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THEORY OF THE ELECTRON STRUCTURE

G. M. STÉPHAN

26, Chemin de Quo Vadis 22730, Tregastel
France
e-mail: gmstephan@wanadoo.fr

Abstract

A theory of the electron which is based on the covariant derivative of the potential is developed. It is shown that the particle is represented by a tensor field which groups all the measured quantities mass, charge, spin and magnetic moment of the electron. The theory gives the equations which relate these global quantities to the local elements of the tensors.

1. Introduction

Up to now the electron is represented as a point-like particle of matter and a unit of unbreakable electrical charge associated with a wave. These paradigms are very powerful to describe atomic, molecular and solid state phenomena. However, they become useless to study the distribution of mass or electric charge inside an electron. The theory presented here proposes a new solution to the electron structure together

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*Retired from University de Rennes, Unité 6082 associée au Centre National de la recherche scientifique, 6, rue de Kerampont, 22305 Lannion.

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with its associated characteristics.

The long quest to understand the electron structure began at the end of the XIXth century [1]. In his book published in 1909 [2] Hendrik A. Lorentz "*ascribes to each electron certain finite dimensions*". However, the history [3] of this particle shows that the different models [4, 5, 6, 7] to describe the inside of an electron were unsuccessful.

The use of Maxwell's equations was the basis of the early works, but a turning point was the introduction by Max Born and Leopold Infeld [8] of the idea of non-linear electromagnetism in the year 1934. Instead of starting from Maxwell's equations, they built a tensor $[a_{ik}]$ which was the sum of the (antisymmetric) electromagnetic tensor $[F_{ik}]$ and the (symmetric) metric tensor $[g_{ik}]$. Then they developed the properties of $[a_{ik}]$ to describe the electron. However, the elements F_{ik} have the dimensions of a magnetic field while g_{ik} has no dimension. Even if $[F_{ik}]$ is properly normalized one can question the validity of adding it to $[g_{ik}]$. The new tensor $[a_{ik}]$ should be the sum of $[F_{ik}]$ and another symmetric tensor of a similar physical nature. But what is this tensor?

The classical electromagnetic potential [9, 10, 11, 12] is a 4-components vector in Minkowski's spacetime and its 16 partial derivatives at an event M are the components of a tensor $D(A_i)$. The antisymmetric part of $D(A_i)$ is the widely used electromagnetic tensor $[F_{ik}]$. By contrast the symmetric part $[S_{ik}]$ does not appear in classical textbooks nor in specialized literature. The answer to the preceding question is evidently $[S_{ik}]$ and the tensor whose properties have to be analyzed is $[a_{ik}] = D(A_i)$. We imagine yoctoscopic observers at M where the scale is small as compared to the size of the particle. Such observers use the potential as the base paradigm and $D(A_i)$ as the fundamental object. Then they apply standard classical notions to develop their properties. In the course of the theory, we will distinguish between *local* properties (accessible to our yoctoscopic observers) and *global* properties accessible to us.

$[S_{ik}]$ has two fundamental characteristics which do not appear in $[F_{ik}]$: (1) being symmetric, it can be diagonalized, and (2) the corresponding mixed tensor S^i_k has a trace which is invariant in a coordinate transformation. These two properties result [13] in a Helmholtz equation whose solutions show concentrations of potential (without any divergence) around the origin. These solutions are characterized by three quantum (integer) numbers n, ℓ, m starting at $n = \ell = m = 0$. When replacing the potential in $D(A_i)$ by these solutions, one obtains a tensor at each event M in spacetime. The field of these tensors describes a particle $(n,\ \ell,\ m)$.

As the determinant of S^i_k is also an invariant, we have associated it with a local Lagrangian density $\mathcal{L}$ in the elementary volume around M. This quantity can be integrated over the whole volume to give a global invariant which allows the calculation of the potential energy, or the total mass of the particle. The determinant of $[a^i_k]$ can also be used in Lagrange's equations which shows that the electric charge can be deduced from the potential. One finds that only the solution $n = 1$, $\ell = m = 0$ can be globally electrically charged. We naturally named it "the electron". This particle is the most known among elementary particles. It can be isolated and trapped [14]. It is characterized by four measurable quantities which are its mass, its electric charge, its angular momentum (spin) and its magnetic moment. Its Compton wavelength is a fifth characteristic quantity which appears when it diffuses Gamma rays. Up to now, these quantities seem to be independent of each other. The aim of the present study is essentially to show that they are all linked together and to express the relations between them and the potential.

We first briefly describe the tensors, the way to obtain the basic Helmholtz equation and its solutions. We write the tensor field of the electron in the static (non rotating) case. The description of the spatial distribution of energy and electric density is given. Then the spinning particle is described as a vortex with a spatial distribution of angular frequency. The angular momentum and the magnetic moment are obtained. Finally the theoretical parameters are numerically computed.

The main results of the following classical theory are the description of the structure of the "point-like particle", the explanation of the de Broglie's pilot wave, the agreement with Wheeler-Feynman's absorber theory and the numerical calculation of the electron characteristics.

2. Generalities

2.1. Notations

The standard notation of the 4-potential vector is $A^i = (\phi/c,\ A^x,\ A^y,\ A^z)$ in the Cartesian coordinates frame of Minkowski's spacetime spanned by the normalized basis vectors $(\vec{e}_t,\ \vec{e}_x,\ \vec{e}_y,\ \vec{e}_z)$. ϕ / c is the scalar potential and the set $\overrightarrow{A} = (A^x,\ A^y,\ A^z)$ is the vector potential at an event M defined by its coordinates $(x^i = ct,\ x,\ y,\ z)$. The covector will be denoted by $\widetilde{A} = (A_x,\ A_y,\ A_z)$. We choose the convention $(+, -, -, -)$ for the metric g_{mn}. It is important to realize that the elementary volume surrounding M is small as compared to the electron volume: The particle will thus be characterized by vector and tensor fields. The covariant derivative $D(A_i)$ at point M is the particle tensor $D(A_i) = [a_{ik}] = [\partial A_i / \partial x^k]$. $D(A_i)$ is divided into its symmetric and antisymmetric parts [15]:

$$[s_{ik}] = \frac{1}{2}([a_{ik}] + [a_{ki}]), \tag{1}$$

$$[f_{ik}] = \frac{1}{2}([a_{ik}] - [a_{ki}]), \tag{2}$$

i and k represent, respectively, the line and the column indices in the square table representation. $[f_{ik}]$ is the usual electromagnetic tensor [16]. $[s_{ik}]$ can be named the matter (or mass) tensor. The particle tensor is the sum of the matter and the field tensors. They keep their symmetric or antisymmetric properties in a coordinate change. It is this association which explains the matter-field duality.

2.2. Helmholtz equation

$[s_{ik}]$ is a symmetric tensor: it can be diagonalized provided its determinant is not zero. It follows that an eigenframe exists where elements $s_{1k} = s_{k1} = 0$ with $k = 2, 3, 4$. Some details are given in Appendix. This is expressed by the equation:

$$\overrightarrow{\text{grad}}\, \phi = \frac{\partial \widetilde{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t}. \tag{3}$$

Now the mixed tensors $[a_k^i], [s_k^i], [f_k^i]$ are introduced. They are characterized by invariants in coordinates changes. Invariants are the coefficients of their characteristic polynomials. The most well known are the trace and the determinant. The trace [17] of $[s_k^i]$ remains the same in a time translation:

$$\frac{\partial s_i^i}{c\partial t} = \frac{\partial}{c\partial t}\left(\frac{\partial(\phi / c)}{c\partial t} + \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z}\right) = 0. \tag{4}$$

Associating Equations (3) and (4) gives the Helmholtz equation:

$$\frac{\partial^2 \phi / c}{c^2 \partial t^2} - \Delta\phi / c = 0, \tag{5}$$

where Δ represents the Laplacian. The permanent solutions oscillating at an angular freqency ω obey:

$$\frac{\omega^2}{c^2}\phi / c + \Delta\phi / c = 0. \tag{6}$$

This equation is a tensor equation which remains the same in any system of coordinates and we will use it in the spherical reference frame attached to M. This frame is defined by its proper time t and the geometrical coordinates (r, θ, φ) such that:

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta. \tag{7}$$

The advantage of the (r, θ, φ) system is that it makes use of the spherical symmetry of the electron. Its disadvantage with respect to the Cartesian system is that the formulation of $D(A_i)$ is more complicated in the general case.

Solutions of Equation (6) have been studied in the context of the hydrogenoid atoms [18]. Coupled radial-angular solutions are found from the ansatz:

$$\phi / c = R(r)\,\Phi(\varphi)\,\Theta(\theta). \tag{8}$$

Decoupled solutions are obtained from the ansatz: $\phi / c = R(r) + \Phi(\varphi) + \Theta(\theta)$.

In the present case, there is no need to do any approximation to write the radial function $R(r)$. Now, we introduce the *normalized* distance to the center: $\mathrm{x} = \omega r / c$. Note the different typography with respect to the coordinate x. This distance will thus be measured in units of the reference length c / ω. The solutions are:

$$\phi_{n,\ell,m}(\mathrm{x}, \theta, \phi) = \mathcal{A}_{n,\ell,m}\, J_n(\mathrm{x})\, Y_\ell^m(\theta, \varphi) \cos(\omega t + \alpha). \tag{9}$$

$\mathcal{A}_{n,\ell,m}$ is the amplitude of the solution. We will choose $\alpha = 0$ or $-\pi / 2$ for reasons which will appear later. $J_n(\mathrm{x})$ is the spherical Bessel function of order n and $Y_\ell^m(\theta, \varphi)$ is the ℓ, m spherical harmonic.

2.3. The tensor field of the electron

The solution corresponding to the electron is characterized by $n = 1$, $\ell = m = 0$. The reason is that the global electric charge which is associated to a solution vanishes for all of them except for this special one [13]. This point is developed in Section 4. The following developments will concern this case only. The scalar potential is expressed as an even or an odd function of ωt (resp., upper and lower lines). One passes from the first to the second solution by removing $\pi / 2$ from ωt:

$$\phi_{1,0,0} = \frac{\mathcal{A}}{\sqrt{4\pi}} J_1 \begin{cases} \cos \omega t, \\ -\sin \omega t, \end{cases} \tag{10}$$

where J_1 is the spherical Bessel function of order 1. This solution is spherically symmetric. The potential vector has only one radial component given by Equation (3):

$$A_r = \frac{\mathcal{A}}{c\sqrt{4\pi}} J_1' \begin{cases} \sin \omega t, \\ \cos \omega t, \end{cases} \tag{11}$$

where J_1' stands for the derivative of J_1 with respect to x.

