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TILING THE PLANE WITH k-GONS

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Abstract

We present a way to tile the plane by k-gons for a fixed k. We use usual regular 6-gons by putting some in a row and fill them with k-gons. We use only one or two or four different k-gons.

1. Introduction

It is a widespread opinion that one can tile the plane $\mathbb{R}^2$ only with triangles, squares and regular 6-gons. This is wrong. A further possibility is to put regular 6-gons in a row.

We think that it is useful to repeat the definition of a *simple polygon*.

A simple polygon with k vertices consists of k different points of the plane (x_1, y_1), (x_2, y_2), ... (x_{k-1}, y_{k-1}), (x_k, y_k), called *vertices*, and the straight lines between (x_i, y_i) and (x_{i+1}, y_{i+1}) for $1 \leq i \leq k-1$, called *edges*. Also the straight line between (x_k, y_k) and (x_1, y_1) belongs to

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the polygon. We demand that it is homeomorphic to a circle and that there are no three consecutive collinear points (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}), for $1 \le i \le k-2$. Also the three points (x_k, y_k), (x_1, y_1), (x_2, y_2) and (x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1) are not collinear.
We call this just described simple polygon a k-*gon*.

Definition 1. Let t be any natural number. We call a simple polygon a t row 6-gon, if t regular 6-gons are put in a row. Two neighboring regular 6-gons were glued at a common edge.
See the example in Figure 2. There we show a 5 row 6-gon.
Note that a 1 row 6-gon is just a regular 6-gon.

Proposition 1. *One can tile the plane with t row 6-gons for all natural numbers t.*

Proof. Trivial.

Proposition 2. *A t row 6-gon has $2 + 4 \cdot t$ vertices.*

Proof. Easy.

2. Tiling

Theorem 1. *Let k be a natural number larger than 2. There exists for all k a tiling of $\mathbb{R}^2$ with k-gons.*

Proof. For $k = 3$ and $k = 4$ and $k = 6$ the theorem is well-known. For $k = 5$ please see Figure 1. We also can take a regular 6-gon instead of a rectangle. We cut it into congruent halves.
Now let k be a natural number larger than 6.

Lemma 1. *It holds $k - 2 \equiv p \bmod 4$, where $p \in \{0, 1, 2, 3\}$.*

Proof. Well-known.

We discuss the four possibilities.

Possibility 1. $p = 0$. In this easy case we take the polygon t row 6-gon as a k-gon. We get t from the equation $k - 2 = 4 \cdot t$.
The sequence of the numbers of k is 10, 14, 18, ... By Proposition 2 the number of vertices of a t row 6-gon is $2 + 4 \cdot t$. This is k.

Possibility 2. $p = 1$. In this case we had to calculate. We take a $(4 \cdot t + 1)$ row 6-gon. It is filled with four k-gons. We use the vertices of the $(4 \cdot t + 1)$ row 6-gon as vertices of the four k-gons. See Figure 2. Note that the four k-gons have three edges in common. Therefore, we have to subtract 6 from the number of the vertices.
We get t from the equation $k - 2 = 4 \cdot t + 1$.
The number of vertices both for a $(4 \cdot t + 1)$ row 6-gon and also 4 k-gons -6 is $16 \cdot t + 6$. The sequence of the numbers of k is 7, 11, 15, 19, ...

Possibility 3. $p = 2$. We take a $(2 \cdot t + 1)$ row 6-gon. It is filled with two k-gons. We get t from $k - 2 = 4 \cdot t + 2$.
The sequence of the numbers of k is 8, 12, 16, 20, ... Two k-gons alltogether have $8 + 8 \cdot t$ vertices. See Figure 3. Note that if two neighboring k-gons tile a polygon a pair of vertices is canceled, since the k-gons have a common edge. Therefore, they have $6 + 8 \cdot t$ vertices.
This is also the number of vertices of a $(2 \cdot t + 1)$ row 6-gon.

Possibility 4. $p = 3$. We take a $(4 \cdot t + 3)$ row 6-gon. It is filled with four k-gons. We get t from $k - 2 = 4 \cdot t + 3$.
The common number of vertices is $16 \cdot t + 14$.
The sequence of the numbers of k is 9, 13, 17, 21, ...

The theorem is proved.
It follows three figures.

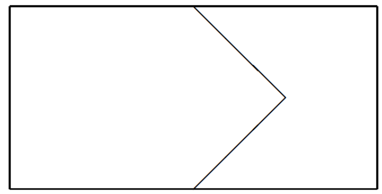

Figure 1.

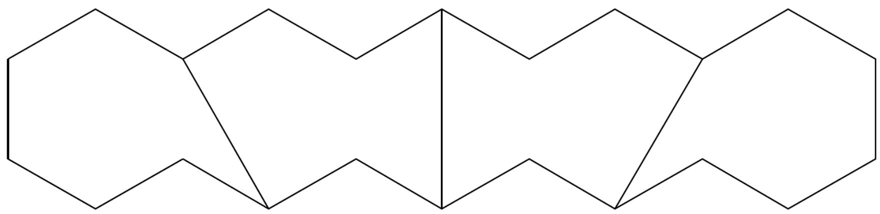

Figure 2. A 5 row 6-gon, which is subdivided in four 7-gons. We see also three edges. Each is a common edge of two 7-gons.

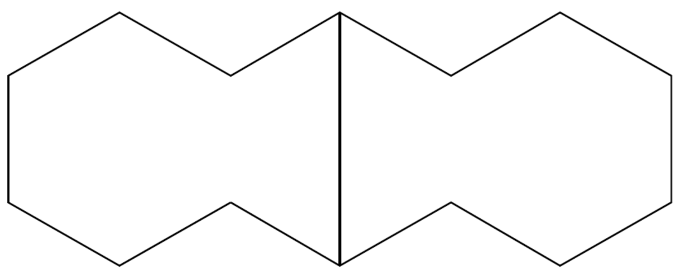

Figure 3. We see a 3 row 6-gon. It consists of two 8-gons.

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