Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 2, Issue 1, 2015, Pages 1-28
This paper is available online at http://www.frdint.com/
Published online March 31, 2015

THE REGULARLY SOLVABLE OPERATORS IN L_w^p SPACES

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Abstract

In this paper, a characterization of all the regularly solvable operators and their adjoints generated by a general quasi-differential expressions $\tau_{p,q}$ in L_w^p spaces is obtained. The domains of these operators are described in terms of boundary conditions involving the square-integrable solutions of the quasi-differential equation $\tau_{p,q}[u] = \lambda uw$ and the adjoint equation $\tau_{q',p'}^{+}[v] = \overline{\lambda} vw\,(\lambda \in \mathbb{C})$. The results herein are extensions of those of Evans and Ibrahim [4] and Hilbert Frentzen [13, 14] for general case in L_w^p spaces. The results include those of Sun Jiong [8] and Zettl [22] concerning self-adjoint realizations of symmetric operators when the minimal operator $T_0(\tau_{p,q})$ has equal deficiency indices, and Evans [2], Race [17] and Zai-Jiu Shang [18] for the special case concerns the *J*-self-adjoint operators, where *J* denotes complex conjugation. If the deficiency

Keywords and phrases: quasi-differential expressions, regular and singular endpoints, minimal and maximal operators, regularly solvable operators, deficiency indices, boundary conditions.

2010 Mathematics Subject Classification: 34B05, 34B24, 47A10, 47E05.

Received January 5, 2015; Accepted February 4, 2015

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indices are unequal the maximal symmetric operators $T(\tau_{p,q})$ are determined by the results herein. The minimal and maximal operators considered as operators from a subspaces of L^p_w into L^q_w, p, q are arbitrary, $w > 0$ be a positive weight function.

1. Introduction

The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression τ are those which are regularly solvable with respect to the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ generated by a general ordinary quasi-differential expression τ and its formal adjoint τ^+ respectively, the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ form an adjoint pair of closed, densely-defined operators in the underlying L^2_w-space, that is $T_0(\tau) \subset [T_0(\tau^+)]^*$. Such an operator S satisfies $T_0(\tau) \subset S \subset [T_0(\tau^+)]^*$ and for some $\lambda \in \mathbb{C}$, $(S - \lambda I)$ is a Fredholm operator of zero index, this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solution space of $(S - \lambda I)u = 0$ and furthermore the solution space of $(S - \lambda I)u = 0$ and $(S^* - \overline{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik [19].

For (formally) symmetric differential expressions τ and L^2_w space much work has been done on the problem of finding self-adjoint differential operator $T(\tau)$ with the aid of boundary conditions as in Naimark [16], Zettl [22] and Weidman [20] to mention only a few. If τ is not symmetric, one can ask for $T(\tau)$ which are Fredholm operators with index zero. For L^2_w space and second order scalar differential expressions this question was answered by Evans [2], for scalar differential expressions of general order on half-open intervals by Evans and Ibrahim [4] and on open intervals by Ibrahim [8].

Our objective in this paper is a generalization of the results of Evans and Ibrahim [4] for general case in L^p_w spaces an analogue of Hilbert Frentzen [13, 14]. A rather general class of quasi-differential expressions $\tau_{p,q}$ with matrix-valued coefficients

and the associated maximal operators $T(\tau_{p,q})$ and minimal operator $T_0(\tau_{p,q})$ as maps of a subspace of L^p_w into L^q_w for arbitrary $p, q \in [1, \infty)$. A characterization of all the regularly solvable operators and their adjoints generated by a general quasi-differential expressions $\tau_{p,q}$ in L^p_w spaces is obtained. The domains of these operators are described in terms of boundary conditions involving the squareintegrable solutions of the quasi-differential equation $\tau_{p,q}[u] = \lambda uw$ and the adjoint equation $\tau^+_{q',p'}[v] = \bar{\lambda} vw\, (\lambda \in \mathbb{C})$ (see [5]). The results herein include those of Sun Jiong [12], Race [17] and Zai-Jiu Shang [18] but also gives a description of the domain of the maximal symmetric extensions of $T_0(\tau_{p,q})$ in the case when $T_0(\tau_{p,q})$ is a symmetric operator of unequal deficiency indices. Another noteworthy special case of our result is that of D. Race [17], Zai-jiu Shang [18] and Zhikhar [21] concerning the *J*-self-adjoint extensions of a *J*-symmetric differential operators $T_0(\tau_{p,q})$, where *J* denotes complex conjugation.

2. Notation and Preliminaries

We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [3, Chapter III], [4] and [8-11]. The domain and range of a linear operator *T* acting in a Hilbert space *H* will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of *T*, written $nul(T)$, is the dimension of $N(T)$ and the deficiency of *T*, written $def(T)$, is the co-dimension of $R(T)$ in *H*; thus if *T* is densely defined and $R(T)$ is closed, then $def(T) = null(T^*)$. The Fredholm domain of *T* is (in the notation of [2]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \in \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where *I* is the identity operator in *H*. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $ind(T - \lambda I) = nul(T - \lambda I) - def\ (T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators *A* and *B* acting in a Hilbert space *H* are said to form an adjoint pair if $A \subset B^*$ and, consequently, $B \subset A^*$; equivalently,

$(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(.,.)$ denotes the inner-product on H.

Definition 2.1. The field of regularity $\prod(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that

$$\|(A - \lambda I)x\| \geq K(\lambda)\|x\| \text{ for all } x \in D(A), \tag{2.1}$$

or, equivalently, on using the Closed Graph Theorem, $mul(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

The joint field of regularity $\prod(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \prod(A)$, $\overline{\lambda} \in \prod(B)$ and both $def(A - \lambda I)$ and $def(B - \overline{\lambda} I)$ are finite. An adjoint pair A and B is said to be compatible if $\prod(A, B) \neq \phi$.

Definition 2.2. A closed operator S in Hilbert space H is said to be *regularly solvable* with respect to the compatible adjoint pair of A and B if $A \subset S \subset B^*$ and $\prod(A, B) \cap \Delta_4(S) \neq \phi$, where $\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), \ ind\ (S - \lambda I) = 0\}$.

If A is symmetric operator, then with $B = A$, the operators S which conform to Definition 2.1 are the self-adjoint or maximal symmetric extensions of A. In this case when A is J-symmetric relative to complex conjugation J, A and $B = JAJ$ form an adjoint pair with $\prod(A, B) = \prod(A)$; any J-self-adjoint extension of A whose resolvent set is nonempty, is regularly solvable with respect to A and JAJ. For the above results, see [3, Chapter III].

The basic result in the theory developed in [2-4] for the operators which are regularly solvable with respect to a compatible adjoint pair is the following.

Theorem 2.2 (cf. [3, Theorem III.3.1])**.** *Let T be a closed, densely-defined operator in H with $\prod(A) \neq \phi$. Then for any closed extension S of T and $\lambda \in \prod(T)$ we have $D(S) = D(T) + N([T^* - \overline{\lambda} I][S - \lambda I])$. If $\lambda \in \prod(T) \cap \Delta_4(S)$ and $def\ (T - \lambda I) < \infty$, then*

$$\dim\{D(S) / D(T)\} = ind(S - \lambda I) + def(T - \lambda I).$$

The result has the following consequences

Theorem 2.3 (cf. [3, Theorem III.3.2])**.** *Let A, B be a compatible adjoint pair*

of closed, densely-defined operators in H. Then if $\lambda \in \prod (A, B)$,

$$D(B^*) = D(A) \dotplus N([A^* - \overline{\lambda} I][B^* - \overline{\lambda} I])$$

$$D(A^*) = D(B) \dotplus N([B^* - \lambda I][A^* - \overline{\lambda} I])$$

and

$$\dim\{D(B^*) / D(A)\} = \dim\{D(A^*) / D(B)\} = def(A - \lambda I) + def(B - \overline{\lambda} I).$$

Hence, $def(A - \lambda I) + def(B - \overline{\lambda} I)$ *is constant for all* $\lambda \in \prod(A, B)$

Throughout this paper, let $\mathbb{K}$ denote either $\mathbb{R}$, the field of real numbers, or $\mathbb{C}$, the field of complex numbers. For a positive integers k amd $\mathbb{M}_{k,m}$ denote the vector space of $k \times m$ matrices with $\mathbb{K}$-valued entries and GL_m the subset of $\mathbb{M}_m := \mathbb{M}_{m,m}$ consisting of all nonsingular matrices. For $A \in \mathbb{M}_{k,m}$, let A^T denote the transpose and A^* the adjoint, i.e., the complex conjugate transpose of A.