The tensor which will represent the electron at point M is explicitly written in its eigenframe for the even solution:

$$[a_{ik}] = \begin{bmatrix} \frac{\partial(\phi / c)}{c\partial t} & \frac{\partial(\phi / c)}{\partial r} & 0 & 0 \\ -\frac{\partial A_r}{c\partial t} & -\frac{\partial A_r}{\partial r} & 0 & 0 \\ 0 & 0 & -\frac{A_r}{r} & 0 \\ 0 & 0 & 0 & -\frac{A_r}{r} \end{bmatrix}$$

$$= \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \begin{bmatrix} -\sin\omega t\, J_1 & \cos\omega t\, J_1' & 0 & 0 \\ -\cos\omega t\, J_1' & -\sin\omega t\, J_1'' & 0 & 0 \\ 0 & 0 & -\sin\omega t\, \frac{J_1'}{x} & 0 \\ 0 & 0 & 0 & -\sin\omega t\, \frac{J_1'}{x} \end{bmatrix}. \tag{12}$$

Terms $A_r / r = (\mathcal{A}\omega)/(\sqrt{4\pi}c^2)\sin\omega t \frac{J_1'}{x}$ originate from Christoffel's coefficients which appear when shifting from the Cartesian to the spherical system of coordinates [15]. The field tensor is:

$$[f_{ik}] = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \begin{bmatrix} 0 & \cos\omega t\, J_1' & 0 & 0 \\ -\cos\omega t\, J_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

The matter tensor is:

$$[s_{ik}] = -\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \sin\omega t \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_1'' & 0 & 0 \\ 0 & 0 & \frac{J_1'}{x} & 0 \\ 0 & 0 & 0 & \frac{J_1'}{x} \end{bmatrix}. \tag{14}$$

These tensors are written in the frame spanned by normalized basis vectors where the metric tensor is also $1, -1, -1, -1$ along the diagonal. It follows that the mixed field tensor is:

$$[f_k^i] = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \begin{bmatrix} 0 & \cos\omega t\, J_1' & 0 & 0 \\ \cos\omega t\, J_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} 0 & -E_2^1/c & 0 & 0 \\ -E_2^1/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

The three components of the electric field are associated to a pseudo vector $\vec{E}$. The radial component $E_2^1 = E_1^2$ is related to the potential by the relation:

$$\frac{E_2^1}{c} := \frac{1}{2}\left(-\frac{\partial(\phi/c)}{\partial r} - \frac{\partial A^r}{c\partial t}\right)$$

$$= -\frac{\partial(\phi/c)}{\partial r} = -\frac{\partial A^r}{c\partial t} = -\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\cos\omega t\, J_1'. \tag{17}$$

The factor $1/2$ follows directly from Equation (2). The standard relation [11] does not include this factor. Note that the magnetic field vanishes in the eigenframe of coordinates.

The mixed matter tensor is:

$$[s_k^i] = \frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\sin\omega t \times \begin{bmatrix} -J_1 & 0 & 0 & 0 \\ 0 & J_1'' & 0 & 0 \\ 0 & 0 & \frac{J_1'}{\mathrm{x}} & 0 \\ 0 & 0 & 0 & \frac{J_1'}{\mathrm{x}} \end{bmatrix} \tag{18}$$

2.4. The far-field tensor

The behavior of $D(A_i)$ is very different in the regions close to or far from the origin of coordinates which is the center of the particle. Expressions of J_1, J_1' and J_1'' are:

$$J_1 = -\frac{\cos\mathrm{x}}{\mathrm{x}} + \frac{\sin\mathrm{x}}{\mathrm{x}^2},$$

$$J_1' = \frac{\sin x}{x} + 2\frac{\cos x}{x^2} - 2\frac{\sin x}{x^3},$$

$$J_1'' = \frac{\cos x}{x} - 3\frac{\sin x}{x^2} - 6\frac{\cos x}{x^3} + 6\frac{\sin x}{x^4}. \tag{19}$$

At large distances from the origin, only terms in $1/x$ survive and J_1, J_1' and J_1'' reduce to:

$$J_1 \simeq -\frac{\cos x}{x},$$

$$J_1' \simeq \frac{\sin x}{x},$$

$$J_1'' \simeq \frac{\cos x}{x}. \tag{20}$$

The field becomes:

$$E_1^2 = -\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c}\cos\omega t\,\frac{\sin x}{x}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c}\frac{1}{x}(\sin(\omega t + x) - \sin(\omega t - x)). \tag{21}$$

The two remaining terms in s_k^i are:

$$s_1^1 = -s_2^2 = \frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\sin\omega t\,\frac{\cos x}{x}$$

$$= \frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\frac{1}{x}(\sin(\omega t + x) + \sin(\omega t - x)). \tag{22}$$

These expressions show travelling advanced waves $\sin(x + \omega t)$ and retarded waves $\sin(x - \omega t)$ which correspond exactly to those described in Wheeler-Feynman's absorber theory [19]. We have thus adopted their interpretation which is based on causality: the outgoing waves are emitted by the particle and are absorbed by the surrounding medium which acts also as the emitter of waves which are absorbed by the particle. The equilibrium is obtained when both incoming and outgoing waves have the same energy. This is the condition for the stability of the particle and existence of permanent solutions. It follows that the amplitude $\mathcal{A}$ is fixed by this condition.

The two terms s_1^1 and s_2^2 are interpreted as “matter waves” and should correspond to the de Broglie’s pilot wave [20]. They intervene in self-interference effects in particle diffraction experiments. Their physical interpretation is different:

- s_1^1 originates from the derivative of the scalar potential ϕ / c. The standing wave is scalar.

- s_1^2 originates from the derivative of the radial potential A^r. The standing wave is radially polarized.

For future use, we write now the tensor (12) at point M in the far field region (subscript “$_{ff}$” stands for *far field*):

$$[a_k^i]_{ff} = \begin{bmatrix} \frac{\partial(\phi / c)}{c\partial t} & -E_1^2 / c & 0 & 0 \\ -E_1^2 / c & \frac{\partial A_r}{\partial r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

One is led to the conclusion that an electron (or more generally, any particle) is not localized: it manifests itself everywhere in the universe. It follows that one should make the distinction between a theoretical vacuum where the potential and its derivatives cancel everywhere and the real vacuum which is a superposition of electromagnetic and matter waves originating from the multitude of the particles of our universe. This state can be understood as the fundamental noise which is characterized by the quantities ϵ_0, μ_0 and its impedance $Z = 377\Omega$. In a region far from the center of any particle the contribution of one electron is very weak, while inside the electron, the noise can be neglected. We will come back on this subject later.

3. Potential Energy

In this section, we will compute the local energy densities associated to $[a_k^i]$, $[s_k^i]$ and $[f_k^i]$. The Lagrangian density $\delta\mathcal{L}$ at each point M is computed first: It is necessarily associated to an invariant of each of these tensors. The Hamiltonian density can then be computed and its summation in spacetime will give the total potential energy which is one of the macroscopic measurable quantities which characterize the particle. Let us consider successively the three tensors.

3.1. Particle tensor

Leaving aside the factor $(\mathcal{A}\omega)/(\sqrt{4\pi}c^2)$, the characteristic polynomial of $[a_k^i]$ is:

$$P(\lambda) = \left(\sin\omega t\,\frac{J_1'}{x} + \lambda\right)^2 \left((\lambda + \sin\omega t\, J_1)(\lambda + \sin\omega t\, J_1'') - \cos^2\omega t\, J_1'^2\right) \quad (24)$$

and the 4 invariants I_k are:

$$I_0 = -\cos^2\omega t\,\sin^2\omega t\, J_1'^2\,\frac{J_1'^2}{\mathrm{x}^2} + \sin^4\omega t\, J_1\, J_1''\,\frac{J_1'^2}{\mathrm{x}^2}, \quad (25)$$

$$I_1 = \sin\omega t\left(J_1 + J_1'' + 2\,\frac{J_1'}{\mathrm{x}^2}\right), \quad (26)$$

$$I_2 = -\cos^2\omega t\, J_1'^2 + \sin^2\omega t\, J_1\, J_1'' + 2\sin^2\omega t\, J_1\,\frac{J_1'}{\mathrm{x}}$$

$$+ 2\sin^2\omega t\, J_1''\,\frac{J_1'}{\mathrm{x}} + \sin^2\omega t\,\frac{J_1'^2}{\mathrm{x}^2}, \quad (27)$$

$$I_3 = -2\cos^2 \omega t \, \sin\omega t \; J_1'^2 \frac{J_1'}{\mathrm{x}} + 2\sin^3 \omega t \; J_1 \; J_1'' \frac{J_1'}{\mathrm{x}}$$

$$+ \sin^3 \omega t \; J_1 \frac{J_1'^2}{\mathrm{x}^2} + \sin^3 \omega t \; J_1'' \frac{J_1'^2}{\mathrm{x}^2}. \tag{28}$$

We associate [13] the Lagrangian density [21] to the determinant $\delta\mathcal{L}_a \propto I_0$. We write:

$$\delta\mathcal{L}_a = -\mathcal{C}\left(\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\right)^4 I_0. \tag{29}$$

$\delta\mathcal{L}_a$ has the dimensions of an energy density in the 4-volume $(ct\ x\ y\ z)$. The proportionality constant $\mathcal{C}$ is a physical quantity which has the dimensions $[\mathcal{C}] = \mathrm{M}^{-3}\mathrm{L}^{-2}\mathrm{T}^2\mathrm{Q}^4$. The minus sign is introduced to have a negative potential energy. The trace I_1 is the invariant which leads to the Lorentz gauge [13]. I_2 and I_3 are not used here. One should note that any power I_0^n is also an invariant. We keep $n = 1$ which will give an energy proportional to the fourth power of the amplitude $\mathcal{A}$. This property will explain the wide energy range of elementary particles [22].

