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THE QUANTILE *T*-WEIBULL DISTRIBUTION INDUCED BY THE UNIFORM DISTRIBUTION: SOME PROPERTIES WITH APPLICATION

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Abstract

In [1], the T-Weibull $\{Y\}$ family of distributions were introduced, and some of their properties with application were presented. Inspired by quantile generated probability distributions [2], we introduce a so-called q_T-Weibull family of distributions induced by the uniform distribution and present some properties of this new class of distributions. When T follows the standard Power distribution, it is shown that the Quantile standard Power-Weibull distribution induced by the uniform distribution is practically significant in modeling real life data.

Keywords and phrases: quantile generated probability distributions, Weibull distribution, uniform distribution.

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1. The T-$R\{Y\}$ Family of Distributions

This family of distributions was proposed by Alzaatreh et al. [3]. In particular, let T, R, Y be random variables with CDF's $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$, and $F_Y(x) = P(Y \le x)$, respectively. Let the corresponding quantile functions be denoted by $Q_T(p), Q_R(p)$, and $Q_Y(p)$, respectively. Also if the densities exist, let the corresponding PDF's be denoted by $f_T(x), f_R(x)$, and $f_Y(x)$, respectively. Following this notation, the CDF of the T-$R\{Y\}$ is given by

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)\,dt = F_T\{Q_Y(F_R(x))\}$$

and the PDF of the T-$R\{Y\}$ family is given by

$$f_X(x) = \frac{f_R(x)}{f_Y\{Q_Y(F_R(x))\}} f_T\{Q_Y(F_R(x))\}.$$

2. The q_T-X Family of Distributions Induced by V

Definition 2.1. Let V be any function such that the following holds:

(a) $F(x) \in [V(a), V(b)]$,

(b) $F(x)$ is differentiable and strictly increasing,

(c) $\lim_{x\to-\infty} F(x) = V(a)$ and $\lim_{x\to\infty} F(x) = V(b)$.

Then the CDF of the q_T-X family induced by V is given by

$$K(x) = \int_a^{V(F(x))} \frac{1}{r(Q(t))}\,dt,$$

where $\frac{1}{r(Q(t))}$ is the quantile density function of random variable $T \in [a, b]$, for $-\infty \le a < b \le \infty$, and $F(x)$ is the CDF of any random variable X.

Theorem 2.2. *The CDF of the* q_T-X *family induced by* V *is given by*

$$K(x) = Q[V(F(x))].$$

Proof. Follows from the previous definition and noting that $Q' = \frac{1}{r \circ Q}$.

Theorem 2.3. *The PDF of the* q_T-X *family induced by* V *is given by*

$$k(x) = \frac{f(x)}{r[Q(V(F(x)))]} V'[F(x)].$$

Proof. $k = K'$, $Q' = \frac{1}{r \circ Q}$, $F' = f$, and K is given by Theorem 2.2.

3. The q_T-Weibull Family induced by Uniform Distribution

We take $V(F(x)) = F_{WB}(x)$, where $F_{WB}(x)$ is the CDF of the two-parameter Weibull distribution, that is, $F_{WB}(x) = 1 - e^{-(\frac{x}{\lambda})^k}$, where $x, \lambda, k > 0$. Note that the PDF of the two-parameter Weibull distribution is obtained by differentiating the CDF, $F_{WB}(x)$. In this paper, we denote the PDF of the two-parameter Weibull distribution, by $f_{WB}(x)$. Now let Q_T be the quantile function of the random variable T, then from Theorem 2.2, we have the following:

Corollary 3.1. *The CDF of* q_T*-Weibull family of distributions induced by the Uniform distribution is given by*

$$F_M(x) = Q_T(F_{WB}(x)).$$

If r_T is the PDF of the random variable T, and since $(Q_T)' = \frac{1}{r_T \circ Q_T}$, then it follows from Theorem 2.3, that we have the following:

Corollary 3.2. *The PDF of* q_T*-Weibull family of distributions induced by the Uniform distribution is given by*

$$f_M(x) = \frac{f_{WB}(x)}{(r_T \circ Q_T)(F_{WB}(x))}.$$

4. Transformation and Quantile Function

Lemma 4.1. *If T is uniform, then the random variable $X = \lambda[-\log(1-T)]^{\frac{1}{k}}$ belongs to the Quantile Uniform-Weibull family of distributions induced by the uniform distribution.*

Proof. By Corollary 3.1, we must show the CDF of X is $F_{WB}(x)$. Observe T is defined on $[0, 1]$, and since T is uniform, thus Lemma 1(a) of [1], implies X has CDF $F_T\{F_R(x)\} = F_R(x)$, where $F_R(x)$ is CDF of the two-parameter Weibull distribution, that is, $F_R(x) := F_{WB}(x)$.