If Z is a subset of $\mathbb{M}_{k,m}$ and I is an interval, $B(I, Z)$ denotes the set of Lebesgue measurable maps of I into Z and $AC_{loc}(I, Z)$ the set of locally absolutely continuous maps. Measurable maps are regarded as equal if they are equal almost everywhere on I. Further we define

$$L^p(I, Z) := \{y \in B(I, Z) \mid |y|^p \text{ is Lebsgue-integrable}\},$$

$$\|y\|_{p,I} := \left(\int_I |y|^p\right)^{\frac{1}{p}} \text{ for all } \quad y \in L^p(I, Z) \text{ and } p \in [1, \infty),$$

$$L^\infty(I, Z) := \{y \in B(I, Z) \mid y \text{ is essental bounded}\},$$

$$\|y\|_{\infty,I} := \operatorname{ess\,sup}_{x \in I} |y(x)| \text{ for all } y \in L^\infty(I, Z),$$

and

$$L^p_{loc}(I, Z) := \{y \in B(I, Z) \mid y|K \in L^p(K, Z) \text{ for all compact subintervals } K \text{ of } I\} \text{ for all } p \in [1, \infty).$$

If $r \in [1, \infty)$, then $r' \in [1, \infty)$ is always chosen such that $\frac{1}{r} + \frac{1}{r'} = 1$. We

always assume that $p, q \in [1, \infty)$ and $+ \in \{T, *\}$. If $L^p := L^p(I, \mathbb{K}^s)$ for some positive integer s, then $(L^p)^+ = L^{p'}$ for $p \in [1, \infty)$ and L^1 is a subspace of $(L^\infty)^+$. We refer to [13, 14] for more details.

3. Minimal and Maximal Operators Generated by a Quasi-differential Expressions

Let I be an interval with endpoints $a, b\,(-\infty \le a < b \le \infty)$, let n, s be positive integrs and $p, q \in [1, \infty)$. The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F on an interval I.

Definition 3.1 [13]**.** The set $Z_{n,s}^{p,q}(I)$ of Shin-Zettl matrices on I consists of matrices is defined to be the set of all lower triangular matrices $F = \{f_{j,k}\}$ of the form

$$F = \begin{pmatrix} f_{0,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ f_{n,1} & \cdots & f_{n,n+1} \end{pmatrix}$$

whose entries are complex-valued functions on I which satisfy the following conditions:

$$f_{0,1} \in L_{loc}^p(I, \mathbb{M}_s) \text{ and } f_{n,n+1} \in L_{loc}^{q'}(I, \mathbb{M}_s), \tag{3.1}$$

$$f_{j,k} \in L_{loc}^p(I, \mathbb{M}_s) \text{ for all } 1 \le j \le n \text{ and } 1 \le k \le \min\{j+1, n\} \tag{3.2}$$

and

$$f_{j,j+1}(x) \in GL_s \text{ for all } 0 \le j \le n \text{ and } x \in I. \tag{3.3}$$

For $F \in Z_{n,s}^{p,q}(I)$ we define $\tilde{F}$ as the $(n \times n)$ matrix obtained from F by removing the first row and the last column, i.e.,

$$\tilde{F} = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ f_{n-1,1} & f_{n-1,2} & \cdots & f_{n-1,n} \\ f_{n,1} & f_{n-2} & \cdots & f_{n,n} \end{pmatrix},$$

Definition 3.2 [4, 13]**.** For $\widetilde{F} \in Z_{n,s}^{p,q}(I)$, the quasi-derivatives associated with $\widetilde{F}$ are defined by

$$y_{\widetilde{F}}^{[0]} := y_{\widetilde{F}},$$

$$y_{\widetilde{F}}^{[j]} := (f_{j,j+1})^{-1}\left\{\left(y_{\widetilde{F}}^{[j-1]}\right)' - \sum\nolimits_{k=1}^{j} f_{j,k}\, y_{\widetilde{F}}^{[k-1]}\right\}, \quad (1 \le j \le n-1) \tag{3.2}$$

$$y_{\widetilde{F}}^{[n]} := \left\{\left(y_{\widetilde{F}}^{[n-1]}\right)' - \sum\nolimits_{k=1}^{n} f_{j,k}\, y_{\widetilde{F}}^{[k-1]}\right\},$$

where the prime $'$ denotes differentiation.

The quasi-differential expression $\tau_{\widetilde{F}}$ associated with $\widetilde{F}$ is given by:

$$\tau_{\widetilde{F}}[.] := i^n y_{\widetilde{F}}^{[n]}, \quad (n \ge 2), \tag{3.3}$$

this being defined on the set :

$$V(\tau_{\widetilde{F}}) := \left\{y_{\widetilde{F}} : y_{\widetilde{F}}^{[j-1]} \in AC_{loc}(I, \mathbb{K}^s),\ 1 \le j \le n\right\}$$

where $AC_{loc}(I, \mathbb{K}^s)$, denotes the set of functions which are locally absolutely continuous on every compact subinterval of *I*.

$$\text{For } y \in V(\tau_{\widetilde{F}}) \text{ we define } Q_{\widetilde{F}} y := \begin{pmatrix} y_{\widetilde{F}}^{[0]} \\ \vdots \\ y_{\widetilde{F}}^{[n-1]} \end{pmatrix}.$$

Clearly the maps $\tau_{\widetilde{F}} : V(\tau_{\widetilde{F}}) \to B(I, \mathbb{K}^s)$ and $Q_{\widetilde{F}} : V(\tau_{\widetilde{F}}) \to AC_{loc}(I, \mathbb{K}^{n,s})$ are linear.

In analogy to the adjoint and the transpose of a matrix, there are two different "(formal) adjoint" of a quasi-differential expression, we refer to [2-4] and [8-14] for more details.

In the following we always assume that $\widetilde{F} \in Z_{n,s}^{p,q}$ and $\tau_{\widetilde{F}} := \tau_{p,q}$.

The formal adjoint $\tau_{p,q}^{+}$ of $\tau_{p,q}$ is defined by the matrix $\widetilde{F}^{+}$ given by

$$\tilde{F}^{+} = -J_n^{-1}\tilde{F}^{*}J_n \tag{3.5}$$

where $\tilde{F}^{*}$ is the conjugate transpose of $\tilde{F}$ and J_n is the non-singular $(n\times n)$ matrix

$$J_n = \left((-1)^j \delta_{j,n+1-k}\right)_{\substack{1\le j\le n\\ 1\le k\le n}}, \tag{3.6}$$

δ being the Kronecker delta. If $\tilde{F}^{+} = f^{+}_{j,k}$ then it follows that

$$f^{+}_{j,k} = (-1)^{j+k+1}\bar{f}_{n-k+1,\,n-j+1}.$$

The quasi-derivatives associated with the matrix $\tilde{F}+$ in $Z^{p,q}_{n,s}(I)$ are therefore

$$y_{+}^{[0]} := y,$$

$$y_{+}^{[j]} := (\bar{f}_{n-j,n-j+1})^{-1}\left\{\left(y_{+}^{[j-1]}\right)' - \sum\nolimits_{k=1}^{j}(-1)^{j+k+1}\bar{f}_{n-k+1,n-j+1}y_{+}^{[k-1]}\right\}, \tag{3.7}$$

$$y_{+}^{[n]} := \left\{\left(y_{+}^{[n-1]}\right)' - \sum\nolimits_{k=1}^{n}(-1)^{n+k+1}\bar{f}_{n-k+1,1}y_{+}^{[k-1]}\right\},$$

and

$$\tau^{+}_{q',p'}[.] := i^n y_{+}^{[n]}\ (n\ge 2) \text{ for all } y\in V\left(\tau^{+}_{q',p'}\right), \tag{3.8}$$

$$V\left(\tau^{+}_{q',p'}\right) := \left\{y : y_{+}^{[j+1]}\in AC_{loc}(I,\mathbb{K}^s),\ 1\le j\le n\right\}, \tag{3.9}$$

Note that. $(\tilde{F}^{+})^{+} = \tilde{F}$ and so $\left(\tau^{+}_{q',p'}\right)^{+} = \tau_{p,q}$. We refer to [3, 4], [6, 7] and [8-14] for a full account of the above and subsequent results on quasi-differential expressions.

