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THE INTERPOLATIVE BERINDE WEAK MAPPING THEOREM IN CONTROLLED METRIC TYPE SPACES

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Abstract

Motivated by [1], we prove the interpolative Berinde weak mapping theorem of [2] in the setting of controlled metric type spaces.

1. Introduction and Preliminaries

Definition 1.1 ([3])**.** Let X be a nonempty set and $\theta : X \times X \mapsto [1, \infty)$. An extended b-metric is a function $d : X \times X \mapsto [0, \infty)$ such that for all $x, y, z \in X$:

(a) $d(x, y) = 0$ iff $x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, y) \leq \theta(x, y)[d(x, z) + d(z, y)]$.

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Moreover, we say (X, d) is an extended b-metric space.

Definition 1.2 ([1]). Let X be a nonempty set, and $\alpha : X \times X \mapsto [1, \infty)$. The function $d : X \times X \mapsto [0, \infty)$ is called a controlled metric type, if for all $x, y, z \in X$:

(a) $d(x, y) = 0$ iff $x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, y) \leq \alpha(x, z)d(x, z) + \alpha(z, y)d(z, y)$.

Moreover, we say (X, d) is a controlled metric type space.

Example 1.3 ([1]). Take $X = \{0, 1, 2\}$. Consider the function d given as

$$d(0, 0) = d(1, 1) = d(2, 2) = 0,$$

$$d(0, 1) = d(1, 0) = 1,$$

$$d(0, 2) = d(2, 0) = \frac{1}{2},$$

$$d(1, 2) = d(2, 1) = \frac{2}{5}.$$

Take $\alpha : X \times X \mapsto [1, \infty)$ to be symmetric and defined as

$$\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = \alpha(0, 2) = 1,$$

$$\alpha(1, 2) = \frac{5}{4},\ \alpha(0, 1) = \frac{11}{10}.$$

It follows that d is a controlled metric type space, but is not an extended b-metric space, since

$$d(0, 1) = 1 > \frac{99}{100} = \alpha(0, 1)[d(0, 2) + d(2, 1)].$$

Definition 1.4 ([1]). Let (X, d) be a controlled metric type space, and $\{x_n\}_{n\geq 0}$ be a sequence in X.

(a) We say that the sequence $\{x_n\}$ converges to some $x \in X$, if, for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In this case we write $\lim_{n\to\infty} x_n = x$.

(b) We say that the sequence $\{x_n\}$ is Cauchy, if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.

(c) The controlled metric type space (X, d) is called complete, if every Cauchy sequence is convergent.

Definition 1.5 ([1]). Let (X, d) be a controlled metric type space. Let $x \in X$ and $\epsilon > 0$.

(a) The open ball $B(x; \epsilon)$ is

$$B(x; \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

(b) The mapping $T : X \mapsto X$ is said to be continuous at $x \in X$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$T(B(x, \delta)) \subseteq B(Tx, \epsilon).$$

Remark 1.6 ([1]). If T is continuous at x in the controlled metruc type space (X, d), then $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

2. Main Result

Theorem 2.1. *Let* (X, d) *be a complete controlled metric type space. Let* $T : X \mapsto X$ *be a mapping such that*

$$d(Tx, Ty) \leq \lambda d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}}$$

for all $x, y \in X$, $x, y \notin Fix(T) = \{x \in X : Tx = x\}$, $\lambda \in (0, 1)$. *For* $x_0 \in X$, *take* $x_n = T^n x_0$. *Suppose that*

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_{i+1}, x_m)} < \frac{1}{\lambda}.$$

In addition, assume that for every $x \in X$, *we have*

$$\lim_{n \to \infty} \alpha(x_n, x) \text{ and } \lim_{n \to \infty} \alpha(x, x_n) \text{ exist and are finite}.$$

Then the fixed point of T *exist.*

Proof. Consider the sequence $\{x_n = T^n x_0\}$, and observe we have the following

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(T^n x_0, T^{n+1} x_0) \\
&= d(T(T^{n-1} x_0), T(T^n x_0)) \\
&= d(Tx_{n-1}, Tx_n) \\
&\leq \lambda d(x_{n-1}, x_n)^{\frac{1}{2}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{2}} \\
&= \lambda d(x_{n-1}, x_n)^{\frac{1}{2}} d(x_{n-1}, x_n)^{\frac{1}{2}} \\
&= \lambda d(x_{n-1}, x_n).
\end{aligned}$$

Also

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq \lambda d(x_n, x_{n+1})^{\frac{1}{2}} d(x_n, Tx_n)^{\frac{1}{2}} \\
&= \lambda d(x_n, x_{n+1})^{\frac{1}{2}} d(x_n, x_{n+1})^{\frac{1}{2}} \\
&= \lambda d(x_n, x_{n+1}) \\
&\leq \lambda^2 d(x_{n-1}, x_n).
\end{aligned}$$

The two chain of inequalities immediately above implies $d(x_1, x_2) \leq \lambda d(x_0, x_1)$ and $d(x_2, x_3) \leq \lambda^2 d(x_0, x_1)$. By induction we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$$

for all $N \in \mathbb{N}$. Now let $n, m \in \mathbb{N}$ with $n < m$, using the fact that $\alpha(x, y) \geq 1$, observe we have the following

