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# THE EXPONENTIATED GENERALIZED FLEXIBLE WEIBULL EXTENSION DISTRIBUTION

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**Abstract**

In this paper, we introduce a new four-parameters model called the Exponentiated Generalized Flexible Weibull Extension (EG-FWE) distribution which exhibits bathtub-shaped hazard rate. Some of its statistical properties are obtained including ordinary and incomplete moments, quantile and generating functions, reliability and order statistics. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is derived. We illustrate the usefulness of the proposed model by applications to real data.

Keywords and phrases: exponentiated generalized class distribution, exponentiated Weibull, flexible Weibull extension distribution, generalized Weibull, reliability engineering, maximum likelihood estimation.

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## 1. Introduction

In recent years new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub hazard failure rate; see Xie and Lai [21]. Among of these modeling lifetime distributions were exponentiated Weibull family, Mudholkar and Srivastava [18], beta-Weibull distribution, Famoye et al. [6], generalized modified Weibull distribution, Carrasco et al. [3], a flexible Weibull extension, Bebbington et al. [2]. The Flexible Weibull Extention (FWE) distribution, Bebbington et al. [2] has a wide range of applications including life testing experiments, reliability analysis, applied statistics and clinical studies. If $X$ is a random variable, we can say to have the Flexible Weibull Extension (FWE) distribution with parameters $\alpha, \beta > 0$ if its probability density function (pdf) is given by

$$g(x) = \left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x} - e^{\alpha x - \frac{\beta}{x}}\right\},\ x > 0, \tag{1.1}$$

while the cumulative distribution function (cdf) is given by

$$G(x) = 1 - \exp\left\{- e^{\alpha x - \frac{\beta}{x}}\right\},\ x > 0. \tag{1.2}$$

The survival function is given by the equation

$$S(x) = 1 - G(x) = \exp\left\{- e^{\alpha x - \frac{\beta}{x}}\right\},\ x > 0 \tag{1.3}$$

and the hazard function is

$$h(x) = \left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}}. \tag{1.4}$$

Gupta and Kundu [7] proposed a generalization of the exponential distribution named as Generalized Exponential (GE) distribution. The two-parameter GE distribution with parameters $\vartheta, \gamma > 0,$ has the following distribution function

$$F(x; a, b) = (1 - e^{-ax})^b,\ x > 0,\ a > 0,\ b > 0. \tag{1.5}$$

This equation is simply the $b$th power of the standard exponential cumulative distribution. For a full discussion and some of its mathematical properties, see Gupta and Kundu [8]. In a similar manner, Nadarajah and Kotz [9] proposed the exponentiated gamma $(\text{E}\Gamma)$, exponentiated Frechet (EF) and exponentiated Gumbel (EGu) distributions, although the way they defined the cdf of the last two distributions is slightly different.

In this article, we propose a new class of distributions that extend the exponentiated type distributions and obtain some of its structural properties, a new class of univariate continuous distributions called the exponentiated generalized class of distribution [11]. If $G(x)$ is the baseline cumulative distribution function (cdf) of a random variable $X$, we define the exponentiated generalized (EG) class of distributions by

$$F(x) = [1 - \{1 - G(x)\}^{a}]^{b}, \; x > 0, \tag{1.6}$$

where $a > 0$ and $b > 0$ are two additional shape parameters.

The baseline distribution $G(x)$ is clearly a special case of Eq. (1.6) when $a = b = 1.$ Setting $a = 1$ gives the exponentiated type distributions defined by Gupta et al. [10]. Further, the EE and $\text{E}\Gamma$ distributions are obtained by taking $G(x)$ to be the exponential and gamma cumulative distributions, respectively. For $b = 1$ and if $G(x)$ is the Gumbel and Frechet cumulative distributions, we obtain the EGu and EF distributions, respectively, as defined by Nadarajah and Kotz [9]. Thus, the class of distributions Eq. (1.6) extends both exponentiated type distributions. The probability density function (pdf) of the new class has the form

$$f(x) = abg(x)\{1 - G(x)\}^{a-1}[1 - \{1 - G(x)\}^{a}]^{b-1}, \tag{1.7}$$

where $x > 0, a > 0, b > 0$ and $g(x)$ is the probability density function (pdf) of $G(x).$

The EG family of densities Eq. (1.7) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. In this article, we present a new distribution depending on flexible Weibull extension distribution called the *exponentiated generalized-flexible Weibull extension* (*EG-FWE*) distribution by using the class of univariate distributions defined above.

