Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 14, Issue 1, 2020, Pages 1-15
This paper is available online at http://www.frdint.com/
Published online June 27, 2020

THE TRANSMUTED UNIFORM-EXPONENTIAL (GENERALIZED LAMBDA) DISTRIBUTION WITH APPLICATION TO WHEATON RIVER DATA

CLEMENT BOATENG AMPADU

31 Carrolton Road
Boston, MA 02132-6303
USA
e-mail: drampadu@hotmail.com

Abstract

The Uniform-Exponential (Generalized Lambda) distribution [1] have been shown to be practical in modeling real life data, in particular the Wheaton river data, Table 6 [1]. In the present paper, we introduce a so-called Transmuted Uniform-Exponential (Generalized Lambda) distribution, and compare its performance with the Uniform Exponential (Generalized Lambda) distribution in modeling the Wheaton river data. Some properties of the Transmuted Uniform-Exponential (Generalized Lambda) distribution are presented.

Keywords and phrases: transmuted distribution, uniform distribution, exponential distribution, generalized lambda distribution, wheaton river data.

2020 Mathematics Subject Classification: 62Exx.

Received March 3, 2020; Accepted June 1, 2020

1. Uniform Distribution

Recall from [2], the uniform distribution with parameters $-\infty < a < b < \infty$ has PDF

$$f_{a,b}(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b], \\ 0, & \text{if otherwise} \end{cases}$$

and CDF

$$F_{a,b}(x) = \begin{cases} 0, & \text{if } x < a, \\ \frac{x-a}{b-a}, & \text{if } x \in [a, b), \\ 1, & \text{if } x \geq b. \end{cases}$$

2. Exponential Distribution

Recall from [3], the exponential distribution with parameter $\lambda > 0$ and $x \in [0, \infty)$ has PDF

$$f_{\lambda}(x) = \lambda e^{-\lambda x}$$

and CDF

$$F_{\lambda}(x) = 1 - e^{-\lambda x}.$$

3. Generalized Lambda Distribution

According to [1], the four-parameter generalized lambda distribution is defined in terms of its quantile function, this distribution was proposed by Ramberg and Schmeiser [4]. In particular, with parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $0 < u < 1$, the quantile function is given by

$$Q_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}.$$

When $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \lambda_4$, we obtain the Tukey lambda distribution [5].

4. The Uniform-Exponential (Generalized Lambda) Family of Distributions

Recall from [1], the CDF of the Uniform-Exponential (Generalized Lambda) Family of Distributions, for $x \geq 0;\ \theta, \lambda_3, \lambda_4 > 0,$ is given by

$$F_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\left[1 + (1 - e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right]$$

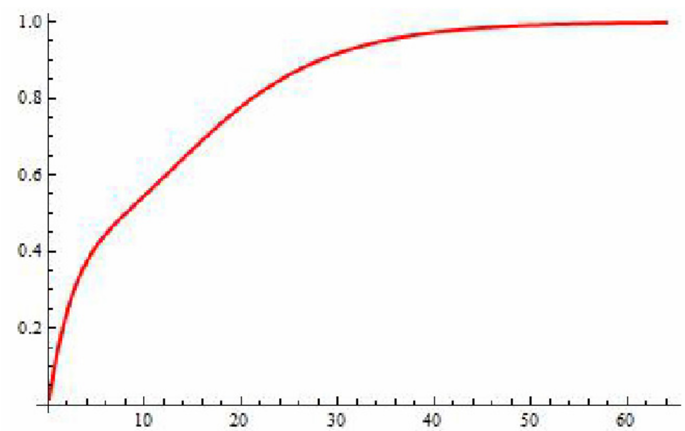

Figure 1. The graph of $F_{0.1134, 5.3192, 3.0133}(x)$.

and the PDF, for $x \geq 0;\ \theta, \lambda_3, \lambda_4 > 0,$ is given by

$$f_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3(1 - e^{-\theta x})^{\lambda_3 - 1} + \lambda_4(e^{-\theta x})^{\lambda_4 - 1}\right].$$