$$\begin{aligned}
d(x_n, x_m) &\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)d(x_{n+1}, x_m) \\
&\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)\alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad + \alpha(x_{n+1}, x_m)\alpha(x_{n+2}, x_m)d(x_{n+2}, x_m) \\
&\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_m)\alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&\quad + \alpha(x_{n+1}, x_m)\alpha(x_{n+2}, x_m)\alpha(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad + \alpha(x_{n-1}, x_m)\alpha(x_{n+2}, x_m)\alpha(x_{n+3}, x_m)d(x_{n+3}, x_m) \\
&\leq \cdots \\
&\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})d(x_i, x_{i+1}) \\
&\quad + \prod_{k=n+1}^{m-1}\alpha(x_k, x_m)d(x_{m-1}, x_m) \\
&\leq \alpha(x_n, x_{n+1})\lambda^n d(x_0, x_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})\lambda^i d(x_0, x_1) \\
&\quad + \prod_{k=n+1}^{m-1}\alpha(x_k, x_m)\lambda^{m-1} d(x_0, x_1) \\
&\leq \alpha(x_n, x_{n+1})\lambda^n d(x_0, x_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(x_j, x_m)\right)\alpha(x_i, x_{i+1})\lambda^i d(x_0, x_1)
\end{aligned}$$

$$+\prod_{k=n+1}^{m-1}\alpha(x_k,\, x_m)\lambda^{m-1}\alpha(x_{m-1},\, x_m)d(x_0,\, x_1)$$

$$=\alpha(x_n,\, x_{n+1})\lambda^n d(x_0,\, x_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha(x_j,\, x_m)\right)\alpha(x_i,\, x_{i+1})\lambda^i d(x_0,\, x_1)$$

$$\leq \alpha(x_n,\, x_{n+1})\lambda^n d(x_0,\, x_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\alpha(x_j,\, x_m)\right)\alpha(x_i,\, x_{i+1})\lambda^i d(x_0,\, x_1).$$

Put

$$S_p=\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\alpha(x_j,\, x_m)\right)\alpha(x_i,\, x_{i+1})k^i.$$

It follows we have

$$d(x_n,\, x_m)\leq d(x_0,\, x_1)[k^n\alpha(x_n,\, x_{n+1})+(S_{m-1}-S_n)].$$

Since

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\alpha(x_{i+1},\, x_{i+2})}{\alpha(x_{i+1},\, x_m)}<\frac{1}{\lambda},$$

then by the ratio test, $\lim_{n\to\infty}S_n$ exists, and hence the real sequence $\{S_n\}$ is Cauchy. Now taking limits in the inequality below

$$d(x_n,\, x_m)\leq d(x_0,\, x_1)[k^n\alpha(x_n,\, x_{n+1})+(S_{m-1}-S_n)],$$

we deduce that

$$\lim_{n,\, m\to\infty}d(x_n,\, x_m)=0.$$

So $\{x_n\}$ is Cauchy in the controlled metric type space $(X,\, d)$. Since $(X,\, d)$ is complete, $\{x_n\}$ converges to some $u\in X$. We show u is a fixed point of T. By using the controlled triangle inequality, we deduce the following

$$d(u, x_{n+1}) \leq \alpha(u, x_n)d(u, x_n) + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}).$$

Since $\sup_{m\geq 1} \lim_{i\to\infty} \dfrac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_{i+1}, x_m)} < \dfrac{1}{\lambda}$, $\lim_{n\to\infty} \alpha(x_n, x)$ and $\lim_{n\to\infty} \alpha(x, x_n)$ exist and are finite, and $\{x_n\}$ is Cauchy, we deduce the following

$$\lim_{n\to\infty} d(u, x_{n+1}) = 0.$$

Using the controlled triangle inequality again we deduce the following

$$\begin{aligned} d(u, Tu) &\leq \alpha(u, x_{n+1})d(u, x_{n+1}) + \alpha(x_{n+1}, Tu)d(x_{n+1}, Tu) \\ &\leq \alpha(u, x_{n+1})d(u, x_{n+1}) + \lambda\alpha(x_{n+1}, Tu)d(x_n, u)^{\frac{1}{2}}d(x_n, Tx_n)^{\frac{1}{2}} \\ &= \alpha(u, x_{n+1})d(u, x_{n+1}) + \lambda\alpha(x_{n+1}, Tu)d(x_n, u)^{\frac{1}{2}}d(x_n, x_{n+1})^{\frac{1}{2}}. \end{aligned}$$

Since $\lim_{n\to\infty} \alpha(x_n, x)$ and $\lim_{n\to\infty} \alpha(x, x_n)$ exist and are finite, and $\lim_{n\to\infty} d(u, x_{n+1}) = 0$, if we take limits in the above inequality as $n \to \infty$, we deduce $d(u, Tu) = 0$, that is $u = Tu$, and the fixed point exists.

3. Open Problem

We begin with the following

Definition 3.1 ([1])**.** Let $T : X \mapsto X$. For some $x_0 \in X$, let $O(x_0) = \{x_0, Tx_0, T^2x_0, \cdots\}$ be the orbit of x_0. A function $H : X \mapsto \mathbb{R}$ is said to be T-orbitally lower semi-continuous at $v \in X$, if for $\{x_n\} \subset O(x_0)$ such that $x_n \to v$, we have

$$H(v) \leq \liminf_{n\to\infty} H(x_n).$$

As a consequence of Theorem 2.1, we have the following

Conjecture 3.2. Let (X, d) be a complete controlled metric type space. Let $T : X \mapsto X$ be a mapping such that

$$d(Tz, T^2 z) \leq \lambda d(z, Tz)^{\frac{1}{2}} d(z, Tz)^{\frac{1}{2}} = \lambda d(z, Tz)$$

for all $z \in O(x_0)$, $z \notin Fix(T)$, $\lambda \in (0, 1)$. Take $x_n = T^n x_0$. Suppose that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_{i+1}, x_m)} < \frac{1}{\lambda}.$$

Then $x_n \to u \in X$ (as $n \to \infty$). Moreover, such u verifies $Tu = u$ iff the functional $x \to d(x, Tx)$ is T-orbitally lower semi-continuous at u.

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