This paper is organized as follows, we define the cumulative, density and hazard functions of the exponentiated generalized flexible Weibull extension (EG-FWE) distribution in Section 2. In Sections 3 and 4, we introduced the statistical properties including, quantile function, median, the mode, skewness and kurtosis, *r*th moments and moment generating function. The distribution of the order statistics is expressed in Section 5. The maximum likelihood estimation of the parameters is determined in Section 6. Real data sets are analyzed in Section 7 and the results are compared with existing distributions. Finally, Section 8 concludes.

## 2. The Exponentiated Generalized Flexible Weibull Extension Distribution

In this section, we study the four parameters exponentiated generalized flexible Weibull extension (EG-FWE) distribution. $G(x)$ and $g(x)$ are used to obtain the cdf and pdf of Eqs. (1.6) and (1.7). The cumulative distribution function cdf of the Exponentiated Generalized Flexible Weibull Extension distribution (EG-FWE) is given by

$$F(x;\, a,\, b,\, \alpha,\, \beta) = \left[1 - \exp\left\{- ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^{b},\ x > 0,\ a,\ b,\ \alpha,\ \beta > 0. \qquad (2.1)$$

The pdf corresponding to Eq. (2.1) is given by

$$f(x;\, a,\, b,\, \alpha,\, \beta) = ab\left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x} - ae^{\alpha x - \frac{\beta}{x}}\right\}$$

$$\times\left[1 - \exp\left\{- ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^{b-1}, \qquad (2.2)$$

where $x > 0,$ and, $\alpha,\ \beta,\ 0$ are two additional shape parameters.

The survival function $S(x),$ hazard rate function $h(x),$ reversed hazard rate function $r(x)$ and cumulative hazard rate function $H(x)$ of $X \sim$ EG-FWE

$(a, b, \alpha, \beta)$ are given by

$$S(x; a, b, \alpha, \beta) = 1 - \left[1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^b, \; x > 0, \tag{2.3}$$

$$h(x; a, b, \alpha, \beta) = \frac{ab\left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x} - ae^{\alpha x - \frac{\beta}{x}}\right\}\left[1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^{b-1}}{1 - \left[1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^b}, \tag{2.4}$$

$$r(x; a, b, \alpha, \beta) = \frac{ab\left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x} - ae^{\alpha x - \frac{\beta}{x}}\right\}}{1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}}, \tag{2.5}$$

and

$$H(x; a, b, \alpha, \beta) = \int_0^x h(t)dt = -\ln\left(1 - \left[1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^b\right), \tag{2.6}$$

respectively, $x > 0$ and $a, b, \alpha, \beta > 0$.

Figures (1-6) display the cdf, pdf, survival, hazard rate, reversed hazard rate function and cumulative hazard rate function of the EG-FWE $(a, b, \alpha, \beta)$ distribution for some parameter values.