Lemma 4.2. *The quantile function for the q_T-Weibull family of distributions induced by the Uniform distribution is given, for $0 < y < 1$, by*

$$Q_M(y) = Q_{WB}[R_T(y)],$$

where Q_{WB} is the quantile function of the two-parameter Weibull distribution, and R_T is the CDF of the random variable T.

Proof. By definition, $Q_M(y) = F_M^{-1}(y)$, where $F_M(y) = Q_T(F_{WB}(y))$. Thus, it is enough to check that $F_M[Q_M(y)] = y$. Now observe we have the following:

$$\begin{aligned} F_M[Q_M(y)] &= F_M[Q_{WB}[R_T(y)]] \\ &= Q_T(F_{WB}(Q_{WB}[R_T(y)])) \\ &= Q_T[R_T(y)] \text{ since } F_{WB}(Q_{WB}(y)) = F_{WB}(F_{WB}^{-1}(y)) = y \\ &= y \text{ since } Q_T(R_T(y)) = R_T^{-1}(R_T(y)) = y. \end{aligned}$$

5. Modes

Observe from Corollary 3.2 that we have the following:

$$\log[f_M(x)] = \log\left[\frac{1}{(r_T \circ Q_T)(F_{WB}(x))}\right] + \log[f_{WB}(x)]$$

$$= -\log[(r_T \circ Q_T)(F_{WB}(x))] + \log[f_{WB}(x)].$$

Now differentiating the above expression with respect to *x*, one obtains

$$\frac{f_M'(x)}{f_M(x)} = -\frac{(r_T \circ Q_T)'(F_{WB}(x))F_{WB}(x)}{(r_T \circ Q_T)(F_{WB}(x))} + \frac{f_{WB}'(x)}{f_{WB}(x)}.$$

Now multiplying both sides of the above expression by $f_M(x)$, one obtains

$$f_M'(x) = -\frac{(r_T \circ Q_T)'(F_{WB}(x))}{(r_T \circ Q_T)(F_{WB}(x))}\frac{f_{WB}(x)^2}{(r_T \circ Q_T)(F_{WB}(x))}$$

$$+\frac{f_{WB}'(x)}{(r_T \circ Q_T)(F_{WB}(x))}.$$

If $k = 1$ in $f_{WB}(x) = 1 - e^{-(\frac{x}{\lambda})^k}$, then let the modified CDF be denoted by

$$F_{WB}^{Mod}(x) := 1 - e^{-\frac{x}{\lambda}}$$

and if the random variable X has CDF $F_{WB}^{Mod}(x)$, write $X \sim WB(\lambda)$. Now we have the following:

Lemma 5.1. *An implicit expression for the modes of the* q_T*-*$WB(\lambda)$ *family of distributions induced by the uniform distribution is given by*

$$x = \lambda \log\left[\frac{(r_T \circ Q_T)'(F_{WB}^{Mod}(x))}{(r_T \circ Q_T)(F_{WB}^{Mod}(x))}\right],$$

where $\lambda > 0$, r_T *is the PDF of the random variable* T, Q_T *is the quantile function* *of the random variable* T, *and* $F_{WB}^{Mod}(x) := 1 - e^{-\frac{x}{\lambda}}$.