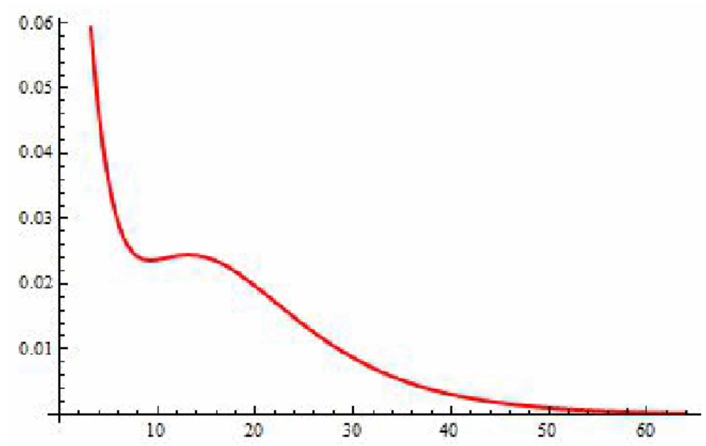

Figure 2. The graph of $f_{0.1134,\,5.3192,\,3.0133}(x)$.

5. The Transmuted Uniform-Exponential (Generalized Lambda) Family of Distributions

Definition 5.1 (Owoloko et al. [6]). A random variable X is said to have a transmuted distribution if its PDF and CDF are, respectively, given by

$$f(x) = g(x)[1 + \xi - 2\xi G(x)],$$

$$F(x) = (1 + \xi)G(x) - \xi[G(x)]^2,$$

where $x > 0$, $-1 \leq \xi \leq 1$ and the baseline distribution has PDF and CDF, respectively, given by, $g(x)$ and $G(x)$.

Definition 5.2. A random variable X is said to have a transmuted uniform-exponential (generalized lambda) distribution if its PDF and CDF are, respectively, given by

$$f_{\theta, \lambda_3, \lambda_4, \xi}(x) = g_{\theta, \lambda_3, \lambda_4}(x)[1 + \xi - 2\xi G_{\theta, \lambda_3, \lambda_4}(x)],$$

$$F_{\theta, \lambda_3, \lambda_4, \xi}(x) = (1 + \xi)G_{\theta, \lambda_3, \lambda_4}(x) - \xi[G_{\theta, \lambda_3, \lambda_4}(x)]^2,$$

where $x,\ \theta,\ \lambda_3,\ \lambda_4\ > 0,\ -1 \le \xi \le 1$ and

$$G_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\left[1 + (1 - e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right]$$

and

$$g_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3(1 - e^{-\theta x})^{\lambda_3 - 1} + \lambda_4(e^{-\theta x})^{\lambda_4 - 1}\right].$$

Remark 5.3. Henceforth a random variable X having the transmuted uniform-exponential (generalized lambda) distribution will be denoted as $X \sim TUEGL(\theta,\ \lambda_3,\ \lambda_4,\ \xi)$.

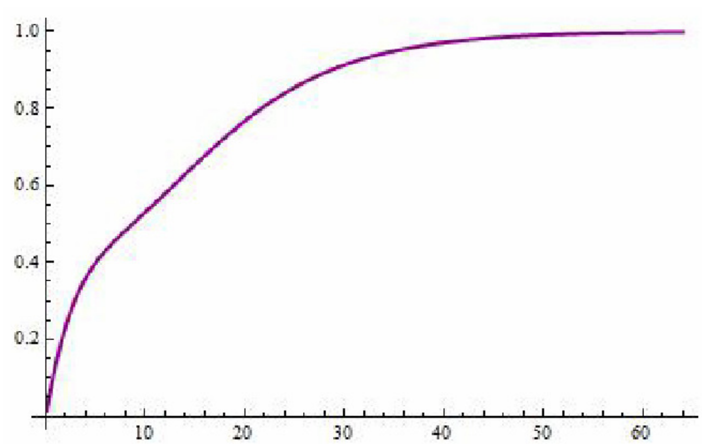

Figure 3. The CDF of $TUEGL\,(0.1134,\ 5.3192,\ 3.0133,\ -0.065271)$.

