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TOPOLOGICAL SUBSPACES; CORRECTIONS AND PROGRESS

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Abstract

The continued investigation of topological properties has revealed the need for corrections in the literature concerning subspace properties and other properties, which are made in this paper, but, fortunately, the needed corrections do not change the main results. Also, subspace and non-subspaces properties are further investigated and additional information is given for each of the two properties.

1. Introduction and Preliminaries

In early studies of topology, it was observed that there are topological properties with the property that for each space with the property, every subspace of the space has the property. Thus subspace properties were introduced and the search for

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subspace properties began.

Definition 1.1. Let P be a topological property. Then P is a subspace property iff a space has property P iff every subspace of the space has property P.

A natural question to ask in the study of subspace properties was whether or not every topological property is a subspace property? To resolve that question, the question of whether or not there is a topological property Q for which there exists a space with property Q with a subspace that does not have property Q was asked? The answer to the second question was yes, and thus there is a topological property that is not a subspace property, making the study of subspace properties meaningful. Examples of such non-subspace properties include normal, compact, and separable. Within this paper, needed corrections are made and applied, subspace properties are further investigated, and topological properties that behave in the same manner as each of normal, compact, and separable with respect to subspaces and non-subspaces are generalized and further investigated as classical non-subspace properties.

Definition 1.2. A topological property P is a classical non-subspace property iff there exists a space with property P with a subspace that does not have property P.

In the study of normal spaces, it was discovered there are spaces for which every subspace is normal, which led to the introduction of the completely normal topological property in 1923 [9].

Definition 1.3. A space has property completely normal iff every subspace of the space is normal.

In this paper, completely normal is generalized to completely P, where P is a classical non-subspace property as given above.

Definition 1.4. A space has property completely P, where P is a classical non-subspace property, iff every subspace of the space has property P.

Great progress was made in the study of subspace and classical non-subspace properties, but there remained natural, unaddressed questions concerning each of the two properties. For example, for subspace properties P and Q, it was unknown whether or not (*P* and *Q*) exists. If so, then for subspace properties *P* and *Q*, (*P* and *Q*) is a subspace property, giving many additional subspace properties. The question of whether or not every space has a subspace property by the definition above was not addressed. If such a subspace property exists, it would be inconsistent with the intended purpose of subspace and classical non-subspace properties, creating a disconnect in the study of those two properties. Also, the question of whether or not there are non-subspace properties other than the classical non-subspaces properties given above was not addressed. The tools and insights needed to address the above questions were simply not known and/or realized at that time. However, that changed with the continued investigation of T_0-identification spaces.

T_0-identification spaces were introduced in 1936 and used to jointly characterize pseudometrizable and metrizable [8].

Definition 1.5. Let (X, T) be a space, let *R* be the equivalence relation on *X* defined by *xRy* iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of *R* equivalence classes of *X*, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map *N*. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

Theorem 1.1. *A space is pseudometrizable iff its T_0-identification space is metrizable.*

T_0-identification spaces were cleverly created to add T_0 to an externally generated, strongly (X, T) related T_0-identification space of (X, T), making T_0-identification spaces a strong, useful topological tool [8], as established in earlier investigations.

Based on their definitions and initial uses, subspaces and T_0-identification spaces appeared to be totally independent of each other.

Moving forward, in a 2015 paper [2], pseudometrizable was generalized to weakly *P*o.

Definition 1.6. Let P be a topological property for which $P\text{o} = (P \text{ and } T_0)$ exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

In the initial investigation of weakly Po spaces and properties, it was shown that for a topological property P for which weakly Po exists, weakly Po is a unique topological property and weakly Po is simultaneously shared by both a space and its T_0-identification space [2].

Within the 2015 paper [2], the search for topological properties that are not weakly Po led to the use of T_0 and "not-T_0". Thus another fundamental role of T_0 in the study of topology was revealed and "not-T_0" proved to be a useful topological property motivating the addition of "not-P", where P is a topological property for which "not-P" exists, to the study of topology [2]. The addition and use of the many new topological properties provided tools not before studied and used in the study of topology and, in a short time period, has exposed a mathematically fertile, never before imagined territory long overlooked within topology that has already changed and expanded the study of topology.