For $u\in V(\tau_{p,q})$, $v\in V\left(\tau^{+}_{q',p'}\right)$ and $\alpha, \beta\in I$, we have Green's formula,

$$\int_{\alpha}^{\beta}\left\{\overline{v}\tau_{p,q}[u] - u\overline{\tau^{+}_{q',p'}[v]}\right\}dx = [u,v](\beta) - [u,v](\alpha), \tag{3.10}$$

where,

$$[u, v](x) = i^n\left(\sum_{r=0}^{n-1}(-1)^{r+n+1}u^{[r]}\overline{v_+^{[n-r-1]}}\right)(x)$$

$$= (-i)^n\left(Q_{\widetilde{F}}^T u\, J_{n\times n} Q_{\widetilde{F}}\overline{\widetilde{v}}\right)(x).$$

$$= (-i)^n(u, u^{[1]}, \ldots, u^{[n-1]})\, J_{n\times n}\begin{pmatrix}\overline{v}\\ \vdots\\ \overline{v}_+^{[n-1]}\end{pmatrix}(x), \tag{3.11}$$

see [2-5], [8] and [16, 22]. Let $w : I \to \mathbb{R}$ be a non-negative weight function with $w \in L^1_{loc}(I)$ and $w > 0$ (for almost all $x \in I$). Then $H^r = L^r_w$ $(I, \mathbb{K}^s)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that

$$\|y\|_{r,I} := \left(\int_I |y|^r w\right)^{\frac{1}{r}} \text{ for all } y \in L^r(I, Z) \text{ and } r \in [1, \infty). \tag{3.12}$$

The equation

$$\tau_{p,q}[u] - \lambda w u = 0 \ (\lambda \in \mathbb{C}) \text{ on } I, \tag{3.13}$$

is said to be *regular* at the left end-point $a \in \mathbb{R}$, if for all $X \in (a, b)$,

$$a \in \mathbb{R},\ w,\ f_{j,k} \in L^1(a, X),\ j, k = 1, 2, \ldots, n,$$

otherwise (3.13) is said to be *singular* at a. If (3.13) is regular at both end-points, then it is said to be regular; in this case we have,

$$a, b \in \mathbb{R},\ w,\ f_{j,k} \in L^1(a, b),\ j, k = 1, 2, \ldots, n, \tag{3.14}$$

We shall be concerned with the case when a_p is a regular end-point of (3.13), the end-point b being allowed to be either regular or singular. Note that, in view of (3.14), an end-point of I is regular for (3.13), if and only if it is regular for the equation

$$\tau^+_{p,q}[v] - \overline{\lambda} w v = 0 \ (\lambda \in \mathbb{C}) \text{ on } I. \tag{3.15}$$

Definition 3.3 [4, 13]**.** (a) The maximal operators corresponding to $\tau_{p,q}$, $\tau^+_{q',p'}$ are defined as operators from a subspaces of L^p_w into L^q_w, p, q are arbitrary.

$$T(\tau_{p,q}) = w^{-1}\tau[u] \text{ for all } u \in D[T(\tau_{p,q})]$$

$$:= \{u \in V(\tau_{p,q}) \cap L_w^p |\ w^{-1}\tau_{p,q}[u] \in L_w^q\},$$

$$T\left(\tau_{q',p'}^{+}\right) = w^{-1}\tau^{+}[v] \text{ for all } v \in D\left[T\left(\tau_{q',p'}^{+}\right)\right]$$

$$:= \left\{v \in V\left(\tau_{q',p'}^{+}\right) \cap L_w^p \middle|\ w^{-1}\tau_{q',p'}^{+}[v] \in L_w^q\right\}.$$

The subspaces $D[T(\tau_{p,q})]$ and $D\left[T\left(\tau_{q',p'}^{+}\right)\right]$ of $L_w^p(I, \mathbb{K}^s)$ are the domains of the socalled maximal operators $T(\tau_{p,q})$ and $T(\tau_{q,p}^{+})$ respectively

(b) For the regular problem the minimal operators $T_0(\tau_{p,q})$ and $T_0\left(\tau_{q',p'}^{+}\right)$ are the restrictions of $w^{-1}\tau_{p,q}[u]$ and $w^{-1}\tau_{q',p'}^{+}[v]$ to the subspaces:

$$\left.\begin{aligned} D_0(\tau_{p,q}) &:= \{u : u \in D(\tau_{p,q}), \quad (Q_{\tilde{F}}u)(a) = (Q_{\tilde{F}}u)(b) = 0\} \\ D_0\left(\tau_{q',p'}^{+}\right) &:= \{v : v \in D\left(\tau_{q',p'}^{+}\right), (Q_{\tilde{F}}v_+)(a) = (Q_{\tilde{F}}v_+)(b) = 0\} \end{aligned}\right\}. \tag{3.16}$$

The subspaces $D_0(\tau_{p,q})$ and $D_0\left(\tau_{q',p'}^{+}\right)$ are dense in $L_w^p(I, \mathbb{K}^s)$ and $T_0(\tau_{p,q})$ and $T_0\left(\tau_{q',p'}^{+}\right)$ are closed operators (see [2], [5], [9, Section 3], [11], [13] and [16]).

In the singular problem we first introduce the operators $T_0'(\tau_{p,q})$ and $T_0'\left(\tau_{q',p'}^{+}\right)$; $T_0'(\tau_{p,q})$ being the restriction of $w^{-1}\tau_{p,q}[.]$ to the subspace

$D_0'(\tau_{p,q}) := \{u : u \in D(\tau_{p,q}) |\ supp(u)$ is compact and contained in the interior of $I\}$.

And with $T_0'\left(\tau_{q',p'}^{+}\right)$ defined similarly. These operators are densely-defined and closable in $L_w^p(I, \mathbb{K}^s)$; and we define the minimal operators $T_0(\tau_{p,q})$ and $T_0\left(\tau_{q',p'}^{+}\right)$ to be their respective closures (see [12], [13], [16] and [19]). We denote the domains of $T_0(\tau_{p,q})$ and $T_0\left(\tau_{q',p'}^{+}\right)$ by $D_0(\tau_{p,q})$ and $D_0\left(\tau_{q',p'}^{+}\right)$ respectively. It can be shown that:

$$\left.\begin{array}{l} u \in D_0(\tau_{p,q}) \Rightarrow (Q_{\tilde{F}} u)(a) = 0 \\ v \in D_0(\tau^+_{p,q}) \Rightarrow (Q_{\tilde{F}} v_+)(a) = 0 \end{array}\right\}, \tag{3.17}$$

because we are assuming that a is a regular end-point. Clearly $T_0(\tau_{p,q})$ and $T(\tau_{p,q})$ are linear operators of L^p_w into L^q_w and $T_0(\tau_{p,q}) \subset T(\tau_{p,q})$.

Moreover, in both regular and singular problems, we have

$$T_0^*(\tau_{p,q}) = T\left(\tau^+_{q',p'}\right),\ T^*(\tau_{p,q}) = T_0\left(\tau^+_{q',p'}\right), \tag{3.18}$$

see [4] and [22, Section 5] in the case when $\tau_{p,q} = \tau^+_{q',p'}$ and compare with treatment in [3, Section III.10.3] and [16] in general case. Also, we refer to [1-4] and [6-14] for more details.

Corollary 3.4 (cf. [13, Corollary 3.10]**.** (*a*) *If* $\tau_{2,2}$ *is symmetric*, *then* $T_0(\tau_{2,2})$ *is symmetric in the Hilbert space sense*.

(*b*) *If* $\tau_{2,2}$ *is J-symmetric*, *then* $T_0(\tau_{2,2})$ *is J-symmetric in the Hilbert space sense*.

4. The Regularly Solvable Operators

We see from (3.18) that $T_0(\tau_{p,q}) \subset T(\tau_{p,q}) = [T_0\left(\tau^+_{q',p'}\right)]^*$ and hence $T_0(\tau_{p,q})$, $T_0\left(\tau^+_{q',p'}\right)$ form an adjoint pair of closed-densely operators in $L^p_w(I, \mathbb{K}^s)$.

Lemma 4.1 [4]**.** *For* $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$,

$def[T_0(\tau_{p,q}) - \lambda I] + def\left[T_0\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right]$ *is constant and*

$$0 \le def[T_0(\tau_{p,q}) - \lambda I] + def\left[T_0\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right] \le 2n.$$

In the problem with one singular end-point,

$$n \le def[T_0(\tau_{p,q}) - \lambda I] + def\left[T_0\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right] \le 2n,$$

for all $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$.

In the regular problem,

$$def[T_0(\tau_{p,q}) - \lambda I] + def\left[T_0\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right] = 2n,$$

for all $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$.

Proof. The proof is similar to that in [2, 4], [8-11] and [22] and therefore omitted.

For $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$, we define r, s and m as follows :

$$\left.\begin{array}{c} r = r(\lambda) := def[T_0(\tau_{p,q}) - \lambda I] = null\ \left[T\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right], \\ s = s(\lambda) := def\left[T_0\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right] = null\ [T(\tau_{p,q}) - \lambda I], \\ m = r + s. \end{array}\right\}, \tag{4.1}$$

then, $0 \le r, s \le n$ and by Lemma 4.1, m is constant on $\prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$ and

$$n \le m \le 2n. \tag{4.2}$$

For $\prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right] \neq \phi$ the operators which are regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0\left(\tau^+_{q',p'}\right)$ are characterized by the following theorem which proved for a general quasi-differential operator in [3, Theorem 10.15].