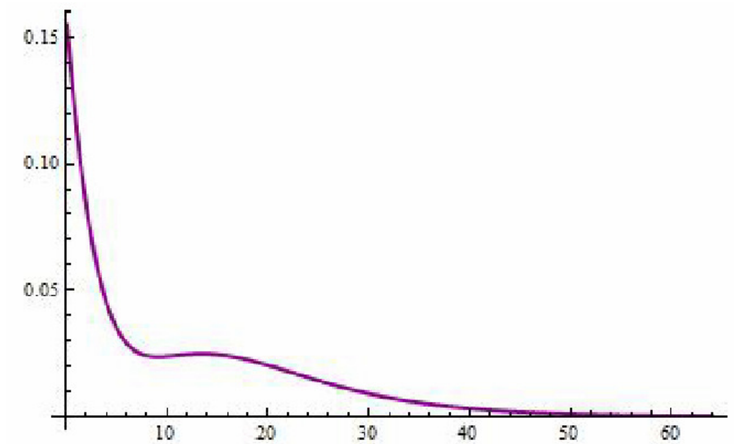

Figure 4. The PDF of $TUEGL(0.1134,\ 5.3192,\ 3.0133,\ -0.065271)$.

6. Application to Wheaton River Data

6.1. The maximum likelihoood estimates in the TUEGL distribution

From Definition 5.2, the likelihood function in the TUEGL distribution is given by

$$L = \prod_{i=1}^{n} \left\{ g_{\theta, \lambda_3, \lambda_4}(x_i)[1 + \xi - 2\xi G_{\theta, \lambda_3, \lambda_4}(x_i)] \right\},$$

where

$$G_{\theta, \lambda_3, \lambda_4}(x_i) = \frac{1}{2}\left[1 + (1 - e^{-\theta x_i})^{\lambda_3} - (e^{-\theta x_i})^{\lambda_4}\right]$$

and

$$g_{\theta, \lambda_3, \lambda_4}(x_i) = \frac{1}{2}\theta e^{-\theta x_i}\left[\lambda_3(1 - e^{-\theta x_i})^{\lambda_3 - 1} + \lambda_4(e^{-\theta x_i})^{\lambda_4 - 1}\right].$$

The estimates in the TUEGL distribution are obtained by solving the following system of equations for $\theta,\ \lambda_3,\ \lambda_4,\ \xi$

$$\frac{\partial \log[L]}{\partial \theta} = 0,$$

$$\frac{\partial \log[L]}{\partial \lambda_3} = 0,$$

$$\frac{\partial \log[L]}{\partial \lambda_4} = 0,$$

$$\frac{\partial \log[L]}{\partial \xi} = 0.$$

6.2. Comparison with empirical distribution and histogram

By modifying the symbolic MLE procedure described in [7] and using Mathematica's *Find Root* procedure, we found upon taking x_i to be the Wheaton river data, Table 6 [1] and using appropriate initial conditions in *Find Root* that the MLE in the TUEGL distribution are given by

$$(\hat{\theta}, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\xi}) = (0.1134, 5.3192, 3.0133, -0.065271).$$

On the other hand the MLE for $\theta, \lambda_3, \lambda_4$ in the Uniform-Exponential (Generalized Lambda) family of distributions are recorded in Table 7 [1]. Thus we have the following