The Hamiltonian is linked to the Lagrangian through a Legendre transform [23]:

$$\delta\mathcal{H} = \sum_k \frac{\partial\delta\mathcal{L}_a}{\partial u_k} u_k - \delta\mathcal{L}_a, \tag{30}$$

where the u_k's are the *independent* terms which appear in $\delta\mathcal{L}_a$ or in $[a_k^i]$. There are only four such terms:

$$u_1 = a_1^1 = \frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2} J_1 \begin{cases} -\sin\omega t, \\ -\cos\omega t, \end{cases} \tag{31}$$

$$u_2 = a_2^2 = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} J_1'' \begin{cases} \sin \omega t, \\ \cos \omega t, \end{cases} \tag{32}$$

$$u_3 = a_1^2 = a_2^1 = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} J_1' \begin{cases} \cos \omega t, \\ -\sin \omega t, \end{cases} \tag{33}$$

$$u_4 = a_3^3 = a_4^4 = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \frac{J_1'}{\mathrm{x}} \begin{cases} \sin \omega t, \\ \cos \omega t. \end{cases} \tag{34}$$

The upper and lower lines in these expressions correspond, respectively, to the even and the odd solutions. The derivatives $\partial \delta \mathcal{L}_a / \partial u_k$ which appear in Equation (30) are the *inductions* corresponding to each u_k.

If we use the expression for the determinant of $[a_k^i]$, one has:

$$\delta \mathcal{L}_a = -\mathcal{C} \, (a_3^3)^2 \, (a_1^1 a_2^2 - (a_2^1)^2), \text{ and Equation (30),}$$

we find:

$$\delta \mathcal{H} = 3 \, \delta \mathcal{L}_a. \tag{35}$$

The Hamiltonian density is now easily obtained:

$$\delta \mathcal{H} = 3 \, \mathcal{C} \frac{1}{(4\pi)^2} \frac{\mathcal{A}^4 \omega^4}{c^8} \begin{cases} \sin^2 \omega t \dfrac{J_1'^2}{\mathrm{x}^2} \left(\sin^2 \omega t \; J_1 \; J_1'' + \cos^2 \omega t \; J_1'^2 \right), \\ \cos^2 \omega t \dfrac{J_1'^2}{\mathrm{x}^2} \left(\cos^2 \omega t \; J_1 \; J_1'' + \sin^2 \omega t \; J_1'^2 \right). \end{cases} \tag{36}$$

$\delta \mathcal{H}$ is the energy included in the 4-volume element $(d(ct) \, dv)$ where $dv = r^2 \, dr \sin \theta \, d\theta \, d\varphi$. The energy density in a geometrical volume dv is obtained by integrating $\delta \mathcal{H}$ over the length $2\pi c / \omega$ on the time axis:

$$d\mathcal{H} = \frac{3\mathcal{C}}{(4\pi)^2} \frac{\mathcal{A}^4 \omega^4}{c^8} \frac{\pi}{4} \frac{c}{\omega} \left(3 \; J_1 \; J_1'' \frac{J_1'^2}{\mathrm{x}^2} + J_1'^2 \frac{J_1'^2}{\mathrm{x}^2} \right). \tag{37}$$

The total energy associated to the electron is obtained from an integration over the whole geometrical volume:

$$\mathcal{H}_{(\text{total})} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 dr \, d\mathcal{H}$$

$$= 4\pi \frac{c^3}{\omega^3} \int_0^{\infty} \mathrm{x}^2 d\mathcal{H} \, d\mathrm{x}. \tag{38}$$

One obtains:

$$\mathcal{H}_{(\text{total})} = \mathcal{C} \frac{3}{16} \frac{\mathcal{A}^4}{c^4} \int_0^{\infty} d\mathrm{x} \; J_1'^2 \left(3 \; J_1 \; J_1'' + J_1'^2\right)$$

$$= \mathcal{C} \frac{3}{16} \frac{\mathcal{A}^4}{c^4} \int_0^{\infty} d\mathrm{x} \frac{d}{d\mathrm{x}} \left(J_1 \; J_1'^3\right) = 0. \tag{39}$$

The total energy of the particle is identically 0: The integral vanishes because $J_{1(\mathrm{x}=0)} = 0$ and the product $J_1 J_1'^3 \propto 1/\mathrm{x}^4 \to 0$ when $\mathrm{x} \to \infty$. Its axial distribution includes a positive and a negative part which cancel each other after integration. This energy is the same for both even and odd solutions; Figure 1 represents the integrant $d(J_1 J_1'^3)/d\mathrm{x}$ as a function of x. The (essentially) negative part $3J_1'^2 \; J_1 \; J_1''$ will represent the mass energy while the positive part $J_1'^4$ is linked to the field energy.

We will keep in mind that $\delta\mathcal{H}$ in Equation (37) includes two terms:

- The first term $-\mathcal{C} \; (a_3^3)^2 a_1^1 \; a_2^2$ is an invariant of the mass tensor, it will represent the potential energy $W_p = m_0 c^2$ that is associated to the mass.

- The second term $\mathcal{C} \; (a_3^3)^2 (a_2^1)^2$ is the difference between two invariants. It is also an invariant which will be related to the field energy.

These invariants will be used to study the spinning electron (Section 5) where mass and field moments appear.

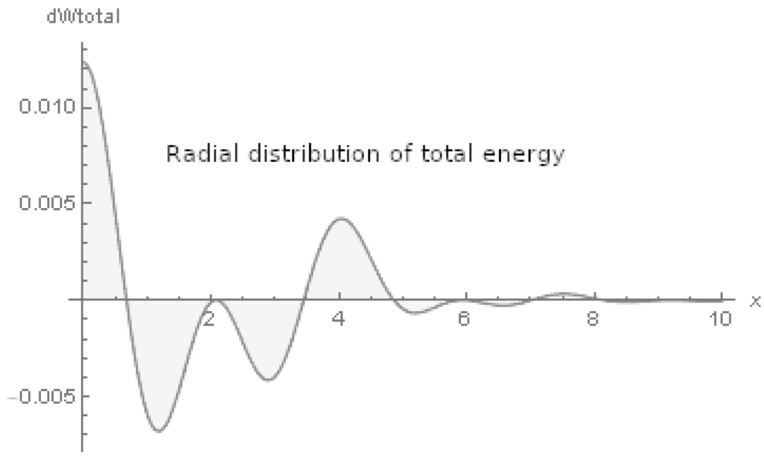

Figure 1. Radial distribution of the total electron rest energy vs. x, distance to the center (arbitrary units).

3.2. Matter tensor

The determinant of the matter tensor (Equation (18)) gives the potential energy density of the particle which is the first term in Equation (37):

$$dW_p = \frac{9\mathcal{C}}{64\pi}\left(\frac{\mathcal{A}\omega}{c^2}\right)^4 \frac{c}{\omega} J_1 \; J_1'' \frac{J_1'^2}{x^2}. \tag{40}$$

Applying Poincaré-Einstein relation $dW_p = dm_0 c^2$, we obtain the element of mass density dm_0 (in the geometrical volume) which we write explicitly below for future use:

$$dm_0 = \frac{9\mathcal{C}}{64\pi}\left(\frac{\mathcal{A}\omega}{c^2}\right)^4 \frac{1}{\omega c} J_1 \; J_1'' \frac{J_1'^2}{x^2}. \tag{41}$$

Note that the potential energy density writes in the 4-volume:

$$ {}_4\delta W_p = \frac{3\mathcal{C}}{(4\pi)^2} \frac{\mathcal{A}^4\omega^4}{c^8} \begin{cases} \sin^4 \omega t \dfrac{J_1'^2}{x^2} J_1 \ J_1'', \\ \cos^4 \omega t \dfrac{J_1'^2}{x^2} J_1 \ J_1''. \end{cases} \tag{42} $$

Integrating dW_p over the 4-volume, one obtains the potential energy:

$$ W_p = \mathcal{C} \frac{3}{16} \frac{\mathcal{A}^4}{c^4} \int_0^\infty d\mathrm{x}\, 3 J_1'^2 \ J_1 \ J_1''. \tag{43} $$

One notes that ω does not appear in this relation.

The electron rest mass m_0 is an invariant, function of the amplitude $\mathcal{A}$, the constant $\mathcal{C}$ and the speed of light:

$$ m_0 = \mathcal{C} \frac{\mathcal{A}^4}{c^6} I_{mo}, \tag{44} $$

where the integral is exactly obtained:

$$ I_{mo} = \int_0^\infty J_1 \ J_1'' \ J_1'^2 d\mathrm{x} = \frac{17\pi}{18480}. \tag{45} $$

In the region close to the origin, $\mathrm{x} \ll 1$ and a series expansion gives:

$$ J_1 = \frac{\mathrm{x}}{3} - \frac{\mathrm{x}^3}{30} + O(\mathrm{x}^4), $$

$$ J_1' = \frac{1}{3} - \frac{\mathrm{x}^2}{10} + O(\mathrm{x}^4), $$

$$ J_1'' = -\frac{\mathrm{x}}{5} + \frac{\mathrm{x}^3}{42} + O(\mathrm{x}^4). \tag{46} $$

The product $J_1'^2 \ J_1 \ J_1''$ converges at $\mathrm{x} = 0$:

$$ J_1'^2 \ J_1 \ J_1'' = -\frac{\mathrm{x}^2}{135} + \frac{86\mathrm{x}^4}{14175} + O(\mathrm{x}^5). \tag{47} $$

Figure 2 represents the distribution of the mass energy along x. The total potential energy which is stored in the particle is the surface under this curve. If we subtract $\mathcal{H}_{(\text{mass})}$ from $\mathcal{H}_{(\text{total})}$, we obtain the field energy which is stored in the particle. The function $J_1'^4$ represents the radial distribution of this energy. It is illustrated in Figure 3.