Theorem 4.2 [4, Theorem 3.2]**.** *For* $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$. *Let r and m be defined by* (4.1), *and let* $\psi_j^{p,q}(j = 1,2,\dots r)$, $\Phi_k^{p,q}(k = r+1,\dots,m)$ *be arbitrary functions satisfying*:

(i) $\psi_j^{p,q}(j = 1, 2, \dots r) \subset D[T(\tau_{p,q})]$ *are linearly independent modulo* $D[T_0(\tau_{p,q})]$ *and* $\Phi_k^{p,q}(k = r+1, \dots, m) \subset D[T(\tau^+_{q',p'})]$ *are linearly independent modulo* $D\left[T_0\left(\tau^+_{q',p'}\right)\right]$.

(ii) $[\psi_j^{p,q}, \Phi_k^{p,q}](b) - [\psi_j^{p,q}, \Phi_k^{p,q}](a) = 0, (j = 1, 2, \dots r; k = r+1, \dots, m)$.

Then the set

$$\{u : u \in D[T(\tau_{p,q})], [u, \Phi_k^{p,q}](b) - [u, \Phi_k^{p,q}](a) = 0,\ k = r+1, \ldots, m\}, \quad (4.3)$$

is the domain of an operator S which is regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0\left(\tau^+_{q',p'}\right)$ and the set

$$\{v : v \in D[T\left(\tau^+_{q',p'}\right)], [\psi_j^{p,q}, v](b) - [\psi_j^{p,q}, v](a) = 0,\ j = 1, 2, \ldots r\} \quad (4.4)$$

is the domain of the operator S^*; moreover $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0\left(\tau^+_{q',p'}\right)$ and $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right] \cap \Delta_4(S)$, then with r and s defined by (4.1) there exist functions $\psi_j^{p,q}(j = 1, 2, \ldots r)$, $\Phi_k^{p,q}(k = r+1, \ldots, m)$ which satisfy (i) and (ii) and are such that (4.3) and (4.4) are the domains of S and S^* respectively.

S is self-adjoint if, and only if, $\tau_{p,q} = \tau^+_{q',p'}$, $r = s$ and $\Phi_k^{p,q} = \psi_{k-r}^{p,q}$ $(k = r+1, \ldots, m)$; S is J–self-adjoint if $\tau_{p,q} = J\,\tau_{p,q}J$ (J is a complex conjugate), $r = s$ and $\Phi_k^{p,q} = \overline{\psi}_{k-r}^{p,q}(k = r+1, \ldots, m)$.

Proof. The proof is entirely similar to that in [4]. We refer also to [8-11] and [22] for more details.

Next, the technical lemmas which are considered for a generalization of ones proved by Sun Jiong [12] and Zettl [22] for a symmetric case, and by Evans and Ibrahim [4] for a general case.

Lemma 4.3 (cf. [15])**.** *Let $T_0(\tau_{p,q})$ and $T(\tau_{p,q})$ be the minimal and maximal differential operators of L_w^p into L_w^q, then*

(*a*) *$R[T_0(\tau_{p,q}) - \lambda I]$ is closed if and only if $R[T(\tau_{p,q}) - \lambda I]$ is closed,*

(*b*) *$R[T_0(\tau_{p,q}) - \lambda I]$ is closed if and only if $R\left[\left[T_0\left(\tau^+_{q',p'}\right) - \overline{\lambda} I\right]\right.$ is closed.*

Proof. The proof is similar to that in [13, Lemma 4.4] and therefore omitted.

For $\lambda \in \prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right]$, define r, s and m as in (4.1), let $\mathfrak{y}_j(j =$

$1, 2, \ldots, r)$, $\Phi_k (k = s+1, \ldots, m)$ be bases for $N[T(\tau_{p,q}) - \lambda I]$ and $N[T(\tau^+_{q',p'}) - \overline{\lambda} I]$ respectively, thus $\psi_j, \Phi_k \in L^p_w(I, \mathbb{K}^s)$ for $(j = 1, 2, \ldots, s;\ k = s+1, \ldots, m)$ and

$$\tau_{p,q}[\psi_j] = \lambda w \psi_j, \quad \tau^+_{q',p'}[\Phi_k] = \overline{\lambda} w \Phi_k \tag{4.5}$$

Since $[T_0(\tau^+_{p,q}) - \overline{\lambda} I]$ has closed range, so does its adjoint $[T(\tau_{p,q}) - \lambda I]$ and moreover $R[T(\tau_{p,q}) - \lambda I]^{\perp} = N[[T_0(\tau^+_{q',p'}) - \overline{\lambda} I] = \{0\}$. Hence $R[T(\tau_{p,q}) - \lambda I] = L^q_w(I, \mathbb{K}^s)$ and $R[T(\tau^+_{q',p'}) - \overline{\lambda} I] = L^q_w(I, \mathbb{K}^s)$. We refer to [4], [13, Lemma 4.4] and [15] for more details.

We can therefore define the following functions $x_j, y_j, j = 1, 2, \ldots, m$:

$$\left.\begin{aligned} x_j &= \psi_j, (j = 1, 2, \ldots, s) \\ [T(\tau_{p,q}) - \lambda I] x_j &= \Phi_j, (j = s+1, \ldots, m) \end{aligned}\right\}, \tag{4.6}$$

$$\left.\begin{aligned} [T(\tau^+_{q',p'}) - \overline{\lambda} I] y_j &= \psi_j, (j = 1, 2, \ldots, s) \\ y_j &= \Phi_j, (j = s+1, \ldots, m) \end{aligned}\right\}. \tag{4.7}$$

Lemma 4.4 (cf. [4, Lemma 4.1])**.** *The set* $\{x_j : j = 1, 2, \ldots, m\}$ *is a basis of* $N([T(\tau^+_{q',p'}) - \overline{\lambda} I][T(\tau_{p,q}) - \lambda I])$ *and* $\{y_j : j = 1, 2, \ldots, m\}$ *is a basis of* $N([T(\tau_{p,q}) - \lambda I][T(\tau^+_{q',p'}) - \overline{\lambda} I])$.

On applying Theorem 2.3 we obtain the following:

Lemma 4.5 (cf. [4, Corollary 4.2])**.** *Any* $z \in D[T(\tau_{p,q})]$ *and* $z^+ \in D[T(\tau^+_{q',p'})]$ *have a unique representations*

$$z = z_0 + \sum\nolimits_{j=1}^{m} a_j x_j \ (z_0 \in D[T_0(\tau_{p,q})],\ a_j \in \mathbb{C}) \tag{4.8}$$

$$z^+ = z_0^+ + \sum\nolimits_{j=1}^{m} b_j y_j \ (z_0^+ \in D[T_0(\tau^+_{q',p'})],\ b_j \in \mathbb{C}). \tag{4.9}$$

A central rule in the argument is played by

Lemma 4.6 (cf. [4, Lemma 4.3])**.** *Let*

$$E_{m\times m} \coloneqq ([x_j,\ y_j](b))_{\substack{1\le i\le m\\ 1\le j\le m}} \tag{4.10}$$

and

$$E^{1,2}_{s\times r} \coloneqq ([x_j,\ y_j](b))_{\substack{1\le i\le s\\ s+1\le j\le m}}\ . \tag{4.11}$$

Then

$$rank\ E^{1,2}_{s\times r} = rank\ E_{m\times m} = m-n. \tag{4.12}$$

In view of Lemma 4.6 and since $r,\ s \ge m-n,$ we may suppose, without loss of generality, that the matrix

$$E^{1,2}_{(m-n)\times(m-n)} = ([x_j,\ y_k](b))_{\substack{1\le j\le m(m-n)\cdot\\ n+1\le k\le m}} \tag{4.13}$$

satisfies

$$rank\ E^{1,2}_{(m-n)\times(m-n)} = m-n. \tag{4.14}$$

If we partition $E_{m\times m}$

$$E_{m\times m} = \begin{pmatrix} E^{1,1}_{(m-n)\times n} & \vdots & E^{1,2}_{(m-n)\times(m-n)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ E^{2,1}_{n\times n} & \vdots & E^{2,2}_{n\times(m-n)} \end{pmatrix} \tag{4.15}$$

and set

$$\left.\begin{aligned} E^{1}_{(m-n)\times m} &= E^{1,1}_{(m-n)\times n} \oplus E^{1,2}_{(m-n)\times(m-n)} \\ E^{2}_{n\times m} &= E^{2,1}_{n\times n} \oplus E^{2,2}_{n\times(m-n)} \end{aligned}\right\} \tag{4.16}$$

$$\left.\begin{aligned} F^{1}_{m\times n} &= E^{1,1}_{(m-n)\times n} \oplus^{T} E^{2,1}_{n\times n} \\ F^{2}_{m\times(m-n)} &= E^{1,2}_{(m-n)\times(m-n)} \oplus^{T} E^{2,2}_{n\times(m-n)} \end{aligned}\right\} \tag{4.17}$$