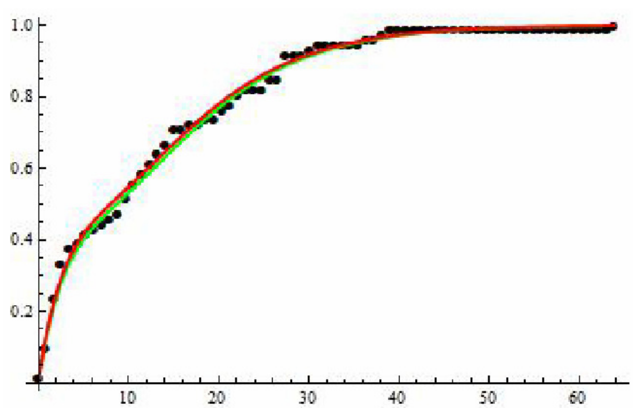

Figure 5. The CDF of $TUEGL(0.1134, 5.3192, 3.0133, -0.065271)$ (green) and $F_{0.1134, 5.3192, 3.0133}(x)$ (red) fitted to the empirical distribution (black) of Table 6 [1].

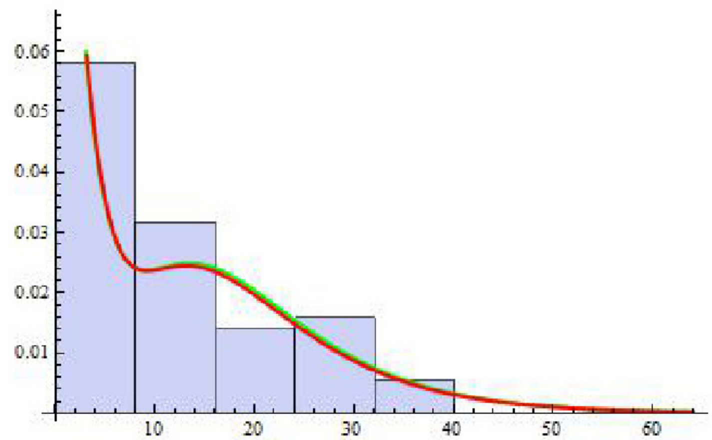

Figure 6. The PDF of $TUEGL\,(0.1134,\ 5.3192,\ 3.0133,\ -0.065271)$ (green) and $f_{0.1134, 5.3192, 3.0133}(x)$ (red) fitted to the histogram of Table 6 [1].

6.3. General observation

From the figures in the previous section, we see the $TUEGL\,(0.1134,\ 5.3192,\ 3.0133,\ -0.065271)$. distribution is equivalent to the UEGL distribution with parameters

$$(\hat{\theta},\ \hat{\lambda}_3,\ \hat{\lambda}_4) = (0.1134,\ 5.3192,\ 3.0133).$$

In particular, both distributions would be suitable for data that are right skewed with long tail.

7. Some Properties of TUEGL

7.1. The survival function

Theorem 7.1. *The survival function of the* $TUEGL(\theta,\ \lambda_3,\ \lambda_4,\ \xi)$ *is given by*

$$S_{\theta, \lambda_3, \lambda_4, \xi}(x) = 1 - F_{\theta, \lambda_3, \lambda_4, \xi}(x),$$

where

$$F_{\theta, \lambda_3, \lambda_4, \xi}(x) = (1+\xi) G_{\theta, \lambda_3, \lambda_4}(x) - \xi [G_{\theta, \lambda_3, \lambda_4}(x)]^2$$

with $x, \theta, \lambda_3, \lambda_4 > 0, \ -1 \le \xi \le 1$ *and*

$$G_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\left[1 + (1 - e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right]$$

and

$$g_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3 (1 - e^{-\theta x})^{\lambda_3 - 1} + \lambda_4 (e^{-\theta x})^{\lambda_4 - 1}\right].$$