For example, in the paper [3], the use of "not-T_0" and "not-P", where "not-P" exists, not only provided needed tools to prove the existence of the never before imagined least of all topological properties L, but, also, provided the needed tools for a quick, easily understood proof of its existence.

Theorem 1.2. *L, the least of all topological properties, is given by $L = (T_0$ or "not-T_0"$) = (P$ or "not-P"$)$, where P is a topological property for which "not-P" exists.*

Having the right tools and the knowledge of those tools can make what once appeared to be an impossible task not only possible, but doable.

Within the paper [3], it was shown that every space has property L. Thus each space and each of its subspaces simultaneously share property L, even if the space has properties that are known not subspace properties, and, by the definition above, L is a subspace property, a reality far different than the initial intent of subspace and classical non-subspace properties.

Thus the discovery of L in the study of weakly Po created a disconnect in the study of subspace and classical non-subspace properties and, if possible, needed fixing. A quick, easy fix to restoring continuity in the study of the two properties was the removal of L as a subspace property.

Definition 1.7. Let P be a topological property. Then P is a subspace property iff $P \neq L$ and a space has property P iff each subspace has property P [4].

Within this paper, Definition 1.7 will be used as the definition of subspace properties. Thus, amongst the initially defined subspace properties, L is unique. It is the only initially defined subspace property that needed to be removed to end the discontinuity in the study of subspace and non-subspace properties. Also, in the paper [4], it was thought that L is unique compared to all other topological properties in that it is the only topological property for which its negation does not exist, which was used in the paper [4] to resolve the unaddressed questions in the study of subspace properties given above. However, the continued investigation of topological and subspace properties has revealed that there are topological properties other than L whose negation does not exist. Below, such an example is given.

The discovery of the unexpected least topological property raised the question of a strongest topological property. In the paper [5], a strongest topological property was assumed, a topological property P was selected, "not-P" was thought to exist, and P and "not-P" were used to give a contradiction. However, the existence of a topological property other than L whose negation does not exist can be easily overcome by using a topological property whose negation is known to exist, for example T_0 and "not-T_0". Thus, there is no strongest topological property.

2. An Example and its Impact on Subspace Properties

Example 2.1. In this example "not-T_0" and "not-T_1", both of which exist, are used. Let $P = (T_0$ or "not-T_1"$)$, which exists. Since L is the least topological property, then for each topological property Q, (Q and L) = Q. Since $(P$ and $T_0) = ((T_0$ or "not-T_1"$)$ and $T_0) = (T_0$ or $(T_0$ and "not-T_1"$)) \neq T_0$, then $P \neq L$. Since "not-T_0" implies "not-T_1", "not-P" $= (T_1$ and "not-T_0"$)$ does not exist.

There are other such examples. Thus the results in the paper [4] became suspect and required further consideration. If P in Example 2.1 is a subspace property, then the results in the paper [4] are felonious. If not, then there is a possibility that the results in [4] could be true.

With the many, varied known subspace properties and the example above, the task of determining if the negation of each subspace property exists is, with the known properties in classical topology, at best, a challenging undertaking. Perhaps, as above, there is a tool unknown in classical topology that can make the task not only doable, but easily doable.

In an introductory topology class, the topological space with a single element is used as a quick, easily seen, and understood topological space. However, recent further considerations of singleton set spaces led to the introduction and investigation of the singleton set topological property (SSTP), which has been established as a powerful topological property [4].

Definition 2.1. A space (X, T) has the SSTP property iff X is a singleton set.

In the paper [4], it was shown that the SSTP is the strongest of the subspace properties, revealing another non-classical tool available for use in trying to resolve the unanswered questions above. Below subspace properties P and Q for which $(P$ and "not-Q"$)$ exists are used; an example of which would be $(T_0$ and "not-T_1"$)$.