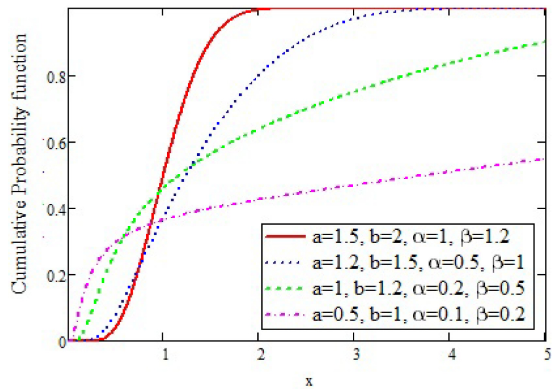


**Figure 1.** The cdf of the EG-FWE for different values of parameters.

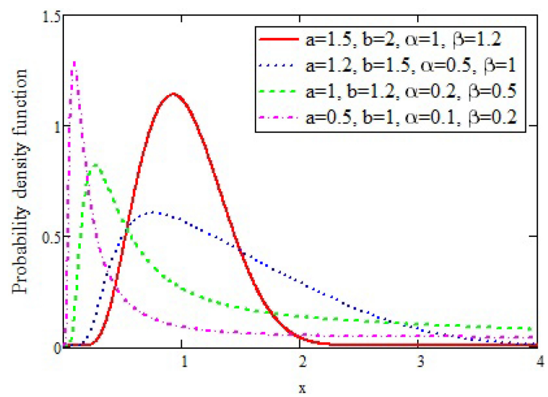


**Figure 2.** The pdf of the EG-FWE for different values of parameters.

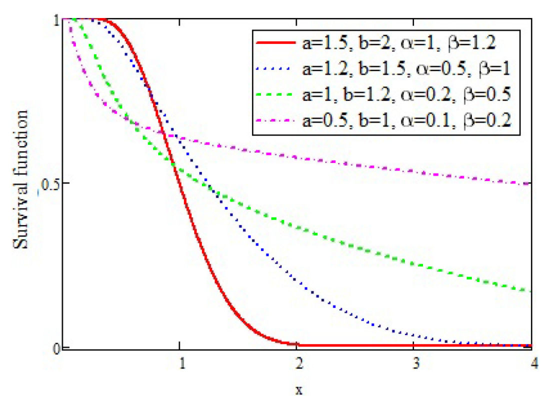


**Figure 3.** The survival function of the EG-FWE for different values of parameters.

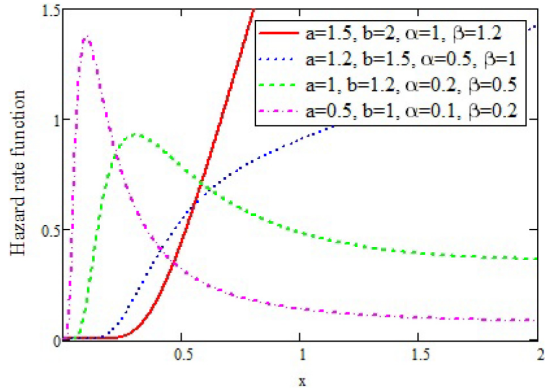


**Figure 4.** The hazard rate function of the EG-FWE for different values of parameters.

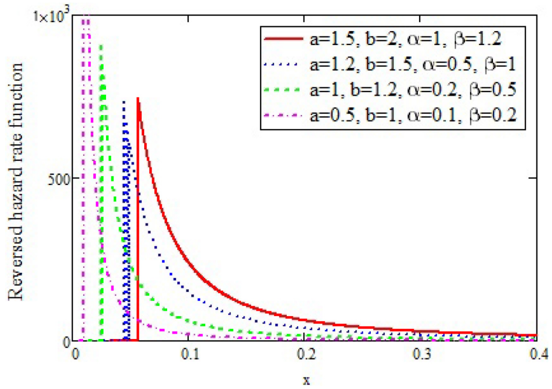


**Figure 5.** The reversed hazard rate function of the EG-FWE for different values of parameters.

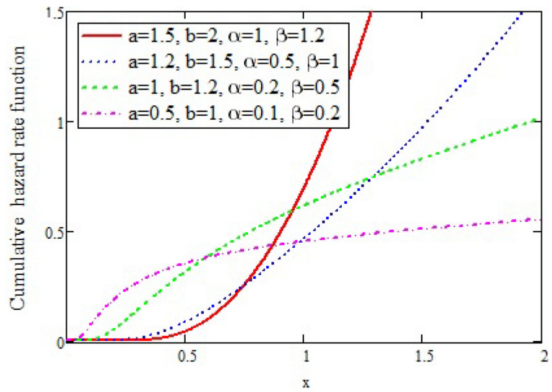


**Figure 6.** The cumulative hazard rate function of the EG-FWE for different values of parameters.

## 3. Statistical Properties

In this section, we study the statistical properties for the EG-FWE distribution, specially quantile function and simulation median, skewness, kurtosis and moments.