3.3. Field tensor

The characteristic polynomial of $[f_k^i]$ is:

$$P(\lambda) = \lambda^2(\lambda^2 - (E_1^2)^2 / c^2). \tag{48}$$

The field invariants are thus 0 and $(E_1^2)^2$. In the general case where the magnetic field $\vec{H} \neq 0$, they are $H^2 - E^2 = \text{inv.}$ and $\vec{E}\,\vec{H} = \text{inv.}$

The local energy density of the field is obtained from the product of the field E_1^2 and the induction $\mathcal{D}_2^1 = \delta\mathcal{L}_a / \partial E_1^2$:

$$\begin{aligned} d\mathcal{H} &= E_1^2 \mathcal{D}_2^1 \\ &= 2\mathcal{C}\left(\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\right)^4 \sin^2\omega t \cos^2\omega t \,\frac{J_1'^4}{\mathrm{x}^2}. \end{aligned} \tag{49}$$

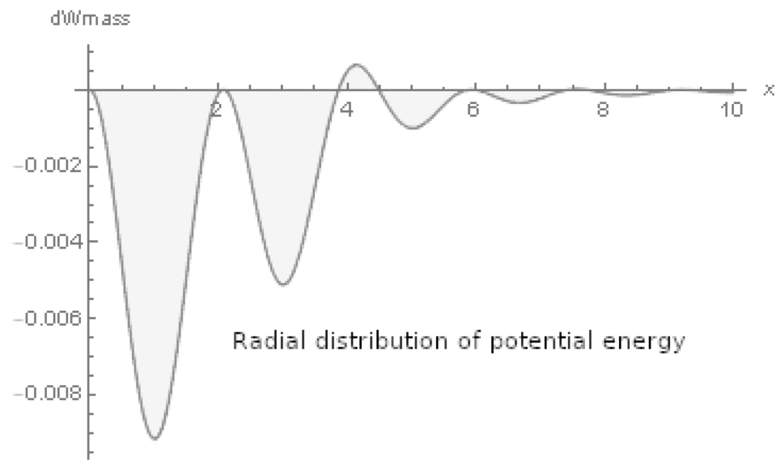

Figure 2. Radial distribution of potential (mass) energy for the electron vs. x, normalized distance to the center (arbitrary units).

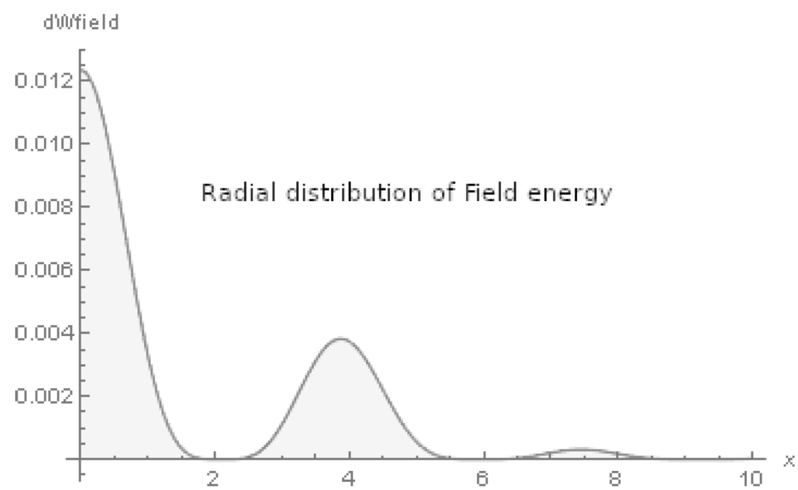

Figure 3. Radial distribution of field energy for the electron vs. x, normalized distance to the center (arbitrary units).

After integrating over time, it leads to the field term in Equation (39).

As the particle manifests itself at large distances by terms in $1/x$, the potential, the field or the energy stored in an element dv around a point M are very small in this region as compared to the noncoherent sum of the corresponding quantities originating from the multitude of particles of the universe. One can thus characterize this region (the "vacuum") by a noise tensor whose components n_k^i are incoherent and with a modulus large as compared to those originating from a single test-particle. We write the local tensor as a sum of this noise tensor and the test-particle tensor (Equation ((23)). The Lagrangian density is the determinant:

$$\mathcal{L}_{ff} = -\mathcal{C} \left\| n_k^i + a_{k_{ff}}^i \right\|. \tag{50}$$

The local induction due to the field (E_1^2) is the derivative of $\mathcal{L}_{ff}$ with respect to $E_1^2 / c = a_1^2$. Developing the determinant and performing the derivation gives the electric induction:

$$D_{2_{ff}}^1 = \frac{\partial \mathcal{L}_{ff}}{\partial a_{2_{ff}}^1} = -2\mathcal{C}\,(n_3^3 n_4^4 - n_4^3 n_3^4)(E_1^2)/c. \tag{51}$$

This relation will be used in the next section.

4. Electrical Charge

This section deals with the electric charge Q associated to the electron. In traditional electrostatics, Q is the time-independent source for the field. However, we have seen in Subsection (2.4) that at large distances, the field is a longitudinally-oscillating vector (which prevents instantaneous action). In the present theory, Q should thus be an oscillating quantity: we will write $\widetilde{Q} = Q \sin \omega t$ (or $\cos \omega t$) to display this time dependence.

The two fundamental equations which are related to Q in classical electromagnetism are:

• Maxwell's equation: div $\overrightarrow{D} = \rho$ ($\overrightarrow{D}$: dielectric induction, ρ: charge density).

• The Lorentz force $\vec{F} = Q\vec{E}$ or, equivalently the electrostatic energy $W_{el} = QV$ acquired by a charge Q embedded in an external potential V.

We study now the relation between the first of these equations and the tensor field of the electron.

4.1. Maxwell equation

Maxwell's equation of classical electrostatics:

$$\text{div}\overrightarrow{D} = \rho \tag{52}$$

relates the charge density ρ in the small geometrical volume dv around a point M to the electric displacement $\overrightarrow{D}$ with $\overrightarrow{D} = \epsilon_0 \overrightarrow{E}$. It is tempting to use this relation to **define** ρ as a function of the tensor components. The integral of this equation over the whole volume would give the total electric charge associated to the particle. However, Equation (52) is established for the far field region and makes reference to vacuum properties through ϵ_0. It follows that the tensor field of the particle alone cannot lead to Equation (52). In other words, while $\overrightarrow{E}$ is created by the particle, ϵ_0 originates from the vacuum fields (the "noise") and $\overrightarrow{D}$ contains the properties of both components. The situation is completely different inside the particle (close to the origin) where the amplitude of the vacuum fields becomes small and negligible as compared to the a^i_j's. We are thus led to use expression (51) in the far field and to introduce the vacuum permittivity ϵ_0 in the next paragraph.

4.2. Lagrange equation

Lagrange's equations are applied to the potential components A^j which are considered to be the generalized coordinates. For each component A^j:

$$\sum_i \left[\frac{\partial \mathcal{L}_a}{\partial A^j} - \frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial A^j}{\partial x^i} \right)} \right) \right] = 0. \tag{53}$$

$\mathcal{L}_a$ is the Lagrangian density corresponding to the particle embedded in noise, where the components n_j^i are negligible inside the particle and preponderant outside. Terms $\partial \mathcal{L}_a / \partial(\partial A^j / \partial x^i)$ are the components of the 4-vector induction ${}_4\widetilde{D}$. This equation is again a tensor equation which we use in the spherical frame of coordinates. Its dimensions are $\mathrm{QL^{-3}T^{-1}}$ which shows that an electric charge can be recovered after an integration over time and geometrical space. Dimensions of $\vec{D}$ (defined in the 3-dimensional space) in Equation (52) are $\mathrm{QL^{-2}}$.

When Equation (53) is written for the scalar potential $A^0 = \phi / c$, the first term cancels because $\mathcal{L}_a$ does not depend on ϕ / c explicitly. The second term expresses the 4-divergence of the 4-vector induction ${}_4\widetilde{D}$ which can be split into its temporal and spatial parts:

$$ {}_4\widetilde{D} = \left(\frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial(\phi / c)}{\partial(ct)} \right)}, \vec{\widetilde{D}} \right). \tag{54}$$

The spatial term has only one (radial) component:

$$\vec{\tilde{D}} = \left(\frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial(\phi / c)}{\partial r} \right)},\ 0,\ 0 \right). \tag{55}$$

Equation (53) gives:

$$\frac{\partial}{\partial(ct)} \frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial(\phi / c)}{\partial(ct)} \right)} + \operatorname{div} \vec{\tilde{D}} = 0. \tag{56}$$

Here the divergence reduces to a single term:

$$\operatorname{div} \vec{\tilde{D}} = \frac{\partial}{\partial r} \frac{\partial(\phi/c)}{\partial r} = \frac{\partial \widetilde{D^r}}{\partial r}. \tag{57}$$

Now expression (56) is integrated over a sphere with a radius R large as compared to the particle size. Applying Gauss's theorem, one obtains:

$$\int dv \frac{\partial}{\partial(ct)} \frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial(\phi/c)}{\partial(ct)} \right)} + 4\pi R^2 \widetilde{D^r}\,(\mathrm{R}) = 0. \tag{58}$$

The electric induction in the far field region is given by Equation (51)

$$\widetilde{D^r}\,(\mathrm{R}) = D^r_{ff} = -2\mathcal{C} n_3^3\ n_4^4 - n_4^3\ n_3^4\ \frac{E_1^2(\mathrm{R})}{c}$$

$$= 2\mathcal{C} \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} (n_3^3\ n_4^4 - n_4^3\ n_3^4)\ J_1' \begin{cases} \cos \omega t, \\ -\sin \omega t. \end{cases} \tag{59}$$

The first term in Equation (56) is computed now:

$$\frac{\partial}{\partial(ct)} \frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial(\phi / c)}{\partial(ct)} \right)} = \frac{\partial}{\partial(ct)} \frac{\partial \mathcal{L}_a}{\partial a_1^1} = -\mathcal{C} \frac{\partial}{\partial(ct)} ((a_3^3)^2 a_2^2)$$

$$= -3\,\mathcal{C} \frac{\omega}{c} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \right)^3 \frac{J_1'^2}{\mathrm{x}^2} J_1'' \begin{cases} \sin^2 \omega t\ \cos \omega t, \\ -\cos^2\ \omega t\ \sin \omega t. \end{cases} \tag{60}$$

Integration over a spherical volume with $R \gg c / \omega$ is equivalent to an integration over an infinite volume as in Equation (38):

$$\int dv \frac{\partial}{\partial(ct)} \frac{\partial \mathcal{L}_a}{\partial\left(\frac{\partial(\phi / c)}{\partial(ct)}\right)} \sim -12\pi\, \mathcal{C} \frac{c^2}{\omega^2} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2}\right)^3$$

$$\times \int_0^\infty J_1'^2 \; J_1'' dx \begin{cases} \sin^2 \omega t \; \cos \omega t, \\ -\cos^2 \omega t \; \sin \omega t. \end{cases} \tag{61}$$

The integral in this equation is:

$$\int_0^\infty J_1'^2 \; J_1'' \; dx = \frac{1}{3} \left[J_1'^3\right]_0^\infty = \frac{1}{3^4}. \tag{62}$$

Now one uses Equations (59) and (61) in Equation (58), simplifies both terms by $\sin \omega t$ or $\cos \omega t$ ($\widetilde{D^r}$ reduces to D^r), and integrates over a period of time $2\pi / \omega$ to obtain:

$$\int_0^{2\pi/\omega} 4\pi\, R^2 |D^r(\mathrm{R})|\, dt = \frac{12\pi^2}{3^4} \mathcal{C} \frac{c^2}{\omega^3} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2}\right)^3. \tag{63}$$

Maxwell's Equation (52) gives the interpretation of these results: the l.h.s. gives the electric induction in the geometrical volume and the r.h.s. is the electric charge:

$$e = \mp \frac{12\pi^2}{3^4} \mathcal{C} \frac{c^2}{\omega^3} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2}\right)^3. \tag{64}$$

One notes the $\mp$ signs which appear in Equation (61) for the different parities of the solutions. They correspond to the electron and the positron.