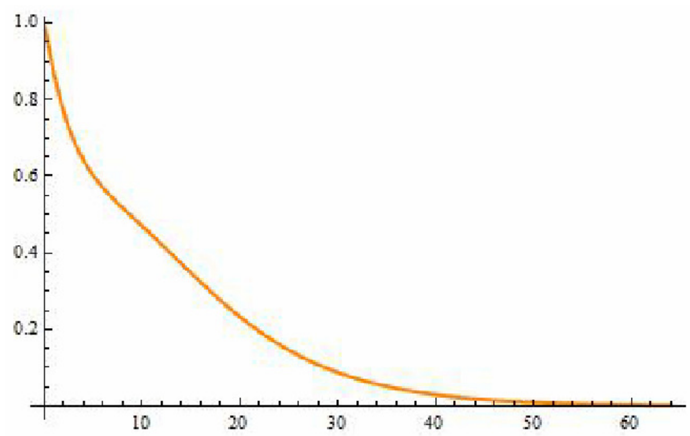

Figure 7. The survival function of $TUEGL(0.1134, 5.3192, 3.0133, -0.065271)$.

7.2. The hazard rate function

Theorem 7.2. *The hazard rate function of the* $TUEGL(\theta, \lambda_3, \lambda_4, \xi)$ *is given by*

$$H_{\theta, \lambda_3, \lambda_4, \xi}(x) = \frac{f_{\theta, \lambda_3, \lambda_4, \xi}(x)}{1 - F_{\theta, \lambda_3, \lambda_4, \xi}(x)},$$

where

$$f_{\theta, \lambda_3, \lambda_4, \xi}(x) = g_{\theta, \lambda_3, \lambda_4}(x)[1 + \xi - 2\xi G_{\theta, \lambda_3, \lambda_4}(x)],$$

$$F_{\theta,\lambda_3,\lambda_4,\xi}(x) = (1+\xi)G_{\theta,\lambda_3,\lambda_4}(x) - \xi[G_{\theta,\lambda_3,\lambda_4}(x)]^2$$

with $x,\ \theta,\ \lambda_3,\ \lambda_4 > 0,\ -1 \le \xi \le 1$ *and*

$$G_{\theta,\lambda_3,\lambda_4}(x) = \frac{1}{2}\left[1 + (1 - e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right],$$

and

$$g_{\theta,\lambda_3,\lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3(1 - e^{-\theta x})^{\lambda_3 - 1} + \lambda_4(e^{-\theta x})^{\lambda_4 - 1}\right].$$

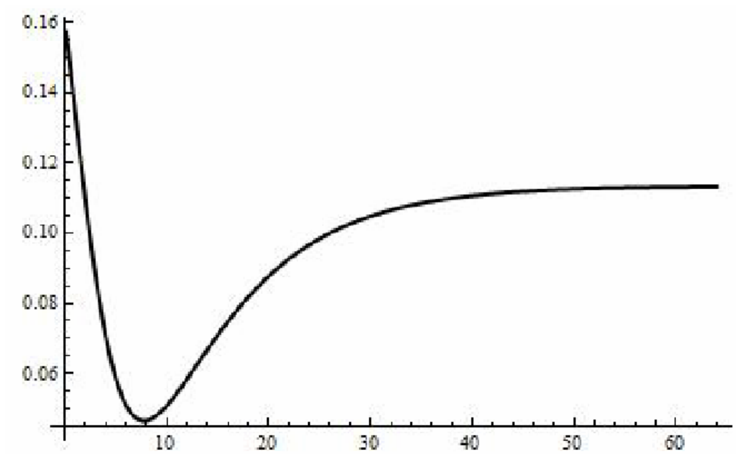

Figure 8. The hazard rate function of $TUEGL(0.1134,\ 5.3192,\ 3.0133,\ -0.065271)$.