Then (4.14) yields the results

$$rank\ E^{1}_{(m-n)\times m} = rank\ F^{2}_{m\times(m-n)} = m-n. \tag{4.18}$$

Lemma 4.7 (cf. [4, Lemma 4.5]). *Let* $z_i (i = 1, 2, \ldots, n)$ *be a function in* $D[T(\tau_{p,q})]$ *which satisfy*

$$\left.\begin{array}{c} z_i^{[k-1]}(a) = \delta_{ik},\ z_i^{[k-1]}(c) = 0 \\ z_i(t) = 0 \quad for \quad t \geq c. \end{array}\right\} \tag{4.19}$$

and suppose that (4.14) *is satisfied. Then* $x_i (i = m-n+1, \ldots, m)$ *has a unique representation*

$$x_i = x_{i0} + \sum\nolimits_{j=1}^{n} b_{ij} z_j + \sum\nolimits_{j=1}^{m-n} c_{ij} x_j \tag{4.20}$$

where $x_{i0} \in D[T_0(\tau_{p,q})]$, *and the* b_{ij} *and* cij *are complex constants.*

Lemma 4.8 (cf. [4, Lemma 4.6]). *Let* $D_1[T(\tau_{p,q})]$ *be the linear span of* $\{z_i (i = 1, 2, \ldots, n)\}$ *where* $z_i \in D[T(\tau_{p,q})]$ *satisfy* (4.19) *and let* $D_2[T\ (\tau_{p,q})]$ *be the linear span of* $\{x_i (i = 1, 2, \ldots, m-n)\}$ *with* (4.14) *satisfied. Then*

$$D[T(\tau_{p,q})] = D[T_0(\tau_{p,q})] \dot{+} D_1[T(\tau_{p,q})] \dot{+} D_2[T(\tau_{p,q})].$$

If $z_i^{+} (i = 1, 2, \ldots, n)$ are function in $D[T(\tau^{+}_{q',p'})]$ which for $k = 1, 2, \ldots, n$ and some $c \in I$, satisfy

$$\left.\begin{array}{c} (z_i^{+})_{+}^{[k-1]}(a) = \delta_{ik},\ (z_i^{+})_{+}^{[k-1]}(c) = 0 \\ z_i^{+}(t) = 0 \quad \text{for } t \geq 0. \end{array}\right\} \tag{4.21}$$

and (4.14) is satisfied, then we have that each $y_i (i = 1, 2, \ldots, n)$ has a unique representation

$$y_i = y_{i0} + \sum\nolimits_{j=1}^{n} d_{ij} z_j^{+} + \sum\nolimits_{j=n+1}^{m} e_{ij} x_j, \tag{4.21}$$

where $y_{i0} \in D[T_0(\tau^{+}_{q',p'})]$, and the d_{ij} and e_{ij} are complex constants. Also, if $D_1[T(\tau^{+}_{q',p'})]$ and $D_2[T(\tau^{+}_{q',p'})]$ are the linear spans of $\{z_i^{+} : i = 1, 2, \ldots, n\}$ and $\{y_i : i = n+1, \ldots, m\}$ respectively then

$$D\left[T\left(\tau^{+}_{q',\,p'}\right)\right] = D[T_0\left(\tau^{+}_{q',\,p'}\right)] \dot{+} D_1[T\left(\tau^{+}_{q',\,p'}\right)] \dot{+} D_2[T\left(\tau^{+}_{q',\,p'}\right)] \tag{4.22}$$

5. The Main Theorems

We shall now characterize all the operators which are regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0\left(\tau^{+}_{q',\,p'}\right)$ in terms of boundary conditions featuring square-integrable solutions of the equations in (4.5). This generalizes an analogue of H. Frentzen [13, 14] a results of Evans and Ibrahim [4] for $p = q = 2$, Sun Jiong's result [12] in the special case when $T_0(\tau_{p,q})$ is a symmetric operator with equal deficiency indices and Zai-ju Shang, s result [18] when $T_0(\tau_{p,q})$ is *J*-symmetric.

Theorem 5.1. *Let* $\lambda \in \prod\left[T_0(\tau_{p,q}),\, T_0\left(\tau^{+}_{q',\,p'}\right)\right]$, *let r, s and m be defined by* (4.1) *and let* $x_i (i = 1, 2, \ldots, m)$, $y_j (j = 1, 2, \ldots, m)$ *be defined by* (4.6) *and* (4.7) *and arranged to satisfy* (4.14). *Let* $K^{p,q}_{r\times n}$, $L^{p,q}_{r\times(m-n)}$, $M^{p,q}_{s\times n}$ *and* $N^{p,q}_{s\times(m-n)}$ *be numerical matrices which satisfy the following conditions*:

$$(i) \qquad rank\,\left(K^{p,q}_{r\times n} \oplus L^{p,q}_{r\times(m-n)}\right) = r,\ rank\,\left(M^{p,q}_{s\times n} \oplus N^{p,q}_{s\times(m-n)}\right) = s,$$

$$(ii)\ L^{p,q}_{r\times(m-n)} E^{1,2}_{(m-n)\times(m-n)} \left(N^{p,q}_{s\times(m-n)}\right)^{T} + (-i)^n K^{p,q}_{r\times n} J_{n\times n} \left(M^{p,q}_{s\times n}\right)^{T} = 0_{r\times s}.$$

Then the set of all $u \in D[T(\tau_{p,q})]$ such that

$$M^{p,q}_{s\times n}(Q_{\tilde{F}}u)(a) - N^{p,q}_{s\times(m-n)}\left([u,\, y_{n+1}]\right)(b), \ldots, [u,\, y_m]\right)(b))^{T} = 0_{s\times 1} \tag{5.1}$$

Is the domain of an operator *S* which is regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0\left(\tau^{+}_{q',\,p'}\right)$ and $D(S^*)$ is the set of all $v \in D[T\left(\tau^{+}_{q',\,p'}\right)]$ which are such that

$$K^{p,q}_{r\times n}(Q_{\tilde{F}}\bar{v}))(a) - L^{p,q}_{r\times(m-n)}([x_1,\, v](b), \ldots, [x_{m-n},\, v](b))^{T} = 0_{r\times 1}. \tag{5.2}$$

Proof. Let

$$M^{p,q}_{s\times n} J^{-1}_{n\times n} = -i^n (\alpha_{jk})_{\substack{r+1\le j\le m \\ 1\le k\le n}},\ N^{p,q}_{s\times(m-n)} = (\beta_{jk})_{\substack{r+1\le j\le m \\ n+1\le k\le m}} \tag{5.3}$$

and set

$$g_j = \sum_{k=n+1}^{m} \overline{\beta}_{jk}\, y_k \quad (j = 1, 2, \ldots, m). \tag{5.4}$$

Then $g_j \in D[T(\tau^{+}_{q', p'})]$ and, by [22, Theorem 8], we may choose $\varphi_j (i = r+1, \ldots, m) \in D\left[T(\tau^{+}_{q', p'})\right]$ such that for $k = 1, 2, \ldots, n$ and some $c \in I$,

$$\left.\begin{array}{c} (\varphi_j)_{+}^{[k-1]}(a) = \delta_{jk},\ (\varphi_j)_{+}^{[k-1]}(c) = (g_j)_{+}^{[k-1]}(c) \\ \varphi_j = g_j \text{ on } [c, b). \end{array}\right\}, \tag{5.5}$$

This gives

$$M^{p,q}_{s\times n}\left((Q_{\tilde{F}}u)(a)\right) = -i^n\left((\overline{\varphi}_j)_{+}^{[k-1]}(a)\right)_{\substack{r+1\le j\le m \\ 1\le k\le n}} J_{n\times n}\left((Q_{\tilde{F}}u)(a)\right)$$

$$= ([u, \varphi_{r+1}](a), [u, \varphi_{r+2}](a), \ldots, [u, \varphi_m](a))^T$$

by (3.11). Also, since $\varphi_j = g_j$ on $[c, b)$,

$$N^{p,q}_{s\times(m-n)} \begin{pmatrix} [u, y_{n+1}](b) \\ \vdots \\ [u, y_m](b) \end{pmatrix} = ([u, \sum_{k=n+1}^{m} \overline{\beta}_{r+1,k}\, y_k](b), \ldots, [u, \sum_{k=n+1}^{m} \overline{\beta}_{m,k}\, y_k](b))^T$$

$$= ([u, \varphi_{r+1}](b), \ldots, [u, \varphi_m](b))^T.$$

The boundary conditions (5.1) therefore coincides with that in (4.3). Similarly (5.2) coincides with that in (4.4) on making the following choices :

$$K^{p,q}_{r\times n} J^{-1}_{n\times n} = -i^n (\gamma_{jk})_{\substack{1\le j\le\le r \\ 1\le k\le n}},\ L^{p,q}_{r\times(m-n)} = (\varepsilon_{jk})_{\substack{1\le j\le r \\ 1\le k\le m-n}}, \tag{5.6}$$

$$h_j = \sum_{k=1}^{m-n} \varepsilon_{jk}\, x_k \quad (j = 1, 2, \ldots, r) \tag{5.7}$$

and $\psi_j (j = 1, 2, \ldots, r) \in D[T(\tau_{p,q})]$ such that for $k = 1, 2, \ldots, n$ and some $c \in I$,

$$\left.\begin{array}{c} \psi_j^{[k-1]}(a) = \gamma_{jk},\ \psi_j^{[k-1]}(c) = h_j^{[k-1]}(c) \\ \psi_j = h_j \text{ on } [c, b). \end{array}\right\}. \tag{5.8}$$