The l.h.s. in Equation (56) relates the components of the noise tensor to the vacuum permittivity ϵ_0:

$$\epsilon_0 = 2 \frac{\mathcal{C}}{c} \int_0^{2\pi/\omega} (n_4^3 \; n_3^4 - n_3^3 \; n_4^4) dt. \tag{65}$$

Dimensions of this equation are: $\mathrm{M^{-1}Q^2L^{-3}T^2}$.

In passing, one notes that the integral which appears in Equation (62) nullifies if $n \neq 1$ ($J'_{n(\mathrm{x}=0)} = 0$ if $n \neq 1$). The space integral vanishes also for solutions containing Y_ℓ^m with m and $n \neq 0$. It follows that particles corresponding to solutions other than that for the electron are not electrically charged. However, they are still characterized by a field (term a_2^1 of the tensor). The conclusion is that it is the gravitational field and that the potential A^i contains also the gravitational potential.

The fundamental unit charge e of the electron can thus be expressed in terms of the constant $\mathcal{C}$, the potential A and the speed of light:

$$e = \frac{12\pi^2}{3^4}\frac{c^2}{\omega^3}\mathcal{C}\left(\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\omega}{c^2}\right)^3 = \frac{\sqrt{\pi}}{54}\frac{1}{c^4}\mathcal{C}\mathcal{A}^3. \tag{66}$$

Note that ω disappears again in this formula. Note also that the term $\frac{\partial}{\partial(ct)}\frac{\partial\mathcal{L}_a}{\partial\left(\frac{\partial(\phi/c)}{\partial(ct)}\right)}$ in Equation (60) is not an invariant. It follows that the charge e will not be the same for the spinning electron.

The integrand $J_1'^2\, J_1''$ represents the radial distribution of the charge. It is represented in Figure 4.

A series expansion in the vicinity of $\mathrm{x} = 0$ shows the convergence of this expression around the origin:

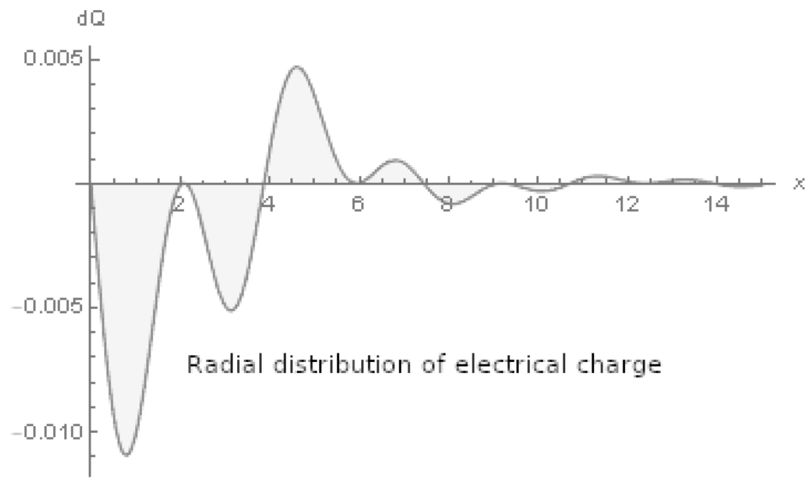

Figure 4. Radial distribution of electric charge for the electron vs. x, normalized distance to the center (arbitrary units).

$$J_1'^2 \; J_1'' \sim -\frac{\mathrm{x}}{45} + \frac{151\mathrm{x}^3}{9450} + \dots \,. \tag{67}$$

The absolute value of the local charge density at point M is explicitly written below for future use:

$$\rho = 3\pi\, \mathcal{C}\, \frac{1}{c} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \right)^3 \frac{J_1'^2}{\mathrm{x}^2} J_1''. \tag{68}$$

It is obtained from (60) after the removal of $\sin \omega t$ or $\cos \omega t$, and the integration over a period of time $2\pi / \omega$.

5. Spinning Electron

In this section, we will consider the particle in rotation around the z axis in the inertial frame of the laboratory. We first compute the tensor in this frame. Each volume element around a point M rotates with a tangential velocity which we will deduce from the principle of energy conservation. In a preliminary study, we have studied a Maxwell-

Boltzman distribution function for the angular velocity. However, this is only a phenomenological model. The theory should rest on more fundamental laws: we use below the principle of energy conservation which allows a harmonious transition with the preceding sections.

5.1. Tensors in the laboratory frame

The point M is subject to a rotation around the z axis as seen by an observer O_i in the inertial system of the laboratory. We compute now the expressions for the components of $[a_{ij}]$ in the system of fixed (overlined) coordinates of O_i. We consider the local infinitesimal coordinates cdt, dr, $r\,d\theta$, $r \sin\theta\, d\varphi$ which define the volume element dv around M and which are measured by the observer attached to M. These coordinates become $c\overline{dt}$, $\overline{dr}$, $\overline{r\,d\theta}$, $\overline{r \sin\theta d\varphi}$ for O_i. Both sets of coordinates are linked by a local, tangential, Lorentz transformation. The motion is along the $\vec{\varphi}$ axis and the two other local axes $\vec{r}$ and $\vec{\theta}$ are perpendicular to it. It follows that the coordinates dr and $r\,d\theta$ are not affected by the rotation. Only the length element $r \sin\theta\, d\varphi$ and the time element $c\,dt$ at event M are subject to the Lorentz transformation:

$$c\,\overline{dt} = \gamma c\,dt + \gamma\beta r \sin\theta\, d\varphi, \quad \bar{r} = r,$$

$$\bar{\theta} = \theta, \quad r \sin\theta\,\overline{d\varphi} = \gamma r\, \sin\theta\, d\varphi + \gamma\beta\, c\, dt. \tag{69}$$

The standard notations for the relative tangential velocity β, and for the Lorentz factor γ are used:

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{70}$$

β and γ depend upon coordinates and we will see that the rotation is completely different from that of a rigid body.

Note that the temporal phase ωt (which is a true scalar) is invariant in a Lorentz transformation: $\omega t = \overline{\omega}\bar{t}$. Note also that the factor c/ω which appears in the expression of the potential or its derivatives is the normalization parameter which transforms the invariant radial coordinate r into the non-dimensioned quantity x. This parameter is not modified here.

The Jacobian of the transformation is the Lorentz matrix:

$$\mathbb{J}_L = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{71}$$

which gives the components $\overline{b}_{pm}$ of the tensor in the laboratory frame:

$$\overline{b}_{pm} = (\mathbb{J}_L)_{hp}(\mathbb{J}_L)_{km} a_{hk}. \tag{72}$$

Now we write expression (12) for $[a_{hk}]$ under the compressed form:

$$[a_{hk}] = \begin{bmatrix} a_{11} & E_r/c & 0 & 0 \\ -E_r/c & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{33} \end{bmatrix}. \tag{73}$$

Applying transformation (72) gives the tensor expressed at M in the laboratory frame:

$$[a_{ik}]_{lab} = \begin{bmatrix} \gamma^2(a_{11} + a_{33}\beta^2) & \gamma E_r/c & 0 & \beta\gamma^2(a_{11} + a_{33}) \\ -\gamma E_r/c & a_{22} & 0 & -\beta\gamma E_r/c \\ 0 & 0 & a_{33} & 0 \\ \beta\gamma^2(a_{11} + a_{33}) & \beta\gamma E_r/c & 0 & \gamma^2(a_{33} + a_{11}\beta^2) \end{bmatrix}. \tag{74}$$

The antisymmetric part is:

$$[f_{ik}]_{lab} = \begin{bmatrix} 0 & \gamma E_r / c & 0 & 0 \\ -\gamma E_r / c & 0 & 0 & -\beta \gamma E_r / c \\ 0 & 0 & 0 & 0 \\ 0 & \beta \gamma E_r / c & 0 & 0 \end{bmatrix}. \tag{75}$$

As expected one notes the appearance of the magnetic field $B_\theta = \beta \gamma E_r / c$. It is oriented along the local axis $\vec{\theta}$. The behavior of B_θ is governed by β which is studied in the next paragraph.

The symmetric part is:

$$[s_{ik}]_{lab} = \begin{bmatrix} \gamma^2 (a_{11} + a_{33}\beta^2) & 0 & 0 & \beta \gamma^2 (a_{11} + a_{33}) \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ \beta \gamma^2 (a_{11} + a_{33}) & 0 & 0 & \gamma^2 (a_{33} + a_{11}\beta^2) \end{bmatrix}. \tag{76}$$

Expression (74) can be used in the far field region together with the noise tensor to compute the magnetic induction and the magnetic permeability μ_0. The calculation is similar to the computation of ϵ_0 (Equation (65)).

5.2. Energy conservation

The aim of this section is to apply the principle of energy conservation to the system in rotation. The idea is to consider a first, or initial state in the laboratory frame where the potential is flat everywhere. We associate a zero energy to this state. The second, or final state, corresponds to the creation of potential wells of negative energy as illustrated above (Figure 2) in the rest frame. The first and the second state should have the same energy [25]: the decrease of potential energy W_m is used to create the rotational kinetic energy $W_k : (W_k + W_m)_{\text{initial}} = (W_k + W_m)_{\text{final}} = 0$. This situation is ideal in the sense that we neglect the noise. We develop this idea below.

