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THE SINGLETON SET PROPERTY, AND OPEN AND CONTINUOUS OPEN IMAGE PROPERTIES

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Abstract

Within this paper, the singleton set, open images, and continuous open images properties continue to be investigated using new, fundamental topological properties and inverse point finite images.

1. Introduction and Preliminaries

Within the study of classical topology, questions about topological properties simultaneously shared by a space and open images or continuous open images of the space have been addressed and progress has been made in resolving the questions. In this paper, the investigations continue with the use of inverse point finite mages and recently introduced fundamental properties given below.

Definition 1.1. Let P be a topological property. Then "not-P" is the negation of P, provided the negation exists [1].

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Theorem 1.1. $L = (T_0$ or "not-T_0") *is the least of all topological properties* [1].

Within this paper, L will be used to denote the least topological property.

Theorem 1.2. *For a topological property* P, *the following are equivalent*: (a) "*not-P*" *exists*, (b) "*not-P*" *is a topological property*, P *is stronger than* L, *and* $P \neq$ "*not-P*", (c) $P \neq L$ *and* $P \neq$ "*not-P*", (d) P *is stronger than* L, *and* (e) "*not-P*" *is stronger than* L [1].

Theorem 1.3. *Let* P *be a topological property different from* L. *Then* $L = (P$ *or* "*not-P*") [1].

In two follow up papers; [2] and [3], the new properties given above proved to be useful and needed in answering several questions that naturally arise in the study of topology. However, since the existence of the least topological property L had not even been considered prior to the paper cited above [1], its existence created problems. Within the paper [2], it was proven that L is a product property and that every product space exhibits the product property L, which is far from the intent of the creators of product properties and created a disconnect in the study of product properties. Thus, to maintain continuity between past studies of product properties and any future studies of product properties, a correction was needed in the definition of product properties with the removal of L as a product property [2]. Likewise, the existence of L created a problem in the study of subspace properties, leading to a needed change in the definition for subspace properties in which L was excluded [3]. Within the paper on subspaces [3], the continued investigation of the new properties as related to subspaces led to an unexpected discovery. Singleton set topological spaces have an important role in answering new, never before asked subspace questions and it was proven that singleton set topological spaces simultaneously satisfies all subspace properties. It was well known that singleton set spaces satisfies many properties, including properties that are not subspace properties, but the connection between singleton set spaces and subspace properties was unknown prior to the new subspace property paper [3]. Within a follow up paper [4], it was proven that the singleton set property is both a subspace property and a product property, and is the strongest of the continuous image properties.

Definition 1.2. A space (X, T) has singleton set property iff X is a singleton set [3].

Definition 1.3. Let P be a topological property. Then P is a continuous image property if $P \neq L$, and for each space with property P, each continuous image of the space has property P [4].

In this paper, open images and continuous open images are further investigated, and additional roles of singleton set spaces in the study of topology are given.

2. Open Image, Continuous Open Images, and Singleton Set Properties

Since the least topological property L was unknown and its existence not even considered in prior studies of open image and continuous open image properties, its existence could, and, in fact, does create a disconnect in the study of open and continuous open images. Since every space has property L [2], then every space has property L iff each open (continuous open) image of the space has property L. Thus for there to be continuity between past studies and future studies of open (continuous open) images, L would have to be removed from consideration.

Definition 2.1. Let P be a topological property different from L. Then P is an open image property if for each space with property P, each open image of the space has property P.

Definition 2.2. Let P be a topological property different from L. Then P is a continuous open image property if for each space with property P, each continuous open image of the space has property P.

Theorem 2.1. *The open image property and the continuous open image property are topological properties.*

The straightforward proof is omitted.

Theorem 2.2. (a) *A space* (X, T) *has open image* (*continuous open image*) *property P iff* (b) *every open image* (*continuous open image*) *of* (X, T) *has property* P.

Proof. By definition (a) implies (b).

(b) implies (a): Since the identity function defined on (X, T) is continuous, open, and onto, then (X, T) has property P.

In the paper [4], it was proven that (a) (X, T) has singleton set property iff (b) for each set Y and each topology U on X and each topology V on Y, $f : X \to Y$ is onto iff $f : (X, U) \to (Y, V)$ is a homeomorphism, which is used below.