Theorem 2.1. *Let P and Q be subspace properties for which $(P$ and "not-Q"$)$ exists. Then $(P$ and "not-Q"$)$ is not a subspace property.*

Proof. Assume $(P$ and "not-Q"$)$ is a subspace property. Let (X, T) be a space with property $(P$ and "not-Q"$)$. Let Y be a singleton set subset of X. Then (Y, T_Y) has the SSTP property, which implies (Y, T_Y) has property $((P$ and "not-Q"$)$ and $Q)$, but $((P$ and "not-Q"$)$ and $Q)$ does not exist, which is a contradiction. Thus $(P$ and "not-Q"$)$ is not a subspace property.

Theorem 2.2. *Let P and Q be subspace properties. Then* (P *and* Q) *exists.*

Proof. Suppose (P and Q) does not exist. Since $P = ((P$ and $Q)$ or $(P$ and "not-Q"$))$, then $P = (P$ and "not-Q"$)$, but then P is both a subspace property and a non-subspace property, which is a contradiction. Hence, (P and Q) exists.

Theorem 2.3. *Let P and Q be subspace properties. Then* (P *and* Q) *is a subspace property.*

The proof is straightforward and omitted.

Theorem 2.4. *Let P be a subspace property. Then* "*not-P*" *exists and is a non-subspace property.*

Proof. Suppose "not-P" does not exist. Let Q be a subspace property. Then $Q = ((Q$ and $P)$ or $(Q$ and "not-P"$)) = (Q$ and $P)$, and Q is stronger than or equal to P. Thus P is the least subspace property. Since T_0 is a subspace property, then $(P$ and $T_0)$ exists, since P is the least subspace property, then T_0 is stronger or equal to P, and since "not-T_0" exists, T_0 is stronger than P and $(P$ and $T_0) = T_0$. Then $P = ((P$ and $T_0)$ or $(P$ and "not-T_0"$)) = (T_0$ or $(P$ and "not-T_0"$))$. Since $P \neq T_0$, then $P = (P$ and "not-T_0"$)$, but, then P is both a subspace and a non-subspace property, which is a contradiction. Thus "not-P" exists.

Assume "not-P" is a subspace property. Let (X, T) be a space with property "not-P". Let Y be a singleton set subset of X. Then (Y, T_Y) has property $(P$ and "not-P"$)$, which does not exist and is a contradiction. Thus "not-P" exists and is a non-subspace property.

Hence L is unique compared to the subspace properties in the sense that amongst all those properties, including L, L is the only one whose negation does not exist.

As given above, the subspace properties have a strongest property; SSTP. Does the subspace properties have a least property?

Theorem 2.5. *Let* $\mathcal{P}_0 = \{(P \text{ and } T_0) \mid p \text{ is a subspace property}\}$. *Then* $\mathcal{P}_0$ *has strongest element SSTP and least element* T_0.

Proof. Since for each $(P \text{ and } T_0) \in \mathcal{P}_0$, $(P \text{ and } T_0)$ implies T_0, and $T_0 = (T_0 \text{ and } T_0) \in \mathcal{P}_0$, then T_0 is the least element in $\mathcal{P}_0$. Since SSTP is the strongest subspace property and (SSTP and T_0) = SSTP, then SSTP $\in \mathcal{P}_0$ and is the strongest element in $\mathcal{P}_0$.

Theorem 2.6. *Let* $\mathcal{P} = \{P \mid P \text{ is a subspace property}\}$ *has strongest element SSTP and no least element.*

Proof. As given above SSTP is the strongest element of $\mathcal{P}$. Suppose $\mathcal{P}$ has least element P. Then T_0 is stronger than or equal to P, $(P \text{ and } T_0) = T_0$, and $P \neq T_0$. In 1943 [7], the T_1 separation axiom was generalized to the R_0 separation axiom. A space (X, T) is R_0 iff for each closed set C and each $x \notin C$, $Cl(\{x\}) \cap C = \phi$. In a 1961 paper [1], it was shown that a space is T_1 iff it is $(R_0 \text{ and } T_0)$. Since R_0 is a subspace property and $R_0 = ((R_0)$o or $(R_0$ and "not-T_0")) = $(T_1$ or $(R_0$ and "not-T_0")), where both T_1 and $(R_0$ and "not-T_0") exist, and $(R_0$ and "not-T_0") does not imply T_0, then $P \neq T_0$. Thus $P = ((P \text{ and } T_0)$ or $(P$ and "not T_0")) = $(T_0$ or $(P$ and "not-T_0")), where both T_0 and $(P$ and "not-T_0") exist, and since $P \neq T_0$, then $P = (P$ and "not-T_0"), which contradicts P is a subspace property. Hence $\mathcal{P}$ has no least element.