The theory of relativity tells us that mass and field are characterized each by a momentum-energy 4-vector. The time component is the energy divided by c and the spatial components are those of the momentum. Momenta are null in the eigenframe and appear in the laboratory frame. The Lorentz transformation changes the energy term and brings elementary momenta associated (1) to the mass element and (2) to the field (Poynting vector). The difference between the new energies and the proper energies (which are invariants) is the kinetic energy for the mass and a change of the field energy as well. The energy-momentum 4-vectors are separately written for the mass and the field parts because there is no energy exchanges between both parts of the tensor $[a_{ij}]$. After integrations the local changes will result in global quantities which can be compared with experimental figures.

The potential energy $\delta\mathcal{H}$ stored in the volume element around M has been computed in the preceding sections (Equation (36)) in the rest frame of the particle where the 4-momentum density vector ${}_4\vec{P}$ for the mass and the field is:

$${}_4\vec{P} = (\delta\mathcal{H} / c,\ 0,\ 0,\ 0). \qquad (77)$$

Applying a Lorentz transformation to this vector gives the energies and the momenta in the laboratory frame. The kinetic energy dW_k is the difference between energies in the eigenframe and the laboratory frame. After integration over time it writes:

$$dW_k = \frac{\gamma - 1}{c} \frac{3\mathcal{C}}{(4\pi)^2} \frac{\mathcal{A}^4 \omega^4}{c^8} \frac{\pi}{4} \frac{c}{\omega} \left(3\, J_1\ J_1'' \frac{J_1'^2}{\mathrm{x}^2} + J_1'^2 \frac{J_1'^2}{\mathrm{x}^2} \right). \qquad (78)$$

The mass part corresponds to the well-know formula: $dW_k(\text{mass}) = dm_0\, c^2\, (\gamma - 1)$. This kinetic energy is acquired at the expense of potential energy as indicated above. The principle of conservation of energy for the whole particle results in the equations for the mass and the field parts (see Equation (39)):

$$\int_0^\infty dx\, J_1'^2 (3\ J_1\ J_1'') = \int_0^\infty (\gamma - 1) J_1'^2\ (3\ J_1\ J_1'') dx, \tag{79a}$$

$$\int_0^\infty dx\, J_1'^2 \left(3\ J_1\ J_1'' + J_1'^2\right) = \int_0^\infty (\gamma - 1) J_1'^2 \left(3\ J_1\ J_1'' + J_1'^2\right) dx. \tag{79b}$$

The simplest solution to these equations is $\gamma = 2$. More generally, γ could be a function of x. A previous study used a Maxwell-Boltzman distribution for the angular velocity. However, this model gives a magnetic field which vanishes in the far field which contradicts the existence of the vacuum magnetic permeability μ_0. We have thus abandoned this model.

The ideal situation which is studied above deals with a lone electron in a perfect vacuum with no noise, i.e., without any interaction with other particles. This interaction can change γ a little bit: we will thus keep γ in the equations and consider it as an adjustable parameter whose exact value will be obtained from a comparison with experimental values.

Let us write now the expressions for the spinning charge $\bar{e}$, the angular momentum L and magnetic moment μ as functions of the theoretical parameters $\mathcal{C}$, $\mathcal{A}$, $\overline{\varpi}$ and γ. These expressions will be written in the geometrical space. The experimental values of $m, \bar{e}, L$ and μ should give the numerical values of the parameters.

5.3. Spinning charge

The density of charge ρ has been computed in Subsection 4.2 in the eigenframe of the tensor. For this purpose, Lagrange's equation was written for the scalar potential and the temporal derivative was shown to give the charge density ρ. However, we cannot use this expression for ρ in the case of the spinning electron because ρ is not invariant in a coordinates change. One should write Maxwell's equation in the form: $\mathrm{div}\overline{\vec{D}} = \overline{\rho}$ where the notation for the charge density in the laboratory frame is $\overline{\rho}$. In this frame, the potential is expressed as $\overline{A^i} = (\gamma\phi / c,\ A^r,\ 0,\ \gamma\beta\phi / c)$ and the elementary coordinates are given by Equations (69): $du^i = (c\,\overline{dt},\ r,\ r\theta,\ r\sin\theta\,\overline{d\varphi})$. The components of the potential derivative are $\partial A^i / \partial u^j$ and the mixed tensor can be expressed in terms of the components of $[a^i_k]$ in the eigenframe:

$$[a^i_k]_{lab} = \begin{bmatrix} \gamma^2(a_{11} + a_{33}\beta^2) & \gamma E_r / c & 0 & -\beta\gamma^2(a_{11} + a_{33}) \\ \gamma E_r / c & -a_{22} & 0 & -\beta\gamma E_r / c \\ 0 & 0 & -a_{33} & 0 \\ \beta\gamma^2(a_{11} + a_{33}) & \beta\gamma E_r / c & 0 & -\gamma^2(a_{33} + a_{11}\beta^2) \end{bmatrix}. \tag{80}$$

The determinant of $[a^i_k]_{lab}$ being the same as that of $[a^i_k]$, it follows that the derivative of the Lagrangian with respect to $c\bar{t}$ is the minor relative to $[a^1_1]_{lab}$. Lagrange's equation gives:

$$\mathrm{div}\,_4\vec{D} = \frac{\partial}{c\partial\bar{t}} \begin{Vmatrix} -a_{22} & 0 & -\beta\gamma E_r / c \\ 0 & -a_{33} & 0 \\ \beta\gamma E_r / c & 0 & -\gamma^2(a_{33} + a_{11}\beta^2) \end{Vmatrix}$$

$$= -\gamma^2 \frac{\partial}{c\partial\bar{t}}(a_{33}(a_{33}(a_{22} + \beta^2(a_{11}a_{22} + E_r / c)^2))). \tag{81}$$

It remains to replace the (a_{ij})'s and E_r / c by their expressions, simplify both members by $\sin\overline{\omega}\bar{t}$ or $\cos\overline{\omega}\bar{t}$ to have the modulus and to integrate over a period of time $\overline{T} = 2\pi / \overline{\omega}$ to obtain the electrical density $\overline{\rho}$. We recopy below the expressions of the (a_{ij})'s for the even solution (from Equation 12):

$$a_{11} = -\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \sin \omega t \; J_1,$$

$$a_{12} = E_r / c = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \cos \omega t \; J_1',$$

$$a_{22} = -\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \sin \omega t \; J_1'',$$

$$a_{33} = -\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\omega}{c^2} \sin \omega t \frac{J_1'}{\mathrm{x}}. \tag{82}$$

The temporal derivative is:

$$\frac{\partial}{c\partial\bar{t}} \left(\frac{\partial \mathcal{L}_a}{\partial \left(\frac{\partial \bar{\phi} / c}{c\partial \bar{t}} \right)} \right) = \frac{\gamma^2 \mathcal{A}^3 \mathcal{C} \, \omega^3 \overline{\omega} \, J_1' \cos(\overline{\omega}\, \bar{t}\,)}{16 c^7 \pi^{3/2} \mathrm{x}^2}$$

$$\times (3J_1' \; J_1'' + (J_1'^2 - 3J_1 \; J_1'')\mathrm{x}\,\beta^2 + 3(J_1' \; J_1''$$

$$+ (-J_1'^2 + J_1 J_1'')\mathrm{x}\,\beta^2) \cos(2\overline{\omega}\bar{t}\,)). \tag{83}$$

The integral of the modulus over a period of time $2\pi / \overline{\omega}$ gives the charge density:

$$\overline{\rho} = -\frac{\gamma^2 \, \omega^3 \, \mathcal{A}^3 \, \mathcal{C}}{8 c^7 \sqrt{\pi}\, \mathrm{x}^2} J_1' \; (-3J_1' \; J_1'' + (J_1'^2 - 3J_1' \; J_1'')\, \mathrm{x}\, \beta^2). \tag{84}$$

The total electric charge of the electron in the laboratory frame is obtained after replacing J_1, J_1' and J_1'' by their expressions (19) and integrating over the whole volume. The radial integrals are exactly computed:

$$I_{e1} = \frac{\sqrt{\pi}}{2}\int_0^\infty J_1' \,(-3J_1' \;\; J_1'')dx = \frac{\sqrt{\pi}}{54},$$

$$I_{e2} = \frac{\sqrt{\pi}}{2}\int_0^\infty J_1' x \left(J_1'^2 - 3J_1 \;\; J_1''\right)dx$$

$$= \frac{3\sqrt{\pi}}{160}\, 81(-28 + 23\mathrm{Log}\,[3]) \tag{85}$$

and the total spinning charge is:

$$\bar{e} = (I_{e1} + I_{e2}\beta^2)\gamma^2 \,\frac{\mathcal{A}^3\mathcal{C}}{c^4}$$

$$= \frac{\sqrt{\pi}\gamma^2(80 + 81\beta^2(\;-28 + 23\mathrm{Log}\,[3]))}{4320}\,\frac{\mathcal{A}^3\mathcal{C}}{c^4}. \tag{86}$$

For $\gamma = 2$, one finds:

$$\bar{e} = 0.1410807\ldots\frac{\mathcal{A}^3\mathcal{C}}{c^4}. \tag{87}$$

5.4. Angular momentum

The angular momentum $\mathbf{L}$ with respect to O is computed now. It is obtained from the sum of the elementary contributions $\mathbf{L}(M)$ at points M. It is a pseudovector in the geometrical space. The local velocity is oriented along $\overrightarrow{\varphi^0}$. We keep the component $d\mathbf{L}_\theta$ which is oriented along the axis $\overrightarrow{\theta^0}$:

$$d\mathbf{L}_\theta = r(\gamma\, dm_0)\,(\beta c).$$

The other component $d\mathbf{L}_r$ which is oriented along the axis $\overrightarrow{r^0}$ will give a null contribution after the volume integration.