It remains to show that the above functions $\varphi_k (k = r+1, \ldots, m)$ and

$\psi_j (j = 1, 2, \ldots, r)$ satisfy conditions (i) and (ii) in Theorem 4.2. First, suppose that $\psi_j (j = 1, 2, \ldots, r)$ is linearly independent modulo $D[T_0(\tau_{p,q})]$, that is, there exist constants $c_1, \ldots, c_r$ not all zero, such that $u = \sum_{j=1}^{r} c_j \psi_j \in D[T_0(\tau_{p,q})]$. Then, from (3.16), (3.17), (5.6) and (5.8),

$$0_{n\times 1} = \left((Q_{\tilde{F}} u)(a)\right) = (\gamma_{kj})_{\substack{1\le j\le r \\ 1\le k\le n}} \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} = i^n J_{n\times n} (K^{p,q}_{r\times n})^T \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix}$$

on noting that $J^{-1}_{n\times n} = (-1)^{n+1} J_{n\times n} = J^T_{n\times n}$. But $J_{n\times n}$ has rank n and so we infer that

$$(c_1, \ldots, c_r) K^{p,q}_{r\times n} = 0_{1\times n}. \tag{5.9}$$

Since $u \in D[T_0(\tau_{p,q})]$, we have that $[u, v](b) = 0$ for all $v \in D[T_0(\tau^+_{q',p'})]$. Hence

$$\begin{aligned} 0_{1\times m} &= ([u, y_1](b), \ldots, [u, y_m](b)) \\ &= ([\sum_{j=1}^{r} c_j \sum_{k=1}^{m-n} \varepsilon_{jk} x_k, y_1](b), \ldots, [\sum_{j=1}^{r} c_j \sum_{k=1}^{m-n} \varepsilon_{jk} x_k, y_m](b)) \\ &= (c_1, \ldots, c_r) L^{p,q}_{r\times(m-n)} E^1_{(m-n)\times m} \end{aligned}$$

on using the notation in (4.16). In view of (4.18) we conclude that

$$(c_1, \ldots, c_r) L^{p,q}_{r\times(m-n)} = 0_{1\times(m-n)}. \tag{5.10}$$

We obtain from (5.9) and (5.10) that

$$(c_1, \ldots, c_r)\left(K^{p,q}_{r\times n} \oplus L^{p,q}_{r\times(m-n)}\right) = 0_{1\times m}$$

which contradicts the assumption that $K^{p,q}_{r\times n} \oplus L^{p,q}_{r\times(m-n)}$ has rank r. It follows similarly that $\{\varphi_k (k = r+1, \ldots, m)\} \subset D[T(\tau^+_{q',p'})]$ is linearly independent modulo $D[T_0(\tau^+_{q',p'})]$.

To prove (ii) in Theorem 4.2 we first observe that

$$\left([\psi_j, \varphi_k](a)\right)_{\substack{1\le j\le r\\ r+1\le k\le m}} = (-i)^n \left(\psi_j^{[k-1]}(a)\right)_{\substack{1\le j\le r\\ r+1\le k\le n}} J_{n\times n} \left(\overline{\varphi}_k^{[j-1]}(a)\right)_{\substack{1\le j\le n\\ r+1\le k\le m}}$$

$$= (-i)^n (\gamma_{jk})_{\substack{1\le j\le\le r\\ 1\le k\le m}} J_{n\times n} (\alpha_{kj})_{\substack{1\le j\le\le n\\ r+1\le k\le m}}$$

by (5.5) and (5.8),

$$= -(-i)^n K^{p,q}_{r\times n} J_{n\times n} (M^{p,q}_{s\times n})^T.$$

Next, we see that

$$[\psi_j, \varphi_k](b) = [\sum\nolimits_{\ell=1}^{m-n} \varepsilon_{j\ell} x_\ell, \sum\nolimits_{p=n+1}^{m} \overline{\beta}_{k,p} y_p](b),$$

whence

$$\left([\psi_j, \varphi_k](b)\right)_{\substack{1\le j\le r\\ r+1\le k\le m}} = (\varepsilon_{jk})_{\substack{1\le j\le r\\ 1\le k\le m-n}} \left([x_j, y_k](b)\right)_{\substack{1\le j\le m-n\\ r+1\le k\le m}} (\beta_{jk})^T_{\substack{r+1\le j\le m\\ n+1\le k\le m}}$$

$$= L^{p,q}_{r\times(m-n)} E^{1,2}_{(m-n)\times(m-n)} \left(N^{p,q}_{s\times(m-n)}\right)^T. \tag{5.12}$$

From (ii), (5.11) and (5.12) it follows that condition (ii) in Theorem 4.2 is satisfied. The proof is therefore complete.

The following is the converse of Theorem 5.1:

Theorem 5.2. *Let the operator S be a regularly solvable with respect to $T_0(\tau_{p,q})$ and $T_0\left(\tau^+_{q',p'}\right)$, let $\lambda \in \prod \left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right] \cap \Delta_4(S)$, let r, s and m be defined by* (4.1) *and suppose that* (4.14) *is satisfied. Then there exist numerical matrices $K^{p,q}_{r\times n}$, $L^{p,q}_{r\times(m-n)}$, $M^{p,q}_{s\times n}$ and $N^{p,q}_{s\times(m-n)}$ such that the conditions* (*i*) *and* (*ii*) *in Theorem* 5.1 *are satisfied and $D(S)$ is the set of all $u \in D[T(\tau_{p,q})]$ satisfying* (5.1) *while $D(S^*)$ is the set of all $v \in D[T\left(\tau^+_{q',p'}\right)]$ satisfying* (5.1).

Proof. Let $\{\psi_j (j = 1, 2 \ldots, r)\} \subset D[T(\tau_{p,q})]$, $\{\varphi_k (k = r+1, \ldots, m)\} \subset D\left[T\left(\tau^+_{q',p'}\right)\right]$ satisfy the second part of Theorem 4.2. From (4.22)

$$\varphi_j = y_{j0} + \sum\nolimits_{k=1}^{n} \eta_{jk} z_j^+ + \sum\nolimits_{k=n+1}^{m} \beta_{jk} y_k \quad (j = r+1, \ldots, m) \tag{5.13}$$

for some $y_{j0} \in D[T_0(\tau^+_{q',p'})]$ and complex constants η_{jk} and β_{jk}. Let

$$M^{p,q}_{s\times n} = -i^n\left(\overline{\varphi}^{[k-1]}_j(a)\right)_{\substack{r+1\le j\le m\\ 1\le k\le n}} J_{n\times n},\ N^{p,q}_{s\times(m-n)} = (\overline{\beta}_{jk})_{\substack{r+1\le j\le m\\ n+1\le k\le m}} \tag{5.14}$$

Then

$$\begin{pmatrix} [u,\varphi_{r+1}](a)\\ \vdots\\ [u,\varphi_m](a)\end{pmatrix} = \left\{(-i)^n(u(a), u^{[1]}(a), \ldots, u^{[n-1]}(a))J_{n\times n}\left(\overline{\varphi}^{[j-1]}_k(a)\right)_{\substack{1\le j\le n\\ r+1\le k\le m}}\right\}^T$$

$$= M^{p,q}_{s\times n}\left((Q_{\tilde{F}}u)(a)\right)$$

on recalling that $J^{-1}_{n\times n} = J^T_{n\times n} = (-1)^{n+1}J_{n\times n}$. Moreover, for all $u \in D[T(\tau_{p,q})]$, $[u, y_{j0}](b) = [u, z^+_k](b) = 0$ $(j = r+1, \ldots, m;\ k = 1, \ldots, n)$, and hence, from (5.13)

$$\begin{pmatrix} [u,\varphi_{r+1}](b)\\ \vdots\\ [u,\varphi_m](b)\end{pmatrix} = \begin{pmatrix} [u,\sum_{k=n+1}^m \beta_{r+1,k}y_k](b)\\ \vdots\\ [u,\sum_{k=n+1}^{m-n}\beta_{m,k}y_k](b)\end{pmatrix} = N^{p,q}_{s\times(m-n)}\begin{pmatrix} [u,y_{n+1}](b)\\ \vdots\\ [u,y_m](b)\end{pmatrix}.$$