7.3. The *r*th moment of $TUEGL(\theta,\ 2,\ 1,\ 0)$

Theorem 7.3. *The rth moment of* $TUEGL(\theta,\ 2,\ 1,\ 0)$ *is given by*

$$\mu_r = 2^{-1-r}(-1 + 3 \cdot 2^r)\theta^{-r}\Gamma(1+r),$$

where $\theta > 0,\ r = 1,\ 2,\ 3,\ \cdots,$ *and* $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$

Proof. It is obtained by evaluating the following integral

$$\mu_r = \int_0^\infty x^r \frac{1}{2}\theta e^{-2\theta x}(-2 + 3e^{\theta x})dx.$$

7.4. The moment generating function of $TUEGL(\theta, 2, 1, 0)$

Theorem 7.4. *Assuming* $X \sim TUEGL(\theta, 2, 1, 0)$, *then the moment generating function is given by*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} 2^{-1-r}(-1 + 3 \cdot 2^r)\theta^{-r}\Gamma(1 + r),$$

where $\theta > 0$, *and* $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$

Proof. Given μ_r from Theorem 7.3, and the fact that $f_{\theta,2,1,0}(x) = \frac{1}{2}\theta e^{-2\theta x}$ $(-2 + 3e^{\theta x})$, we deduce the following

$$\begin{aligned}
M_X(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx} f_{\theta,2,1,0}(x)dx \\
&= \int_0^\infty \left\{1 + tx + \frac{(tx)^2}{2!} + \cdots\right\} f_{\theta,2,1,0}(x)dx \\
&= \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f_{\theta,2,1,0}(x)dx \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r
\end{aligned}$$

and the result follows.

7.5. Order statistics of the $TUEGL(\theta, \lambda_3, \lambda_4, \xi)$ **distribution**

First we recall the following, for example, see [8]:

Definition 7.5. If $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \cdots X_n$ from a continuous population with CDF $F_X(x)$ and PDF $f_X(x)$, then the PDF of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}.$$

Thus from the above, it follows that we have the following:

Theorem 7.6. *The PDF of the jth order TUEGL random variable* $X_{(j)}$ *is given by*

$$k_{X_{(j)}}(x; \theta, \lambda_3, \lambda_4, \xi)$$

$$= \frac{n!}{(j-1)!(n-j)!} f_{\theta,\lambda_3,\lambda_4,\xi}(x)[F_{\theta,\lambda_3,\lambda_4,\xi}(x)]^{j-1}[1-F_{\theta,\lambda_3,\lambda_4,\xi}(x)]^{n-j},$$

where

$$f_{\theta,\lambda_3,\lambda_4,\xi}(x) = g_{\theta,\lambda_3,\lambda_4}(x)[1+\xi-2\xi G_{\theta,\lambda_3,\lambda_4}(x)],$$

$$F_{\theta,\lambda_3,\lambda_4,\xi}(x) = (1+\xi)G_{\theta,\lambda_3,\lambda_4}(x) - \xi[G_{\theta,\lambda_3,\lambda_4}(x)]^2,$$

where $x, \theta, \lambda_3, \lambda_4 > 0, -1 \le \xi \le 1$ *and*

$$G_{\theta,\lambda_3,\lambda_4}(x) = \frac{1}{2}\left[1 + (1-e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right]$$

and

$$g_{\theta,\lambda_3,\lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3(1-e^{-\theta x})^{\lambda_3-1} + \lambda_4(e^{-\theta x})^{\lambda_4-1}\right].$$

From the above theorem we have the following:

Corollary 7.7. *The PDF of the* 1*st order TUEGL random variable* $X_{(1)}$ *is given by*

$$k_{X_{(1)}}(x;\ \theta,\ \lambda_3,\ \lambda_4,\ \xi) = nf_{\theta, \lambda_3, \lambda_4, \xi}(x)[1 - F_{\theta, \lambda_3, \lambda_4, \xi}(x)]^{n-1},$$

where

$$f_{\theta, \lambda_3, \lambda_4, \xi}(x) = g_{\theta, \lambda_3, \lambda_4}(x)[1 + \xi - 2\xi G_{\theta, \lambda_3, \lambda_4}(x)],$$

$$F_{\theta, \lambda_3, \lambda_4, \xi}(x) = (1 + \xi)G_{\theta, \lambda_3, \lambda_4}(x) - \xi[G_{\theta, \lambda_3, \lambda_4}(x)]^2,$$

where $x,\ \theta,\ \lambda_3,\ \lambda_4 > 0,\ -1 \le \xi \le 1$ *and*

$$G_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\left[1 + (1 - e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right]$$

and

$$g_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3(1 - e^{-\theta x})^{\lambda_3 - 1} + \lambda_4(e^{-\theta x})^{\lambda_4 - 1}\right].$$