Theorem 2.3. *The singleton set property is the strongest of all open image properties and of all continuous open image properties.*

Proof. Let (Y, S) have the singleton set property with $Y = \{y\}$. Since the singleton set property is not L, then it is a candidate for an open image property. Let P be an open image property. Let (X, T) be a space with property P. Let $f : (X, T) \to (Y, S)$ be the constant function $f(x) = y$ for all $x \in X$. Then (Y, S) is an open image of (X, T) and has property P. Hence the singleton set property implies each open image property.

Let (X, T) have the singleton set property. Let $X = \{x\}$. Let (Y, S) be an open image of (X, T) and let $f : (X, T) \to (Y, S)$ be open and onto. Then $f : X \to Y$ is onto and, by the result above, $f : (X, T) \to (Y, S)$ is a homeomorphism. Since the singleton set property is a topological property [3], then (Y, S) has the singleton set property. Thus the singleton set property is an open image property that implies each open image property, which implies the singleton set property is the strongest open image property.

In the same manner, the singleton set property is the strongest of all continuous open image properties.

In the paper [4], it was shown that the indiscrete property is a continuous image property. Thus the indiscrete property is a continuous open image property.

Definition 2.3. A space (X, T) has the indiscrete property iff T is the indiscrete topology on X [4].

The following example shows the indiscrete property is not an open image property.

Example 2.1. Let $X = \{a, b\}$, $a \neq b$, let T be the indiscrete topology on X, let S be the topology $\{\phi, X, \{a\}\}$ on X, and let $f : X \to X$ be the identity function. Then $f : (X, T) \to (X, S)$ is open, but (X, S) does not have the indiscrete property.

Definition 2.4. A space (X, T) has the discrete property iff T is the discrete topology on X.

Theorem 2.4. *The discrete property is an open image* (*continuous open image*) *property*.

Proof. Let (X, T) be a space with the discrete property. Let (Y, S) be an open image of (X, T) and let $f:(X, T) \to (Y, S)$ be open and onto. Let $y \in Y$. Let $x \in X$ such that $f(x) = y$. Then $\{x\}$ is open in X and, since f is open, $\{y\}$ is open in Y. Thus S is the discrete topology on Y. Hence, the discrete property is an open image property. In a similar manner, the discrete property is a continuous open image property.

Within the paper [4], it was shown that the negation of a continuous image property is not a continuous image property raising a similar question about the negation of open image and continuous open image properties.

Definition 2.5. Let P be a topological property different from L. Then P is a "not-open image property" ("not-continuous open image property") if there exists a space (X, T) with property P and an open image (continuous open image) (Y, S) of (X, T) that is "not-P".

Theorem 2.5. *Let P be an open image* (*continuous open image*) *property*. *Then* "*not-P*" *is a* "*not-open image property*" ("*not*-(*continuous open image property*)").

Proof. Assume "not-P" is an open image property. Then the statements (1) "the singleton set property is the strongest open image property", (2) "the singleton set property implies P", and (3) "the singleton set property implies ("not-P")" are all true, but if (1) and (2) are true, as they are, then (3) is false, which is a contradiction. Thus "not-P" is a "not-open image property".

The proof for "not-continuous open image property" is similar and omitted.

Thus, more important roles of the singleton set property are now known.

Theorem 2.6. *Let P and Q be open image* (*continuous open image*) *properties. Then* $P \neq$ "*not-Q*".

Proof. Suppose $P =$ "not-Q". Then $P \neq Q$. Let (X, T) be a space with property "not-Q" and let (Y, S) be an open image of (X, T) that is ("not-("not-

Q")") $= Q$. Since $P =$ "not-Q" and P is an open image property, then (Y, S) has property P, but $P \neq Q$, which is a contradiction. Hence, $P \neq$ "not-Q".

The proof for "not-continuous open image property" is similar and omitted.

Theorem 2.7. *Let P and Q be open image* (*continuous open image*) *properties. Then* (*P or Q*) *is an open image* (*continuous open image*) *property.*

The straightforward proof is omitted.

Theorem 2.8. *Let P and Q be open image* (*continuous open image*) *properties. Then* (*P and Q*) *exists and is an open image* (*continuous open image*) *property.*

Proof. Suppose P and Q are open image properties and that (P and Q) does not exist. Then P is strictly stronger than "not-Q" and Q is strictly stronger than "not-P". Since "not-Q" = (P or ("not-P" and "not-Q")) and $P \neq$ "not-Q", then "not-Q" = ("not-P" and "not-Q") = "not-(P or Q)", but then $Q =$ (P or Q), which is a contradiction. Hence (P and Q) exists.