Proof. It is obtained by solving the following equation for x

$$0 = -\frac{(r_T \circ Q_T)'(F_{WB}^{Mod}(x))}{(r_T \circ Q_T)(F_{WB}^{Mod}(x))}\frac{\frac{e^{\frac{-2x}{\lambda}}}{\lambda^2}}{(r_T \circ Q_T)(F_{WB}^{Mod}(x))}$$

$$-\frac{\frac{e^{\frac{-x}{\lambda}}}{\lambda^2}}{(r_T \circ Q_T)(F_{WB}^{Mod}(x))}.$$

6. Shannon's Entropy

Theorem 6.1. *If a random variable* M *follows the* q_T*-Weibull family of distributions induced by the uniform distribution, then the Shannon entropy of* M, *call it* S_M *is given by*

$$S_M = -E[\log f_{WB}(Q_{WB}(T))] + \eta_T,$$

where η_T *is the Shannon entropy of the random variable* T.

Proof. By definition

$$S_M = -E[\log f_{WB}(X)] + E\left[-\log\left(\frac{1}{(r_T \circ Q_T)(F_{WB}(X))}\right)\right].$$

However, $T = F_{WB}(X)$ has PDF $\frac{1}{(r_T \circ Q_T)(t)}$, thus, it follows that

$$E[\log f_{WB}(X)] = E[\log f_{WB}(Q_{WB}(T))].$$

On the other hand

$$E\left[-\log\left(\frac{1}{(r_T \circ Q_T)(F_{WB}(X))}\right)\right] = E\left[-\log\left(\frac{1}{(r_T \circ Q_T)(t)}\right)\right] = \eta_T.$$

Hence the conclusion.

7. Moments

Theorem 7.1. *If a random variable* M *follows the* q_T*-Weibull family of distributions induced by the uniform distribution, then the* r^{th} *non-central moments admit the following integral representation*:

$$E[M^r] = \int_0^1 [Q_{WB}(u)]^r \frac{1}{(r_T \circ Q_T)(u)} du.$$

Proof. Follows from using the substitution $u = F_{WB}(m)$.

Lemma 7.2. *Let* Q_{WB} *be the quantile function of the two-parameter Weibull distribution, and* R_T *be the CDF of the random variable* T. *If* X *is uniform on* $[0, 1]$, *then the random variable* $Y = Q_{WB}[R_T(X)]$ *follows the* q_T*-Weibull family of distributions induced by the uniform distribution*.

Proof. By Corollary 3.1, it is enough to show that the CDF of Y is $Q_T(F_{WB}(y))$. Since X is uniform on $[0, 1]$, $P(X \le x) = x$. Now observe we have the following:

$$\begin{aligned} P(Y \le y) &= P[Q_{WB}[R_T(X)] \le y] \\ &= P[R_T(X) \le F_{WB}(y)] \\ &= P[X \le Q_T(F_{WB}(y))] \\ &= Q_T(F_{WB}(y)). \end{aligned}$$

Theorem 7.3. *The* r^{th} *non-central moments of the* q_T*-Weibull family of distributions induced by the uniform distribution admit the following combinatorial sum representation*

$$E[Y^r] = \frac{r\lambda^r}{k} \sum_{i=0}^{\infty} \binom{i - \frac{r}{k}}{i} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{\frac{r}{k} - j} \binom{i}{j} p_{j,i} E[(R_T(X))^{\frac{r}{k}+i}];,$$

where R_T *is the CDF of the random variable* T, X *is uniform on* $[0, 1]$, *and* $p_{j,i}$ *are obtained from equation* (3.19) of [1].

Proof. Using the quantile function of the two-parameter Weibull distribution, it follows from Lemma 7.2 that

$$E[Y^r] = \lambda^r E[(-\log(1 - R_T(X))^{\frac{r}{k}}].$$

From equation (3.18) of [1], we get

$$(-\log(1 - R_T(X))^{\frac{r}{k}} = \frac{r}{k}\sum_{i=0}^{\infty}\binom{i - \frac{r}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{r}{k} - j}\binom{i}{j}p_{j,i}(R_T(X))^{\frac{r}{k}+i}.$$

By taking expectation in the expression immediately above and using it in the expression before it, the result follows.