Corollary 7.8. *The PDF of the nth order TUEGL random variable* $X_{(n)}$ *is given by*

$$k_{X_{(n)}}(x;\ \theta,\ \lambda_3,\ \lambda_4,\ \xi) = nf_{\theta, \lambda_3, \lambda_4, \xi}(x)[F_{\theta, \lambda_3, \lambda_4, \xi}(x)]^{n-1},$$

where

$$f_{\theta, \lambda_3, \lambda_4, \xi}(x) = g_{\theta, \lambda_3, \lambda_4}(x)[1 + \xi - 2\xi G_{\theta, \lambda_3, \lambda_4}(x)],$$

$$F_{\theta, \lambda_3, \lambda_4, \xi}(x) = (1 + \xi)G_{\theta, \lambda_3, \lambda_4}(x) - \xi[G_{\theta, \lambda_3, \lambda_4}(x)]^2,$$

where $x,\ \theta,\ \lambda_3,\ \lambda_4 > 0,\ -1 \le \xi \le 1$ *and*

$$G_{\theta, \lambda_3, \lambda_4}(x) = \frac{1}{2}\left[1 + (1 - e^{-\theta x})^{\lambda_3} - (e^{-\theta x})^{\lambda_4}\right]$$

and

$$g_{\theta,\lambda_3,\lambda_4}(x) = \frac{1}{2}\theta e^{-\theta x}\left[\lambda_3(1-e^{-\theta x})^{\lambda_3-1} + \lambda_4(e^{-\theta x})^{\lambda_4-1}\right].$$

7.6. Random number generation from $TUEGL(\theta, 2, 1, 0)$

Theorem 7.9. *Random numbers from* $TUEGL(\theta, 2, 1, 0)$ *can be obtained from*

$$\frac{Log\left[\frac{-3+\sqrt{1+8u}}{4(-1+u)}\right]}{\theta},$$

where $\theta > 0$ *and* $u \sim U(0, 1)$.

Proof. It follows from solving the following equation for *y*, where $u \sim U(0, 1)$

$$\frac{1}{2}(1-e^{-\theta y} + (1-e^{-\theta y})^2) = u.$$

References

[1] Mahmoud Aldeni, Carl Lee and Felix Famoye, Families of distributions arising from the quantile of generalized lambda distribution, J. Stat. Distrib. App. 4 (2017), 25.

[2] Wikipedia contributors, Uniform distribution (continuous), Wikipedia, The Free Encyclopedia, 13 Oct. 2017, Web 12 Jan. 2018.

[3] Wikipedia contributors, Exponential distribution, Wikipedia, The Free Encyclopedia, 8 Jan. 2018, Web 12 Jan. 2018.

[4] J. S. Ramberg and B. W. Schmeiser, An approximate method for generating asymmetric random variables, Communications of the ACM 17(2) (1974), 78-82.

[5] J. W. Tukey, The practical relationship between the common transformations of percentages of counts and of amounts, Technical Report 36, Princeton University, Statistical Techniques Research Group, 1960.

[6] Enahoro A. Owoloko, Pelumi E. Oguntunde and Adebowale O. Adejumo,

Performance rating of the transmuted exponential distribution: an analytical approach, Springer Plus 4 (2015), 818.

[7] Colin Rose and Murray D. Smith, Symbolic maximum likelihood estimation with mathematica, The Statistician 49(2) (2000), 229-240.

[8] Gokarna R. Aryal and Chris P. Tsokos, Transmuted Weibull distribution: a generalization of the Weibull probability distribution, Euro. J. Pure Appl. Math. 4(2) (2011), 89-102.