Therefore, we have shown that the boundary conditions (5.1) coincide with those in (4.3); similarly (5.2) and the conditions in (4.4) can be shown to coincide if we choose

$$K^{p,q}_{r\times n}J^{-1}_{n\times n} := (-i)^n(\psi^{[k-1]}_j(a))_{\substack{1\le j\le r\\ 1\le k\le n}} J_{n\times n} \text{ and } L^{p,q}_{r\times(m-n)} := (\varepsilon_{jk})_{\substack{1\le j\le r\\ 1\le k\le m-n}}, \tag{5.15}$$

where the ε_{jk} are the constants uniquely determined by the decomposition

$$\psi_j = x_{j0} + \sum_{k=1}^n \xi_{jk}z_j + \sum_{k=1}^{m-n}\varepsilon_{jk}x_k \quad (j = 1, 2, \ldots, r) \tag{5.16}$$

derived from Lemma 4.7.

Next, we prove that (i) and (ii) are consequences of conditions (i) and (ii) in Theorem 4.2. Suppose that $rank\left(K^{p,q}_{r\times n} \oplus L^{p,q}_{r\times(m-n)}\right) < r$. Then there exist constants $c_1, \ldots, c_r$ not all zero, such that

$$(c_1, \ldots, c_r)\left(K^{p,q}_{r\times n} \oplus L^{p,q}_{r\times(m-n)}\right) = 0_{1\times m} \tag{5.17}$$

This implies that

$$0_{1\times n} = (c_1, \ldots, c_r)\, L^{p,q}_{r\times(m-n)} = (-i)^n (c_1, \ldots, c_r)\,(\psi_j^{[k-1]}(a))_{\substack{1\le j\le\le r \\ 1\le k\le n}} J_{n\times n}$$

and as $J_{n\times n}$ is non-singular, it follows that $u = \sum_{j=1}^{r} c_j \psi_j$ satisfies

$$(u(a), u^{[1]}(a), \ldots, u^{[n-1]}(a)) = 0_{1\times n}. \tag{5.18}$$

We also infer from (5.17) that

$$0_{1\times(m-n)} = (c_1, \ldots, c_r) L^{p,q}_{r\times(m-n)} = \left(\sum_{j=1}^{r} c_j \varepsilon_{j,1}, \ldots, \sum_{j=1}^{r} c_j \varepsilon_{j,(m-n)}\right).$$

Consequently, on substituting (5.16), we obtain

$$u = \sum_{j=1}^{r} c_j x_{j0} + \sum_{j=1}^{r}\sum_{k=1}^{n} c_j \xi_{jk} z_k.$$

For arbitrary $v \in D[T(\tau^{+}_{q',p'})]$ it follows that $[u, v](b) = 0$. This fact and (5.18) together imply that $u \in D[T_0(\tau_{p,q})]$ and hence that $\{\psi_j (j = 1, 2, \ldots, r)\}$ is linearly independent modulo $D[T_0(\tau_{p,q})]$, contrary to assumption. We have therefore proved that

$rank\left(K^{p,q}_{r\times n} \oplus L^{p,q}_{r\times(m-n)}\right) = r$. The proof of $rank\left(M^{p,q}_{s\times n} \oplus N^{p,q}_{s\times(m-n)}\right) = s$ is similar. From (5.14)

$$\begin{aligned}
\left([\psi_j, \varphi_k](a)\right)_{\substack{1\le j\le r \\ r+1\le k\le m}} &= (-i)^n \left(\psi_j^{[k-1]}(a)\right)_{\substack{1\le j\le r \\ 1\le k\le n}} J_{n\times n} \left((\overline{\varphi}_k)_{+}^{[j-1]}(a)\right)_{\substack{1\le j\le n \\ r+1\le k\le m}} \\
&= (-i)^n (i^n K^{p,q}_{r\times n} J^{-1}_{n\times n}) J_{n\times n} (-(-i)^n M^{p,q}_{s\times n} J^{-1}_{n\times n})^T \\
&= -(-)^n K^{p,q}_{r\times n} J_{n\times n} (M^{p,q}_{s\times n})^T.
\end{aligned}$$

On using (5.13), (5.16) and the facts that $z_j = z_j^{+} = 0$ on $[c, b)\,(j = 1, 2, \ldots, n)$ and $[u, v](b) = 0$ if either $u \in D[T(\tau_{p,q})]$ and $v \in D[T_0(\tau^{+}_{q',p'})]$ or

$u \in D[T_0(\tau_{p,q})]$ and $v \in D\left[T\left(\tau^+_{q',p'}\right)\right]$, we obtain

$$\left([\psi_j, \varphi_k](a)\right)_{\substack{1\le j\le r\\ r+1\le k\le m}} = \left(\left[\sum_{\ell=1}^{m-n} \varepsilon_{j\ell} x_\ell, \sum_{i=n+1}^{m} \beta_{k,i} y_i\right](b)\right)_{\substack{1\le j\le r\\ r+1\le k\le m}}$$

$$= L^{p,q}_{r\times(m-n)} E^{1,2}_{(m-n)\times(m-n)} \left(N^{p,q}_{s\times(m-n)}\right)^T.$$

The proof is therefore complete.

Corollary 5.3 [4, 12]**.** *Let* $\tau_{p,q}$ *be formally symmetric and suppose that* $T_0(\tau_{p,q})$ *has equal deficiency indices* (ℓ, ℓ)*. Then* $D(S)$ *is the domain of a self-adjoint extension of* $T_0(\tau_{p,q})$ *if and only if* $D(S)$ *is the set of functions* $u \in D[T(\tau_{p,q})]$ *which satisfy* ℓ *boundary conditions*

$$F^{p,q}_{\ell\times n}\left((Q_{\tilde{F}}u)(a)\right) - G^{p,q}_{\ell\times(2\ell-n)}([u, x_1](b), \ldots, [u, x_{2\ell-n}](b))^T = 0_{\ell\times 1}, \quad (5.19)$$

where $F^{p,q}_{\ell\times n}$, $G^{p,q}_{\ell\times(2\ell-n)}$ *are numerical matrices which satisfy*

$$rank\left(F^{p,q}_{\ell\times n} \oplus G^{p,q}_{\ell\times(2\ell-n)}\right) = \ell, \quad (5.20)$$

$$G^{p,q}_{\ell\times(2\ell-n)} B^{p,q}_{(2\ell-n)\times(2\ell-n)} \left(G^{p,q}_{\ell\times(2\ell-n)}\right)^* - (i)^n F^{p,q}_{\ell\times n} J_{n\times n} (F^{p,q}_{\ell\times n})^* = 0_{\ell\times 1}, \quad (5.21)$$

and the L^p_w*– solutions* $x_j\ (j = 1, 2, \ldots, \ell)$ *of the equation* $\tau_{p,q}[u] = iwu$ *are so arranged* (*without loss of generality*) *that*

$$B^{p,q}_{(2\ell-n)\times(2\ell-n)} = \left([x_k, x_j](b)\right)_{\substack{1\le j\le 2\ell-n\\ 1\le k\le 2\ell-n}} \quad (5.22)$$

Satisfies

$$rank\left(B^{p,q}_{(2\ell-n)\times(2\ell-n)}\right) = 2\ell - n. \quad (5.23)$$

Proof. Under the hypothesis, $T_0\left(\tau^+_{q',p'}\right) = T_0(\tau_{p,q})$ and $\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',p'}\right)\right] = \prod[T_0(\tau_{p,q})] \supseteq \mathbb{C}\backslash\mathbb{R}$. Thus Theorems 5.1 and 5.2 apply with $r = s = \ell$ and $m = 2\ell$, the regularly solvable operators being the self-adjoint extensions of