In like manner for continuous open image properties, (P and Q) exists.

The remainder of the proof is straightforward and omitted.

Theorem 2.9. *Let P and Q be open image* (*continuous open image*) *properties such that* (*P and* "*not-Q*") *exists. Then* (*P and* "*not-Q*") *is a* "*not-open image property*" ("*not-continuous open image property*").

Proof. Suppose P and Q are open image properties such that (P and "not-Q") exists and that (P and "not-Q") is an open image property. Then Q and (P and "not-Q") are open image properties and (Q and (P and "not-Q") does not exist, which is a contradiction. Similarly, the statement is true for "not-continuous open image property".

Combining the results above gives the next result.

Corollary 2.1. *The* "*not*-(*singleton set property*)" *is the weakest of all the* "*not*-(*open image properties*)", *of all the* "*not*-(*continuous open image properties*)", *and of all* "*not-continuous image properties*".

The example below shows that the open image (continuous open image) of a T_1 space need not be T_0.

Example 2.2. Let T be the finite complement topology on the set X of natural numbers. Let $Y = \{a, b\}$; $a \neq b$, let S be the indiscrete topology on Y, and let $f : (X, T) \to (Y, S)$ defined by $f\,(\text{even}) = a$ and $f\,(\text{odd}) = b$. Then $f : (X, T) \to (Y, S)$ is continuous, open, and onto, (X, T) is T_1 and (Y, S) is not T_0.

Thus the question of what property, if any, could be added to open and onto (continuous, open, and onto) to make each open image (continuous open image) of a T_0 (T_1) space T_0 (T_1) was raised. Below, this question is addressed.

3. The Inverse Finite Point Property

Definition 3.1. Let X and Y be sets and let $f : X \to Y$ be onto. Then f has inverse finite point property iff for each $y \in Y$, $f^{-1}(y)$ is finite.

Theorem 3.1. *Let (X, T) be T_0, let (Y, S) be an open, inverse finite point image of (X, T), and let $f : (X, T) \to (Y, S)$ be an open, onto, inverse finite point function. Then (Y, S) is T_0.*

Proof. Let u and v be distinct elements of Y. Then $f^{-1}(\{u, v\})$ is finite. Let $f^{-1}(\{u, v\}) = \{x_i \mid i = 1, \cdots, n\}$, where n is a fixed natural number. Since (X, T) is T_0, then there is an open set O in X containing only one of $\{x_i \mid i = 1, \cdots, n\}$ [5] and since f is open, $f(O)$ is open in Y containing only one of x and y. Hence (Y, S) is T_0.

In like manner, the continuous, open, inverse finite point image of a T_0 space is T_0.

Definition 3.2. Let P be a topological property different from L. Then P is an open inverse finite point image property (continuous open inverse finite point image property) iff for each space with property P, each open inverse finite point image (continuous open inverse finite point image) of the space has property P.

Theorem 3.2. *The open inverse finite point image property and the continuous open inverse finite point image property are topological properties.*

The proof is straightforward and omitted.

Corollary 3.1. *T_0 is an open inverse finite point image and a continuous open inverse finite point image property.*

Theorem 3.3. *T_1 is an open inverse finite point image and a continuous open inverse finite point image property.*

Proof. Let (X, T) be T_1, let (Y, S) be an open inverse finite point image of (X, T), and let $f : (X, T) \to (Y, S)$ be an open, onto, inverse finite point function. Let u and v be distinct elements of Y. Let $f^{-1}(u) = \{x_i \mid i = 1, \cdots, n\}$ and let $f^{-1}(v) = \{x_{n+i} \mid i = 1, \cdots, m\}$, where n and m are fixed constants. Since (X, T) is T_1, there exists an open set O containing only x_1 of $\{x_i \mid i = 1, \cdots, n = m\}$ [5] and $f(O)$ is open in Y containing u and not v. Thus (Y, S) is T_1.

The remaining proof is straightforward and omitted.

Example 13.9(b) in the book [6] shows T_2 is neither an open inverse finite point image property nor a continuous open inverse finite point image property.

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