8. Mean Deviations

Lemma 8.1. *Let* $I_c = \int_{-\infty}^{c} x f_M(x)dx$, *where* $f_M(x)$ *is the PDF of the* q_T*-Weibull family of distributions induced by the uniform distribution*, *then*

$$I_c = \frac{\lambda}{k}\sum_{i=0}^{\infty}\binom{i - \frac{1}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{1}{k} - j}\binom{i}{j}p_{j,i}S_u^*(c, 0, \frac{1}{k} + i),$$

where

$$S_{\varphi(u)}^*(c, a, \alpha) = \int_a^{F_{WB}(c)}(\varphi(u))^{\alpha}\frac{1}{(r_T \circ Q_T)(u)}du.$$

F_{WB} *is the CDF of the two-parameter Weibull distribution*, r_T *is the PDF of the random variable* T, Q_T *is the quantile function of the random variable* T, *and* $p_{j,i}$ *are obtained from equation* (3.19) of [1].

Proof. By definition

$$I_c = \int_{-\infty}^{c} \frac{x f_{WB}(x)}{(r_T \circ Q_T)(F_{WB}(x))} dx.$$

Using the substitution $u = F_{WB}(x)$ and the quantile function of the two-parameter Weibull distribution, the above integral can be written as

$$I_c = \int_{a}^{F_{WB}(c)} \lambda(-\log(1-u))^{\frac{1}{k}} \frac{1}{(r_T \circ Q_T)(u)} du.$$

From equation (3.18) of [1], we get

$$(-\log(1-u))^{\frac{1}{k}} = \frac{1}{k}\sum_{i=0}^{\infty}\binom{i-\frac{1}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{1}{k}-j}\binom{i}{j}p_{j,i}u^{\frac{1}{k}+i}.$$

Thus the above integral can be written as

$$I_c = \frac{\lambda}{k}\sum_{i=0}^{\infty}\binom{i-\frac{1}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{1}{k}-j}\binom{i}{j}p_{j,i}\int_{0}^{F_{WB}(c)} u^{\frac{1}{k}+i}\frac{1}{(r_T \circ Q_T)(u)} du$$

$$= \frac{\lambda}{k}\sum_{i=0}^{\infty}\binom{i-\frac{1}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{1}{k}-j}\binom{i}{j}p_{j,i}S_u^*(c, 0, \frac{1}{k}+i),$$

where the notation below has been used in the last equality in the expression immediately above

$$S_{\varphi(u)}^*(c, a, \alpha) = \int_{a}^{F_{WB}(c)} (\varphi(u))^{\alpha} \frac{1}{(r_T \circ Q_T)(u)} du.$$

Theorem 8.2. *For the* q_T*-Weibull family of distributions induced by the uniform distribution, the mean deviation from the mean,* $D(\mu)$*, and the mean deviation from the median,* $D(M)$*, are given, respectively by*

$$D(\mu) = 2\mu F_M(\mu) - 2\frac{\lambda}{k}\sum_{i=0}^{\infty}\binom{i-\frac{1}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{1}{k}-j}\binom{i}{j}p_{j,i}S_u^*(\mu, 0, \frac{1}{k}+i)$$

and

$$D(M) = \mu - 2\frac{\lambda}{k}\sum_{i=0}^{\infty}\binom{i-\frac{1}{k}}{i}\sum_{j=0}^{i}\frac{(-1)^{i+j}}{\frac{1}{k}-j}\binom{i}{j}p_{j,i}S_u^*(M, 0, \frac{1}{k}+i),$$

where F_M *is the CDF of the* q_T*-Weibull family of distributions induced by the uniform distribution*

$$S_u^*(\mu, 0, \frac{1}{k}+i) = S_u^*(c, 0, \frac{1}{k}+i)\big|_{c=\mu},\ S_u^*(M, 0, \frac{1}{k}+i) = S_u^*(c, 0, \frac{1}{k}+i)\big|_{c=M},$$

and

$$S_u^*(c, 0, \frac{1}{k}+i) = \int_0^{F_{WB}(c)} u^{\frac{1}{k}+i}\frac{1}{(r_T \circ Q_T)(u)}du.$$

F_{WB} *is the CDF of the two-parameter Weibull distribution*, r_T *is the PDF of the random variable* T, Q_T *is the quantile function of the random variable* T, *and* $p_{j,i}$ *are obtained from Equation* (3.19) of [1].

Proof. Let F_M be the CDF of the q_T-Weibull family of distributions induced by the uniform distribution. By definition

$$D(\mu) = 2\mu F_M(\mu) - 2I_\mu$$

and

$$D(M) = \mu - 2I_M.$$

So the theorem follows by combining the above definitions with the previous Lemma.