The total angular momentum is oriented along the z axis. It is the sum of the projection on this axis of all elementary contributions $d\mathbf{L}_\theta \sin\theta$. It is obtained after an integration over the geometrical volume in the laboratory frame where the volume element is:

$$\overline{dv} = dr(r\ d\theta)(r\ \sin\theta\,\overline{d\varphi}). \tag{88}$$

Integration gives:

$$\mathbf{L}_z = \gamma\beta\frac{\mathcal{C}\mathcal{A}^4}{c^4}\frac{1}{\omega}\frac{\pi(43 - 76\mathrm{Log}\,[2])}{6400}, \tag{89}$$

where we have included the numerical value of the radial integral:

$$I_L = \int_0^\infty \mathrm{x}\ J_1\ J_1''\ J_1'^2\ d\mathrm{x} = \frac{43 - 76\mathrm{Log}\,[2]}{900}.$$

For $\gamma = 2$, one finds:

$$\mathbf{L}_z = -0.00822942\ldots\frac{\mathcal{A}^4\mathcal{C}}{\omega c^4}. \tag{90}$$

5.5. Magnetic moment

The tensor field in the fixed inertial frame of the laboratory displays a magnetic field, showing that the spinning electron is a little magnet. An elementary electric current di is associated to each volume dv around M. This current is the quantity of charges which crosses the elementary surface $r\ d\theta\ dr$ during the unit of time:

$$di = \overline{\rho}\, r\, d\theta\, dr\, \beta c, \tag{91}$$

and the modulus of the current density is $di_s = \rho\,\beta c$. Its direction is along the axis $\vec{\varphi}$. One applies the definition of the differential magnetic moment:

$$\overrightarrow{d\mu} = \frac{1}{2}\vec{r}\wedge\overrightarrow{j_s}dv = \frac{1}{2}\frac{c}{\omega}\mathrm{x}\,\overline{\rho}\,\beta c\vec{\theta}. \tag{92}$$

The component $d\mu_z$ along the z axis writes:

$$d\mu_z = \frac{1}{2}\frac{c}{\omega}\,\mathrm{x}\sin\theta\,\bar{\rho}\,\beta c. \tag{93}$$

Integrating over the whole volume gives the total magnetic moment:

$$\mu_e = \int_{\mathrm{Vol}} d\mu_z \frac{c^3}{\omega^3} x^2 d\mathrm{x} \sin\theta\, d\bar{\varphi}. \tag{94}$$

Using expression (84) in $d\mu_z$, one finally obtains:

$$\mu_e = \frac{\pi^{5/2}\beta\,\gamma^2(-3+140\beta^2)}{15360}\frac{\mathcal{C}\mathcal{A}^3}{\omega c^2}. \tag{95}$$

For $\gamma = 2$, one finds:

$$\mu_e = 0.402415\ldots\frac{\mathcal{A}^3\mathcal{C}}{\omega c^2}. \tag{96}$$

The calculations are performed with a mathematical software [26].

6. Physical Constants

The parameters which we have introduced in the theory are:

- the amplitude $\mathcal{A}$ and the frequency ω of the scalar potential ϕ / c,

- the constant $\mathcal{C}$ which relates the energy and the Lagrangian densities,

- the Lorentz parameter γ which we found to be close to 2 to obey the principle of energy conservation.

We have obtained the expressions of the mass (Equation (45)), the electric charge (Equation (87)), the angular momentum (Equation (90)) and the magnetic moment (Equation (95)) as functions of these parameters. We group below these formula (written with $\gamma = 2$) together with the experimental values [27], written with the index exp.

$$m_0 = 0.00289\ldots \mathcal{C}\,\frac{\mathcal{A}^4}{c^6},$$

$$m_{0\,\mathrm{exp}} = 9.1093837015 \times 10^{-31}\,\mathrm{kg}, \tag{97a}$$

$$\bar{e} = 0.1410807\ldots \frac{\mathcal{A}^3\mathcal{C}}{c^4},$$

$$\bar{e}_{\mathrm{exp}} = 1.602176634 \times 10^{-19}\,\mathrm{C}, \tag{97b}$$

$$\mathbf{L}_z = -0.00822942\ldots \frac{\mathcal{A}^4\mathcal{C}}{\omega c^4},$$

$$\mathbf{L}_{z\,\mathrm{exp}} = \frac{\hbar}{2} = 5.27286 \times 10^{-35}\,\mathrm{Js}. \tag{97c}$$

$$\mu_e = 0.402415\ldots \frac{\mathcal{A}^3\mathcal{C}}{\omega c^2},$$

$$\mu_{e\,\mathrm{exp}} = 9.2847647043 \times 10^{-24}\,\mathrm{J/T}. \tag{97d}$$

These expressions are used now to compute the values of the different theoretical parameters.

We first compute the dimensionless Landé factor g of the electron. The definition of g is the ratio of $(\mu_e / \mathbf{L}_z)/(\bar{e}/2m_0)$. For $\gamma = 2$, one finds:

$$g = \frac{\mu_e / \mathbf{L}_z}{\bar{e}/2m_0} = -2.00338317. \tag{98}$$

This value of the Landé factor is a major result of the theory.

The experimental value is $-2.00231930436 2$. The difference between both figures is probably due to the noise, or to the particle environment effect. It is interesting to note that Paul Dirac found also $g = 2$ and that the difference was computed from the properties of the vacuum in the standard model. The value of γ giving the experimental figure $g_{\exp}$ is the solution of the equation:

$$\frac{\mu_e / \mathbf{L}_z}{\bar{e} / 2m_0} = g_{\exp} = -2.00231930436 2. \tag{99}$$

One finds $\gamma = 2.0006639663$ which is very close to 2. For this value, the relative speed $\beta = 0.86612$. The following numbers are computed with this corrected value of γ.

Solving Equation (97a) and Equation (97b) for $\mathcal{A}$ and $\mathcal{C}$ gives:

$$\mathcal{A} = 2.497264943 \times 10^7 \ (\mathrm{MT^{-2}L^2Q^{-1}}), \tag{100a}$$

$$\mathcal{C} = 5.88380001 \times 10^{-7} \ (\mathrm{M^{-3}T^2L^{-2}Q^4}). \tag{100b}$$

Dimensions of $\mathcal{A}$ and $\mathcal{C}$ are given in parenthesis.

The remaining symbol to be evaluated is the normalization factor ω. Inserting the preceding values of γ, $\mathcal{A}$ and $\mathcal{C}$ in the expressions (97c) of $\mathbf{L}_z$ or (97a) of μ_e gives: $\omega = 4.423325945 \times 10^{21}$ rad / s. It follows that the unit on the x axis of the figures is $c / \omega = 6.777510^{-14}\,\mathrm{m}$ which gives an idea of the electron size. We note that the Compton wavelength corresponds to an angular frequency $\omega_c = 7.76\ldots \times 10^{20}$ rad / s. It is outside the scope of the present work to study the Compton diffusion process which leads to the difference between ω and ω_c.

This calculation achieves the primary objective of this theory which was to demonstrate that the physical constants which characterize the electron are not independent. The vortex model based on energy conservation leads to satisfying results.

7. Conclusion

We have developed a theory of the electron which is based on well-known and well-proved classical notions and principles:

• The base paradigm is the 4-dimensional electromagnetic potential in Minkowski's spacetime;

• The second notion is the covariant derivative, or the gradient, of this potential which groups the 16 partial derivatives in a tensor;

• The principles of symmetry conservation and invariance of some quantities belonging to the tensors in a coordinate change are applied;

• The association of a Lagrangian density to one invariant together with the principle of least action allows the calculation of the mass and charge densities;

• The principle of relativity and the Lorentz transformation are applied to describe the spinning electron in the laboratory frame;

• The principle of energy conservation is used to find the rotation speed of the spinning particle;

• The usual standard definitions of the angular momentum and the magnetic moment are used.

The theory is essentially based on the study of the symmetric part $[s_{ik}]$ of the covariant derivative $[a_{ik}] = \partial A_i / \partial x^k$ of the 4-potential A_i in spacetime x^k. $[s_{ik}]$ can be diagonalized and is characterized by an eigenframe of coordinates. A relation between the derivatives of the scalar and the vector potential occurs in this frame. When this relation is associated to the conservation of the trace of $[s_k^i]$ in a time translation, a Helmholtz equation results. Among its solutions, one of them is electrically charged and represents the electron. The tensor field is obtained by replacing the potential by this solution at each point. Each tensor describes local properties such as the electromagnetic field. A Lagrangian density is associated to the determinant of $[s_k^i]$ which allows the calculation of the potential energy, or the mass of the electron. The Lagrangian of $[a_k^i]$ is defined in the same way and used in Lagrange's equations to compute the electric charge. The spinning electron is studied. The principle of energy conservation is applied to find the particle in rotation. This makes the electron looks like a vortex with the same tangential speed everywhere. A Lorentz transformation of local coordinates from the tensor eigenframe to the laboratory frame allows us to find the spinning electric charge, the angular momentum and the magnetic moment. Comparing theoretical formulas with experimental figures gives the numerical values of the four parameters used in the equations. Neglecting noise effects, a value of the Lorentz parameter is found to be 2. This value leads to a Landé factor $g = 2.0033\ldots$ which we consider to be a very satisfactory result. The fundamental peculiarity of the theory is that it starts from local characteristics of the electron at each point of its structure. Integration over the whole volume leads to the measurable global properties.

The different parameters which are used in the equations are (1) the amplitude, (2) the frequency of the basic solution, (3) the proportionality constant between the determinant of the tensor and the energy density and (4) the tangential speed of rotation, or the Lorentz factor γ. We have given the equations which link these parameters to the known characteristics of the electron: mass, electrical charge, angular momentum, magnetic moment. We have found that its fundamental frequency of vibration is close to the Compton frequency. A complete coherency between theoretical symbols and measured values is obtained. A conclusion is that the electron structure as it is computed here fits reasonably the intuitive image of a rotating body with a circulating current. However, it is rather a vortex than a rigid body like a top: the angular frequency decreases from the center to the outside of the particle. Its representation needs a tensor field whose local properties include electromagnetic and matter waves. These are sums of incoming and outgoing waves which extend over the whole universe.

The electron is not a superposition of disconnected quantities: waves, electric and magnetic fields, mass and electrical charge. These are all included in $D(A_i)$, the covariant derivative of the potential. A classical description of any experiment (like diffraction experiments) with electrons should thus be done with $D(A_i)$ as the primary tool. Understanding the electron structure would allow to use its properties in practical devices: for instance mastering its fundamental frequency would lead to extraordinary precise clocks. But maybe the most important conclusion of this theory is that it offers a new way to study the yoctoscopic world of elementary particles. This way is that of a "bottom-up" approach which is different from (and complementary to) ordinary quantum physics which is a "top-down" theory.