$T_0(\tau_{p,q})$. To obtain the form of the result stated in this corollary, we observe that since $y_j \in D\left[T\left(\tau^+_{q',p'}\right)\right] = D[T\ (\tau_{p,q})]$, $y_i = y_{i0} + \sum_{j=1}^{n} c_{ij} z_j + \sum_{t=1}^{2\ell-n} d_{it} x_t$ $(t = 1, 2, \ldots, 2\ell)$ on account of Lemma 4.8. This gives, for $u \in D[T(\tau_{p,q})]$,

$$\begin{pmatrix} [u,\ y_{r+1}](b) \\ \vdots \\ [u,\ y_{2\ell}](b) \end{pmatrix} = \begin{pmatrix} [u,\ \sum_{t=1}^{2\ell-n} d_{n+1,t} x_t](b) \\ \vdots \\ [u,\ \sum_{t=1}^{2\ell-n} d_{2\ell,t} x_t](b) \end{pmatrix} = \overline{D}^{p,q}_{(2\ell-n)\times(2\ell-n)} \begin{pmatrix} [u,\ x_1](b) \\ \vdots \\ [u,\ x_{2\ell}](b) \end{pmatrix}. \tag{5.24}$$

where $D^{p,q}_{(2\ell-n)\times(2\ell-n)} = (d_{jk})_{\substack{n+1\le j 2\ell-n \\ n+1\le k\le 2\ell}}$. Also,

$$E^{1,2}_{(2\ell-n)\times(2\ell-n)} = ([x_j,\ \sum_{t=1}^{2\ell-n} d_{k,t} x_t](b))_{\substack{1\le j\le 2\ell-n \\ n+1\le k\le 2\ell}}$$

$$= \overline{B}^{p,q}_{(2\ell-n)\times(2\ell-n)} \left(D^{p,q}_{(2\ell-n)\times(2\ell-n)}\right)^{*}. \tag{5.25}$$

Thus (4.14) implies that $B^{p,q}_{(2\ell-n)\times(2\ell-n)}$ and $D^{p,q}_{(2\ell-n)\times(2\ell-n)}$ are non-singular. In Theorems 5.1 and 5.2 we now put

$$F^{p,q}_{\ell\times n} = M^{p,q}_{\ell\times n} = \overline{K}^{p,q}_{\ell\times n},\ \ G^{p,q}_{\ell\times(2\ell-n)} = N^{p,q}_{\ell\times(2\ell-n)} \overline{D}^{p,q}_{(2\ell-n)\times(2\ell-n)} = \overline{L}^{p,q}_{\ell\times(2\ell-n)}$$

and Corollary 5.3 follows : to obtain $B^{p,q}_{(2\ell-n)\times(2\ell-n)}$ we have used the fact that $[u, v]$ is skew- Hermitian , i.e., $[u, v] = \overline{[u, v]}$.

Remark 5.4. Note that (5.25) implies that, as we can arrange for (4.14) to be satisfied, the matrix $B^{p,q}_{(2\ell-n)\times(2\ell-n)}$ is non-singular, a fact not highlighted by Sun Joing [12].

Remark 5.5. Sun Joing's result in [12] is expressed in terms of ordinary derivatives, with the consequence that the matrix $-i^n J_{n\times n}$ in (5.21) has to be replaced by $Q^{-1}(a)$, where Q is the matrix associated with the sesquilinear form $[u, v]$ expressed as

$$[u, v](x) = ((Q_{\tilde{F}}\bar{v})^T Q_{n\times n}(Q_{\tilde{F}}u))(x).$$

Also, bear in mined that in Sun Joing's analysis, has functions x_j $(j = \ell+1, \ldots, 2\ell)$ are what we call $y_j (j = \ell+1, \ldots, 2\ell)$.

Remark 5.6. Theorems 5.1 and 5.2 characterize the maximal symmetric extensions of $T_0(\tau_{p,q})$ when $T_0(\tau_{p,q})$ is a symmetric operator with unequal deficiency indices (r, s).

We finally turn our attentions to the case when $\tau_{p,q}$ is formally J-symmetric, that is $\tau^+_{q',\,p'} = J\tau_{p,q}J$, where J is complex conjugation. The operator $T_0(\tau_{p,q})$ is then J-symmetric and $T_0(\tau_{p,q})$ and $T_0\left(\tau^+_{q',\,p'}\right) = JT_0\ (\tau_{p,q})J$ form an adjoint pair with $\prod\left[T_0(\tau_{p,q}), T_0\left(\tau^+_{q',\,p'}\right)\right] = \prod[T_0(\tau_{p,q})]$. Since $\tau_{p,q}[u] = \lambda wu$ if and only if $\tau^+_{q',\,p'}[v] = \bar{\lambda}w\bar{v}$, it follows from (4 ,2) that for all $\lambda \in \prod[T_0(\tau_{p,q})]$, $def[T_0(\tau_{p,q}) - \lambda I] + def\left[T_0\left(\tau^+_{q',\,p'}\right) - \bar{\lambda}I\right]$ is constant ℓ, say, and so in (4.1), $r = s = \ell$ with

$$n/2 \le \ell \le n; \tag{5.26}$$

Note that Frentzen proved in [13] that a formally J-symmetric expression $\tau_{p,q}$ generated by a Shin-Zettl matrix must be of even order. If S is a J-self-adjoint extension of $T_0(\tau_{p,q})$, then $S^* = JSJ$ and consequently $v \in D(S^*)$ if and only if $\bar{v} \in D(S)$.

Corollary 5.7 (cf. [4, 18])**.** *Let* $\tau^+_{q',\,p'} = J\tau_{p,q}J$, *where* J *is complex conjugation, and set* $def[T_0(\tau_{p,q}) - \lambda I] + def\left[T_0\left(\tau^+_{q',\,p'}\right) - \bar{\lambda}I\right] = \ell$. *Then* $D(S)$ *is the domain of a J-self-adjoint extension of* $T_0(\tau_{p,q})$ *which is regularly solvable if and only if* $D(S)$ *is the set of* $u \in D[T(\tau_{p,q})]$ *which satisfy* ℓ *boundary conditions*

$$F^{p,q}_{\ell\times n}\left((Q_{\tilde{F}}u)(a)\right) - G^{p,q}_{\ell\times(2\ell-n)}([u, \bar{x}_1](b), \ldots, [u, \bar{x}_{2\ell-n}](b))^T = 0_{\ell\times 1}, \tag{5.27}$$

where $F^{p,q}_{\ell\times n}$, $G^{p,q}_{\ell\times(2\ell-n)}$ *are numerical matrices which satisfy*

$$rank\ \left(F^{p,q}_{\ell\times n} \oplus G^{p,q}_{\ell\times(2\ell-n)}\right) = \ell, \tag{5.28}$$

$$G^{p,q}_{\ell\times(2\ell-n)} H^{p,q}_{(2\ell-n)\times(2\ell-n)} \left(G^{p,q}_{\ell\times(2\ell-n)}\right)^T + (-i)^n F^{p,q}_{\ell\times n} J_{n\times n} (F^{p,q}_{\ell\times n})^T = 0_{\ell\times 1}, \tag{5.29}$$

and

$$H^{p,q}_{(2\ell-n)\times(2\ell-n)} := \left([x_j, \bar{x}_k](b)\right)_{\substack{1\le j\le 2\ell-n \\ 1\le k\le 2\ell-n}} \tag{5.30}$$

can be chosen (*without loss of generality*) *such that*

$$rank\ H^{p,q}_{(2\ell-n)\times(2\ell-n)} = 2\ell - n. \tag{5.31}$$

Proof. In applying Theorems 5.1 and 5.2 to this case, the critical fact is that $\bar{y}_j (j = 1, 2, \ldots, 2\ell) \in D[T(\tau_{p,q})]$ and hence by Lemma 4.8, $\bar{y}_j = y_{j0} + \sum_{k=1}^{n} c_{jk} z_k + \sum_{t=1}^{2\ell-n} d_{jk} x_j (j = 1, 2, \ldots, 2\ell)$. This gives

$$\begin{pmatrix} [u, y_{n+1}](b) \\ \vdots \\ [u, y_{2\ell}](b) \end{pmatrix} = D^{p,q}_{(2\ell-n)\times(2\ell-n)} \begin{pmatrix} [u, \bar{x}_1](b) \\ \vdots \\ [u, \bar{x}_{2\ell-n}](b) \end{pmatrix}$$

where $D^{p,q}_{(2\ell-n)\times(2\ell-n)} = (d_{jk})_{\substack{n+1\le j\le 2\ell \\ n+1\le k\le 2\ell-n}}$, and

$$E^{1,2}_{(2\ell-n)\times(2\ell-n)} = H^{p,q}_{(2\ell-n)\times(2\ell-n)} \left(D^{p,q}_{(2\ell-n)\times(2\ell-n)}\right)^T.$$

The rest follows as in the proof of Corollary 5.3 on setting

$$F^{p,q}_{\ell\times n} = M^{p,q}_{\ell\times n} = K^{p,q}_{\ell\times n},\ G^{p,q}_{\ell\times(2\ell-n)} = N^{p,q}_{\ell\times(2\ell-n)} D^{p,q}_{(2\ell-n)\times(2\ell-n)} = L^{p,q}_{\ell\times(2\ell-n)}.$$

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