### 3.1. Quantile and median

In this subsection, we determine the explicit formulas of the quantile and the median of EG-FWE distribution. The quantile $x_q$ of the EG-FWE $(a, b, \alpha, \beta)$ is given by

$$F(x_q) = P[x_q \leq q] = q, \; 0 < q < 1. \tag{3.1}$$

From Eq. (2.1), we have

$$\left[1 - \exp\left\{- ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^b = q, \tag{3.2}$$

we obtain $x_q$ by solving the following equation:

$$\alpha x_q^2 - k(q) x_q - \beta = 0, \tag{3.3}$$

where

$$k(q) = \ln\left[\frac{-\ln\left(1 - q^{\frac{1}{b}}\right)}{a}\right].$$

So, the simulation of the EG-FWE random variable is straightforward. Let $U$ be a uniform random variable on unit interval $(0, 1)$. Thus, by means of the inverse transformation method, we consider the random variable $X$ given by

$$X = \frac{k(u) \pm \sqrt{k(u)^2 + 4\alpha\beta}}{2\alpha}. \tag{3.4}$$

Since the median is 50% quantile, so the median of EG-FWE distribution can be obtained by setting $q = 0.5$ in Eq. (3.3).

### 3.2. The mode

In this subsection, we will derive the mode of the EG-FWE distribution by derivation of its pdf with respect to $x$ and equate it to zero. The mode is the solution of the following equation with respect to $x$.

$$f'(x) = 0. \tag{3.5}$$

Since

$$f(x; a, b, \alpha, \beta) = h(x; a, b, \alpha, \beta)S(x; a, b, \alpha, \beta),$$

from Eq. (3.5), we have

$$[h'(x; a, b, \alpha, \beta) - h^2(x; a, b, \alpha, \beta)]S(x; a, b, \alpha, \beta) = 0, \tag{3.6}$$

where $h(x; a, b, \alpha, \beta)$ is hazard function of EG-FWE distribution Eq. (2.4), and $S(x; a, b, \alpha, \beta)$ is survival function of EG-FWE Eq. (2.3).

From Figure 2, the EG-FWE is unimodal distribution. It is not possible to get an analytic solution in $x$ to Eq. (3.6) in the general case. It has to be obtained numerically by using methods such as fixed-point or bisection method.

### 3.3. Skewness and Kurtosis

The analysis of the variability Skewness and Kurtosis on the shape parameters $\alpha, \beta$ can be investigated based on quantile measures. The shortcomings of the classical Kurtosis measure are well-known. The Bowely's skewness based on quartiles is given by, Kenney and Keeping [13],

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}}, \tag{3.7}$$

and the Moors' Kurtosis is based on octiles, Moors [17],

$$K_u = \frac{q_{(0.875)} - q_{(0.625)} - q_{(0.375)} + q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}}, \tag{3.8}$$

where $q_{(.)}$ represents quantile function.

### 3.4. The Moments

In this subsection, we discuss the *r*th moment for EG-FWE distribution. Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

**Theorem 3.1.** *If X has EG-FWE $(a, b, \alpha, \beta)$ distribution, then the rth moments of random variable X, is given by the following*

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j+k} a^{j+1} b (i+1)^j \beta^k}{j!\,k!\,\alpha^{r-k-1}(j+1)^{r-2k-1}} \binom{b-1}{i} \times \left[\frac{\Gamma(r-k+1)}{\alpha(j+1)^2} + \beta\Gamma(r-k-1)\right]. \tag{3.9}$$