In the paper [4], improper logic was used to conclude $\mathcal{P}$ has no least element. Thus, by the work above, it is now known to be true.

With the addition and use of "not-P", where P is a subspace property and "not-P" is the negation of P, there would be a need to define "not-P".

Definition 2.2. Let P be a subspace property. Then a space has property "not-P" iff there exists a subspace with property "not-P" [6].

Below additional properties of subspace properties, non-subspace properties, classical non-subspaces properties, and completely P properties are given.

3. "Not-*P*", where *P* is a Subspace Property, Classical Non-Subspace Properties, and Completely *P* Properties

Theorem 3.1. *Let P be a subspace property. Then a space* (X, T) *has property "not-P", as defined above, iff* (X, T) *is "not-P".*

Proof. If (X, T) is "not-P", then (X, T) has a subspace that is "not-P", namely itself, and (X, T) has property "not-P".

Conversely, suppose (X, T) has property "not-P". Let (Y, T_Y) be a subspace of (X, T) with property "not-P". If (X, T) has property ("not-("not-P")"), then (X, T) has property P and (Y, T_Y) is (P and "not-P"), which is a contradiction. Thus (X, T) is "not-P".

Hence, in future studies of subspace properties and non-subspace properties, both can be simultaneously studied, and known subspace and not subspace properties can be used to quickly give additional subspace and not subspace properties with no required extra work.

Theorem 3.2. *Let P be a subspace property. If a space* (X, T) *has property "not-P", then X has two or more elements.*

Proof. Suppose there exists a space (X, T) with property "not-P" that has only one element. Then (X, T) is "not-P", but, since X is a singleton set, (X, T) has the

SSTP property and (X, T) has property P, which is a contradiction. Thus X has two or more elements.

Theorem 3.3. *Let Q be a classical non-subspace property. Then $Q \neq L$ and "not-Q" exists, and "not-Q" = completely Q.*

Proof. Since Q is a classical non-subspace property, there exists a space with property Q with a subspace that is "not-Q". Hence "not-Q" exists and $Q \neq L$. A space is "not-Q" iff there does not exist a space with property Q with a subspace that is "not-Q" iff for each space with property Q, there is no subspace with property "not-Q" iff for each space with property Q every subspace has property ("not-("not-P")") $= Q$ iff the space is completely Q.

Corollary 3.1. *Let Q be a classical non-subspace property. Then Q = "not-(completely Q)".*

Theorem 3.4. *Let P be a subspace property. Then "not-P" is not a classical non-subspace property.*

Proof. Suppose there exists a subspace property P such that "not-P" is a classical non-subspace property. Then completely "not-P" exists. Let (X, T) be a space for which each subspace of (X, T) has property "not-P". Then (X, T) has property "not-P" and X is not a singleton set. Let $x \in X$. Then $(\{x\}, T_{\{x\}})$ is both P and "not-P", which is a contradiction.

Thus {"not-P"| P is a subspace property} is a new category of non-subspace properties totally distinct from the classical non-subspace properties, which has been added to the study of non-subspace properties.

Theorem 3.5. *Completely Q is a subspace property.*

Proof. Let (X, T) be completely Q. Then, by definition, Q is a classical non-subspace property. Then every subspace of (X, T) has property Q and (X, T) has property Q. Let Y be a nonempty subset of X. Then (Y, T_Y) has property Q. Since every subspace of (Y, T_Y) is a subspace of (X, T), then every subspace of (Y, T_Y) has property Q and (Y, T_Y) has property completely Q. Hence, completely Q is a subspace property.

Thus {completely Q | Q is a classical non-subspace property} is a new category of subspace properties, which are added to the study of subspace properties.

Corollary 3.2. *A space that is "not-completely Q" has two or more elements; and "not-P", where P is a subspace property, and classical non-subspace properties are independent properties.*

Are there other non-subspace properties?