9. Example with Application

9.1. Example

Recall from [4] that the standard Power distribution has the CDF given by

$$F_{\alpha}(t) = t^{\alpha}$$

for $0 < t < 1$ and $\alpha > 0$ and the corresponding PDF is given by

$$f_{\alpha}(t) = \alpha t^{\alpha - 1}$$

for $0 < t < 1$ and $\alpha > 0$. Assuming T follows the standard Power distribution, then we have the following:

Theorem 9.1. *The CDF of the Quantile standard Power-Weibull distribution induced by the uniform distribution is given by*

$$F_M(x) = [1 - e^{-(\frac{x}{\lambda})^k}]^{\frac{1}{\alpha}},$$

where $x, \alpha, \lambda, k > 0$.

Proof. Since T is standard Power, then, $Q_T(t) = t^{\frac{1}{\alpha}}$, where $0 < t < 1$, and $\alpha > 0$. On the other hand, F_{WB} is the CDF of the two-parameter Weibull distribution. Thus the result follows from Corollary 3.1.

Remark 9.2. When X follows the Quantile standard Power-Weibull distribution induced by the uniform distribution, we write $X \sim QSPWU(\lambda, k, \alpha)$.

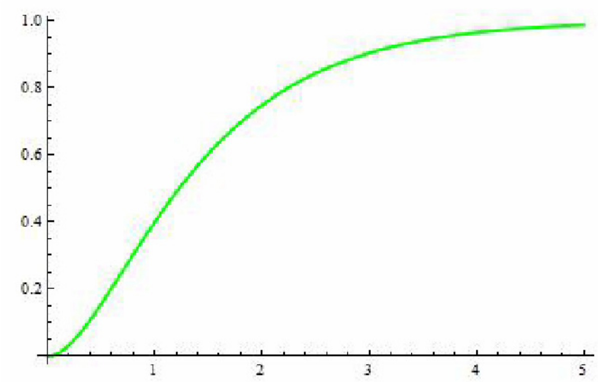

Figure 1. The CDF of *QSPWU* (1, 1, 0. 5).

Theorem 9.3. *The PDF of the Quantile standard Power-Weibull distribution induced by the uniform distribution is given by*

$$f_M(x) = \frac{k e^{-\left(\frac{x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{-1+\frac{1}{\alpha}}\left(\frac{x}{\lambda}\right)^{-1+k}}{\lambda\alpha},$$

where $x, \alpha, \lambda, k > 0.$

Proof. Follows from differentiating the CDF of the Quantile standard Power-Weibull distribution induced by the uniform distribution.

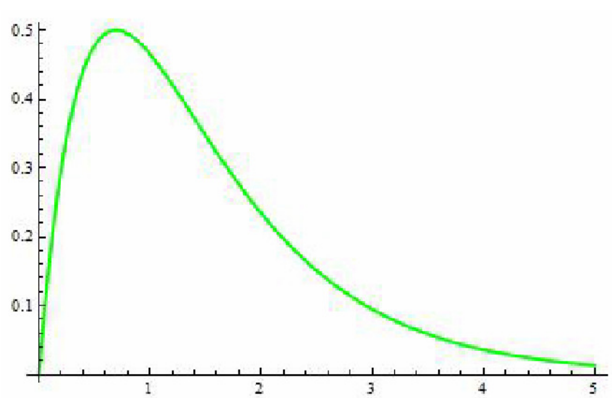

Figure 2. The PDF of *QSPWU* (1, 1, 0.5).

9.2. Application to breaking stress of carbon fibers data

The three-parameter Cauchy-Weibull{*Logistic*} distribution [1] has been shown to be practical in modeling real data, in particular, the breaking stress of carbon fibers data, Table 4 of [1]. In this section, we compare the performance of the Quantile standard Power-Weibull distribution with the three-parameter Cauchy-Weibull{*logistic*} distribution in modeling the breaking stress of carbon fibers data.