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8. Annex

The 16 partial derivatives $\partial A^i / \partial x^k$ of the 4-potential $A^i = (\phi / c,\ A^x,\ A^y,\ A^z)$ in spacetime $x^i = (ct,\ x,\ y,\ z)$ define the components of the mixed tensor $D(A^i)$ in the Cartesian frame:

$$[a_k^i] = \begin{bmatrix} \frac{\partial(\phi/c)}{c\partial t} & \frac{\partial(\phi/c)}{\partial x} & \frac{\partial(\phi/c)}{\partial y} & \frac{\partial(\phi/c)}{\partial z} \\ \frac{\partial A^x}{c\partial t} & \frac{\partial A^x}{\partial x} & \frac{\partial A^x}{\partial y} & \frac{\partial A^x}{\partial z} \\ \frac{\partial A^y}{c\partial t} & \frac{\partial A^y}{\partial x} & \frac{\partial A^y}{\partial y} & \frac{\partial A^y}{\partial z} \\ \frac{\partial A^z}{c\partial t} & \frac{\partial A^z}{\partial x} & \frac{\partial A^z}{\partial y} & \frac{\partial A^z}{\partial z} \end{bmatrix}.$$

The metric tensor is taken to be:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

It follows that the covariant tensor is:

$$[a_{ik}] = \begin{bmatrix} \frac{\partial(\phi / c)}{c\partial t} & \frac{\partial(\phi / c)}{\partial x} & \frac{\partial(\phi / c)}{\partial y} & \frac{\partial(\phi / c)}{\partial z} \\ -\frac{\partial A_x}{c\partial t} & -\frac{\partial A_x}{\partial x} & -\frac{\partial A_x}{\partial y} & -\frac{\partial A_x}{\partial z} \\ -\frac{\partial A_y}{c\partial t} & -\frac{\partial A_y}{\partial x} & -\frac{\partial A_y}{\partial y} & -\frac{\partial A_y}{\partial z} \\ -\frac{\partial A_z}{c\partial t} & -\frac{\partial A_z}{\partial x} & -\frac{\partial A_z}{\partial y} & -\frac{\partial A_z}{\partial z} \end{bmatrix}.$$

This tensor is divided into its symmetric and antisymmetric parts (Equations (1) and (2)). Elements of the first line of the symmetric part are:

$$s_{11} = 2\frac{\partial(\phi / c)}{c\partial t},$$

$$s_{12} = \frac{\partial(\phi / c)}{\partial x} - \frac{\partial A_x}{c\partial t} = s_{21} = -s_1^2,$$

$$s_{13} = \frac{\partial(\phi / c)}{\partial y} - \frac{\partial A_y}{c\partial t} = s_{31} = -s_1^3,$$

$$s_{14} = \frac{\partial(\phi / c)}{\partial z} - \frac{\partial A_z}{c\partial t} = s_{41} = -s_1^4. \tag{101}$$

$[s_{ik}]$ or $[s_k^i]$ can be diagonalized which means that a proper time and a eigen-geometrical frame of coordinates exist where all elements but the diagonal cancel.

Let us describe the series of transformations which will lead to a tensor with elements $\overline{s_2^1} = \overline{s_3^1} = \overline{s_4^1} = 0$. One starts with $[s_k^i]$ and the first operation is a diagonalization of the lower bottom-right 3×3 block of this tensor in the geometrical space. Only the spatial derivatives of the potential vector are concerned by this transformation which gives:

$$[\widehat{s_j^i}] = \mathbf{M}\ s_k^i\ \mathbf{M}^{-1} = \begin{pmatrix} s_1^1 & s_2^1 & s_3^1 & s_4^1 \\ -s_2^1 & -\widehat{s_2^2} & 0 & 0 \\ -s_3^1 & 0 & -\widehat{s_3^3} & 0 \\ -s_4^1 & 0 & 0 & -\widehat{s_4^4} \end{pmatrix}, \tag{102}$$

where the diagonalizing matrix $\mathbf{M}$ has the form:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & m_{32} & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{pmatrix}. \tag{103}$$

Now let us transform $[\widehat{s_j^i}]$ with the Lorentz operator:

$$L_x = \begin{pmatrix} \gamma_x & v_x\gamma_x & 0 & 0 \\ v_x\gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The relative speed of the tensor with respect to the observer along the x axis is noted v_x in units of c and $\gamma_x = 1/\sqrt{1-v_x^2}$.

The transformed tensor is:

$$[\widehat{s_j^i}] = L_x \cdot [\hat{s}_j^i] \cdot L_x^{-1}$$

$$
= \begin{pmatrix}
(s_1^1 + v_x(\hat{s}_2^2 v_x - 2s_2^1))\gamma_x^2 & (s_2^1(v_x^2 + 1) - (s_1^1 + \hat{s}_2^2)v_x)\gamma_x^2 & s_3^1\gamma_x & s_4^1\gamma_x \\
-(s_2^1(v_x^2 + 1) - (s_1^1 + \hat{s}_2^2)v_x)\gamma_x^2 & -(\hat{s}_2^2 + v_x^2(s_1^1 v_x - 2s_2^1))\gamma_x^2 & s_3^1 v_x\gamma_x & s^1 - 4v_x\gamma_x \\
-s_3^1\gamma_x & s_3^1 v_x\gamma_x & -\hat{s}_3^3 & 0 \\
-s_4^1\gamma_x & s_4^1 v_x\gamma_x & 0 & -\hat{s}_4^4
\end{pmatrix}. \tag{104}
$$

Now we choose v_x to be a solution of the equation:

$$
s_2^1(v_x^2 + 1) - (s_1^1 + \hat{s}_2^2)v_x = 0, \tag{105}
$$

i.e.,

$$
v_x = \frac{s_1^1 + \hat{s}_2^2}{2s_2^1} \pm \frac{\sqrt{(s_1^1 + \hat{s}_2^2)^2 - 4(s_2^1)^2}}{2s_2^1}. \tag{106}
$$

Inserting one of these values of v_x in $[\widehat{s_j^i}]$, elements of the first line become:

$$
\begin{aligned}
\hat{\hat{s}}_{11} &= (s_1^1 + v_x(\hat{s}_2^2 v_x - 2s_2^1))\gamma_x^2, \\
\hat{\hat{s}}_2^1 &= 0, \\
\hat{\hat{s}}_3^1 &= s_3^1\gamma_x, \\
\hat{\hat{s}}_4^1 &= s_4^1\gamma_x.
\end{aligned} \tag{107}
$$

One notes also that this manipulation (block diagonalization followed by a Lorentz transformation along *x*) results in the multiplication of the original s_3^1 and s_4^1 by γ_x. It is clear now that the same manipulation done on the y and then on the z axis will allow to cancel terms s_3^1 and s_4^1 in the same way.

After the last Lorentz transformation, the form of the tensor is such that its elements in the first line and the first column are null, with the exception of s_1^1.

Without any further developments, we will accept the theorem following which the tensor and the observer share the same proper time if the terms $s_2^1 = s_3^1 = s_4^1 = s_1^2 = s_1^3 = s_1^4 = 0$. Let us apply this rule to the tensor $D(A_i)$ when it is written in the spherical frame of coordinates:

$$[a_k^i] = \begin{bmatrix} \phi_{,t} & \phi_{,r} & \phi_{,\theta}/r & \dfrac{\phi_{,\varphi}}{r\sin\theta} \\ A^r_{,t} & A^r_{,r} & \dfrac{1}{r}(A^r_{,\theta} - A^\theta) & \dfrac{1}{r\sin\theta}(A^r,\ \varphi - \sin\theta A^\varphi) \\ A^\theta_{,t} & A^\theta_{,r} & \dfrac{1}{r}(A^\theta_{,\theta} + A^r) & \dfrac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta A^\varphi) \\ A^\varphi_{,t} & A^\varphi_{,r} & \dfrac{1}{r}A^\varphi_{,\theta} & \dfrac{1}{r\sin\theta}A^\varphi_{,\varphi} + \dfrac{1}{r}A^r + \dfrac{1}{r}\dfrac{\cos\theta}{\sin\theta}A^\theta \end{bmatrix}. \tag{108}$$

We have used the compressed notation for the derivatives, for instance; $\partial A^r / c\partial t \equiv A^r_{,t}$.

The elements of the mixed symmetric tensor which should cancel in the proper time reference are:

$$s_2^1 = \phi_{,r} + A^r_{,t} = 0,$$

$$s_3^1 = \phi_{,\theta}/r + A^\theta_{,t} = 0,$$

$$s_4^1 = \frac{\phi_{,\varphi}}{r\sin\theta} + A^\varphi_{,t} = 0. \tag{109}$$

Adding s_2^1, s_3^1 and s_4^1 together gives:

$$\frac{\partial \phi / c}{\partial r} + \frac{1}{r}\frac{\partial \phi / c}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \phi / c}{\partial \varphi} = -\frac{\partial A^r}{c\partial t} - \frac{\partial A^\theta}{c\partial t} - \frac{\partial A^\varphi}{c\partial t} \qquad (110)$$

or:

$$\overrightarrow{\text{grad}}\, \phi / c = -\frac{\partial \vec{A}}{c\partial t}. \qquad (111)$$

This is the first formula which is used to obtain the fundamental Helmholtz equation for the elementary particles. The second equation is obtained from the trace of $[a_k^i]$. This trace is invariant in a coordinates change:

$$\text{Trace}\,[a_k^i] = \phi_{,t} + \frac{\partial A^r}{\partial r} + \frac{1}{r}(A^\theta_{,\theta} + A^r)$$
$$+ \frac{1}{r \sin\theta} A^\varphi_{,\varphi} + \frac{1}{r} A^r + \frac{1}{r}\frac{\cos\theta}{\sin\theta} A^\theta.$$

The expression of the divergence of the vector $\vec{A} = (A^r, A^\theta, A^\varphi)$ in spherical coordinates is:

$$\text{div}\,\vec{A} = \frac{\partial A^r}{\partial r} + 2\frac{A^r}{r} + \frac{1}{r}\frac{\partial A^\theta}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A^\varphi}{\partial \varphi} + \frac{\cos\theta}{r \sin\theta} A^\theta \qquad (112)$$

and thus:

$$\text{Trace}\,[a_k^i] = \phi_{,t} + \text{div}\,\vec{A}. \qquad (113)$$

The invariance of $\text{Trace}\,[a_k^i]$ in a time translation leads to the second formula which is used to obtain the fundamental Helmholtz equation.