In the paper [6], it was thought that for subspace properties P and Q, (P or Q) is a subspace property, which is not correct.

Theorem 3.6. *Let P and Q be unequal subspace properties. Then* (*P or Q*) is *not a subspace property.*

Proof. Let (X, T) be a space with property (P or Q). Then (X, T) has property P or (X, T) has property Q. If (X, T) has property P, then, since P and Q are unequal, (X, T) does not have property Q. Similarly, if (X, T) has property Q, then (X, T) does not have property P. Thus (P or Q) is not a subspace property.

Theorem 3.7. *Let P and Q be unequal subspace properties. Then "not-(P or Q)" = (("not-P") and ("not-Q")) exists and is a non-subspace property.*

Proof. Since P and Q are subspace properties, both "not-P" and "not-Q" exist. Suppose (("not-P") and ("not-Q")) does not exist. Since "not-P" = ((("not-P") and Q) or (("not-P") and ("not-Q"))) = (("not-P") and Q), then "not-P is stronger than or equal to Q and, since ("not-P") is not equal to Q, then "not-P" is stronger than Q. Then Q = ((P and Q) or (Q and "not-P")) = ((P and Q) or ("not-P")), and since $Q \neq$ "not-P", then Q = (P and Q). Similarly, P = (P and Q), but, then $P = Q$, which is a contradiction.

Assume (("not-P") and ("not-Q")) is a subspace property. Then W = ((("not-P") and ("not-Q")) and Q) is a subspace property, but W does not exist, which is a contradiction. Hence "not-(P or Q)" = (("not-P") and ("not-Q")) exists and is a non-subspace property.

Hence {(P or Q) | P and Q are unequal subspace properties} and (("not-P") and ("not-Q")) | P and Q are unequal subspace properties} are new, total distinct categories of non-subspace properties added to the non-subspace properties; and spaces with property (("not-P") and ("not-Q")) have two or more elements. Mathematical induction can be used to extend the results above to finitely many unequal subspace properties, giving many more new non-subspace properties.

Theorem 3.8. *Let P and Q be subspace properties that are unequal and neither P nor Q is stronger than the other. Then all of (P and "not-Q"), "not-(P and "not-Q") = (Q or "not-P"), (Q and "not-P"), and "not-(Q and "not-P") = (P or "not-Q") exist and are non-subspace properties; and spaces with properties (P and "not-Q") or (Q and "not-P") contain two or more elements.*

Proof. Since P = ((P and Q) or (P and "not-Q")); and $P \neq Q$ and P is not stronger than Q, then $P \neq$ (P and Q), and (P and "not-Q") exists. Thus, by the result above, (P and "not-Q") is a non-subspace property. Since (P and "not-Q") implies "not-Q", then each space with property (P and "not-Q") has two or more elements. Then "not-(P and "not-Q")" = (Q or "not-P"), which exists, and, by an argument similar to that in Theorem 3.6, (Q or "not-P") is not a subspace property. Similarly, the remainder of Theorem 3.8 can be proven.

Thus four additional new categories are added to the non-subspace properties.

Theorem 3.9. *Let P and Q be subspace properties with one of P and Q stronger than the other, say Q is stronger than P. Then* $(P$ *or* "*not*-Q"$)$ *exists and is a non-subspace property and* $(Q$ *and* "*not*-P"$)$ *does not exist.*

Proof: By an argument similar to that above, $(P$ or "not-Q"$)$ exists and is a non-subspace property. Since $Q = ((Q$ and $P)$ or $(Q$ and "not-P"$))$, and $(Q$ and $P)$ $= Q$, then $(Q$ and "not-P"$)$ does not exist.

Thus an additional new category of non-subspace properties is added to the study of non-subspace properties.

Another possibility for non-subspace properties would be (P or Q), where P is a subspace property and Q is a non-subspace property. If $Q =$ "not-W", where W is a subspace property, then Theorems 3.8 and 3.9 can be used. Otherwise, care should be taken.

As given above, since R_0 is a subspace property and $R_0 = (T_1$ or $(R_0$ and "not-T_0"$))$, then $(T_1$ or $(R_0$ and "not-T_0"$))$ is a subspace property. There are other such properties.

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