9.2.1. The three-parameter Cauchy-Weibull{*Logistic*} distribution

Remark 9.4. We say the random variable T has the Cauchy distribution with parameters $\alpha, \beta,$ and write $T \sim CA(\alpha, \beta)$. We say the random variable R has the Weibull distribution with parameters $k, \lambda,$ and write $R \sim WE(k, \lambda)$. We say the random variable Y has the Logistic distribution with parameters $\theta, \mu,$ and write $Y \sim LO(\theta, \mu)$.

Theorem 9.5. *The CDF of the three-parameter Cauchy-Weibull{Logistic} distribution is given by*

$$F_{CW\{L\}}(x;\, k,\, \lambda,\, \beta) = \frac{1}{\pi}\arctan\left\{\frac{\log[\frac{F_R(x;\, k,\, \lambda)}{1 - F_R(x;\, k,\, \lambda)}]}{\beta}\right\} + \frac{1}{2},$$

where $F_R(x;\, k,\, \lambda) = 1 - e^{-(\frac{x}{\lambda})^k}$, $x \geq 0$ *and* $k, \lambda, \beta > 0$.

Proof. In constructing the three-parameter Cauchy-Weibull{*Logistic*} distribution of Almheidat et al. [1], it is assumed that $Y \sim LO(1, 0)$, $T \sim CA(0, \beta)$, and $R \sim WE(k, \lambda)$. Thus, from equation (2.10) contained in Almheidat et al. [1], we have

$$F_{CW\{L\}}(x;\, k,\, \lambda,\, \beta) = F_T\left\{\log[\frac{F_R(x;\, k,\, \lambda)}{1 - F_R(x;\, k,\, \lambda)}];\, 0,\, \beta\right\},$$

where $F_R(x;\, k,\, \lambda)$ is as stated in the theorem, and

$$F_T\{x;\, 0,\, \beta\} = \frac{1}{\pi}\arctan\left(\frac{x}{\beta}\right) + \frac{1}{2}.$$

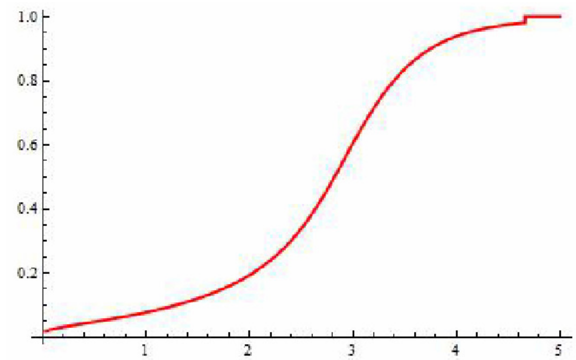

Figure 3. The CDF of $F_{CW\{L\}}(x;\, 7.9321,\, 2.953,\, 2.1437)$.

9.2.2. The Maximum Likelihood Estimates in $QSPWU(\lambda, k, \alpha)$

From Theorem 9.3, the log-likelihood function is given by

$$Log[L] = \sum_{i=1}^{n} Log\left[\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1 - e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}} k\left(\frac{x_i}{\lambda}\right)^{-1+k}}{\alpha\lambda}\right].$$

The maximum likelihood estimates are obtained by solving the following three equations below, simultaneously for λ, k, α

$$0 = \sum_{i=1}^{n} \frac{1}{k} e^{(\frac{x_i}{\lambda})^k}\left(1 - e^{-(\frac{x_i}{\lambda})^k}\right)^{1-\frac{1}{\alpha}} \alpha\lambda\left(\frac{x_i}{\lambda}\right)^{1-k}$$

$$\left[-\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1 - e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}}(-1+k)k\, x_i\left(\frac{x_i}{\lambda}\right)^{-2+k}}{\alpha\lambda^3} \right.$$

$$-\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}}k(\frac{x_i}{\lambda})^{-1+k}}{\alpha\lambda^2}$$

$$+\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}}k^2x_i(\frac{x_i}{\lambda})^{-2+2k}}{\alpha\lambda^3}$$

$$\left.-\frac{e^{-2(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-2+\frac{1}{\alpha}}k^2\left(-1+\frac{1}{\alpha}\right)x_i(\frac{x_i}{\lambda})^{-2+2k}}{\alpha\lambda^3}\right],$$

$$0=\sum_{i=1}^{n}\frac{1}{k}e^{(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{1-\frac{1}{\alpha}}\alpha\lambda\left(\frac{x_i}{\lambda}\right)^{1-k}$$

