T_0"), which together with T_0, equals Qo, and implies weakly Qo is weakly Qo; there are exactly two topological properties, Qo and weakly Qo, stronger than or equal to weakly Qo, which together with T_0, equals Qo; and the only topological properties weaker than weakly Qo, which together with T_0, equals Qo are ((weakly Qo or "not-T_0") or (weakly Qo or F), where F is a topological property that implies "not-T_0" [6]. Applying these results to regular and T_3 gives the following unique, foundational properties for regular and T_3.

Corollary 2.2. *T_3 is the only topological property stronger than regular, which together with T_0, equals T_3.*

Corollary 2.3. *Regular is the only topological property weaker than (regular and "not-T_0"), which together with T_0, equals T_3 and implies regular.*

Corollary 2.4. *T_3 and regular are the only two topological properties stronger than or equal to regular, which together with T_0, equals T_3.*

Corollary 2.5. *(Regular or "not-T_0") and (regular or F), where F is a topological property that implies "not-T_0", are the only topological properties weaker than regular, which together with T_0, equals T_3.*

3. Applications for Weakly Urysohn and Urysohn

Corollary 3.1. *((Weakly Urysohn) or "not-T_0") is the least topological property, which together with T_0, equals Urysohn.*

Corollary 3.2. *Urysohn is the only topological property stronger than weakly Urysohn, which together with T_0, equals Urysohn.*

Corollary 3.3. *Weakly Urysohn is the only topological property weaker than ((weakly Urysohn) and "not-T_0"), which together with T_0, equals Urysohn and implies weakly Urysohn.*

Corollary 3.4. *Urysohn and weakly Urysohn are the only two topological properties stronger than or equal to weakly Urysohn, which together with T_0, equals Urysohn.*

Corollary 3.5. ((*Weakly Urysohn*) *or* "*not*-T_0") *and* ((*weakly Urysohn*) *or* F), *where* F *is a topological property that implies* "*not*-T_0", *are the only topological properties weaker than weakly Urysohn*, *which together with* T_0, *equals Urysohn*.

Thus, as established above, the study of weakly Po spaces and properties has been a productive study not only raising questions not asked before, but, also providing a vehicle for resolution of those questions.

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