$$\left[\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}}(\frac{x_i}{\lambda})^{-1+k}}{\alpha\lambda}\right.$$

$$+\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}}k\,Log[\frac{x_i}{\lambda}](\frac{x_i}{\lambda})^{-1+k}}{\alpha\lambda}$$

$$-\frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}}kLog[\frac{x_i}{\lambda}](\frac{x_i}{\lambda})^{-1+2k}}{\alpha\lambda}$$

$$\left. + \frac{e^{-2(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-2+\frac{1}{\alpha}} k\left(-1+\frac{1}{\alpha}\right) Log[\frac{x_i}{\lambda}](\frac{x_i}{\lambda})^{-1+2k}}{\alpha\lambda}\right],$$

$$0 = \sum_{i=1}^{n} \frac{1}{k} e^{(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{1-\frac{1}{\alpha}} \alpha\lambda\left(\frac{x_i}{\lambda}\right)^{1-k}$$

$$\left[- \frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}} k\,(\frac{x_i}{\lambda})^{-1+k}}{\alpha\lambda}\right.$$

$$\left. - \frac{e^{-(\frac{x_i}{\lambda})^k}\left(1-e^{-(\frac{x_i}{\lambda})^k}\right)^{-1+\frac{1}{\alpha}} k\, Log\left[1-e^{-(\frac{x_i}{\lambda})^k}\right](\frac{x_i}{\lambda})^{-1+k}}{\alpha^3\lambda}\right].$$

9.2.3. Comparison with Empirical Distribution

By modifying the symbolic MLE procedure described in [5] and using Mathematica's *Find Root* procedure, we found upon taking x_i to be the breaking stress of carbon fibers data, Table 4 of [1], and using appropriate initial conditions in *Find Root* that the MLE in the Quantile standard Power-Weibull distribution induced by the uniform distribution, are given by

$$(\hat{\lambda}, \hat{k}, \hat{\alpha}) = (3.23093, 3.91085, 1.24928).$$

Thus we have the following:

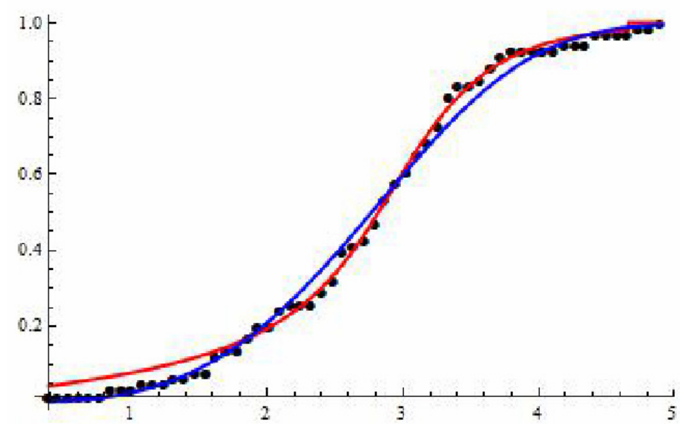

Figure 4. The CDF of $QSPWU(3.23093, 3.91085, 1.24928)$ (blue), and $F_{CW\{L\}}(x;\ 7.9321,\ 2.953,\ 2.1437)$ (red), fitted to the empirical distribution (little circles in black) of Table 4 of [1].

Note that the parameter values in $F_{CW\{L\}}(x;\ 7.9321,\ 2.953,\ 2.1437)$ are the MLE estimates of the three-parameter Cauchy-Weibull$\{Logistic\}$ distribution obtained from Table 4 of [1], and these values are reported in Table 5 of [1].

10. General Observation

From the figure above, it is clear the QSPWU distribution with the parameter values obtained from the MLE procedure is very good in modeling the breaking stress of carbon fibers data, Table 4 of [1], when compared with the parameter values in the three-parameter Cauchy-Weibull$\{Logistic\}$ distribution reported in Table 5 of [1]. The newly discovered Quantile standard Power-Weibull distribution induced by the uniform distribution might be good in modeling data with similar type shape, as that revealed by the histogram of the carbon fibers data, Table 4 of [1].

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