**Proof.** We start with the well known distribution of the *r*th moment of the random variable *X* with probability density function $f(x)$ given by

$$\mu'_r = \int_0^{\infty} x^r f(x; a, b, \alpha, \beta) dx. \tag{3.10}$$

Substituting from Eq. (2.2) into Eq. (3.10), we get

$$\mu_r' = \int_0^\infty x^r ab\left(\alpha + \frac{\beta}{x^2}\right)\exp\left\{\alpha x - \frac{\beta}{x} - ae^{\alpha x - \frac{\beta}{x}}\right\}\left[1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^{b-1} dx, \quad (3.10)$$

since $0 < \left[1 - e^{-ae^{\alpha x - \frac{\beta}{x}}}\right] < 1$ for $x > 0$, the binomial series expansion of $\left[1 - e - ae^{\alpha x - \frac{\beta}{x}}\right]^{b-1}$ yields

$$\left[1 - \exp\left\{-ae^{\alpha x - \frac{\beta}{x}}\right\}\right]^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} e^{-aie^{\alpha x - \frac{\beta}{x}}},$$

we get

$$\mu_r' = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} ab \int_0^\infty x^r \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} e^{-a(i+1)e^{\alpha x - \frac{\beta}{x}}} dx,$$

using series expansion of $e^{-a(i+1)e^{\alpha x - \frac{\beta}{x}}}$, we get

$$e^{-a(i+1)e^{\alpha x - \frac{\beta}{x}}} = \sum_{j=0}^{\infty} \frac{(-1)^j a^j (i+1)^j}{j!} e^{j\left(\alpha x - \frac{\beta}{x}\right)},$$

hence

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \frac{a^{j+1} b(i+1)^j}{j!} \int_0^\infty x^r \left(\alpha + \frac{\beta}{x^2}\right) e^{(j+1)\left(\alpha x - \frac{\beta}{x}\right)} dx$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-1)^{i+j} \binom{b-1}{i} \frac{a^{j+1} b(i+1)^j}{j!} \int_0^\infty x^r \left(\alpha + \frac{\beta}{x^2}\right) e^{(j+1)\alpha x} e^{-(j+1)\frac{\beta}{x}} dx.$$

Using series expansion

$$e^{-(j+1)\frac{\beta}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k (j+1)^k \beta^k}{k!} x^{-k},$$

then

$$\mu'_r = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j+k} ba^{j+1} (i+1)^j (j+1)^k \beta^k}{j!\,k!}$$

$$\times \binom{b-1}{i} \int_0^{\infty} x^{r-k} \left( \alpha + \frac{\beta}{x^2} \right) e^{(j+1)\alpha x} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j+k} ba^{j+1} (i+1)^j (j+1)^k \beta^k}{j!\,k!} \binom{b-1}{i}$$

$$\times \left[ \int_0^{\infty} \alpha x^{r-k} e^{(j+1)\alpha x} + \int_0^{\infty} \beta x^{r-k-2} e^{(j+1)\alpha x} dx \right].$$

By using the definition of gamma function in the form, Zwillinger [23],

$$\Gamma(z) = \theta^z \int_0^{\infty} e^{\theta t} t^{z-1} dt, \; z, \, \theta > 0,$$

finally, we obtain the $r$th moment of EG-FWE distribution in the form

$$\mu'_r = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j+k} ba^{j+1} (i+1)^j (j+1)^k \beta^k}{j!\,k!} \binom{b-1}{i}$$

$$\times \left[ \frac{\Gamma(r-k+1)}{\alpha^{r-k} (j+1)^{r-k+1}} + \frac{\beta \Gamma(r-k-1)}{\alpha^{r-k-1} (j+1)^{r-k-1}} \right].$$

This completes the proof.

## 4. The Moment Generating Function

The moment generating function (mgf) $M_X(t)$ of a random variable $X$ provides the basis of an alternative route to analytic results compared with working directly with the pdf and cdf of $X$.

**Theorem 4.1.** *The moment generating function* (*mgf*) *of EG-FWE distribution is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}a^{j+1}b(i+1)^j\beta^k t^r}{j!\,k!\,r!\,\alpha^{r-k-1}(j+1)^{r-2k-1}}\binom{b-1}{i}$$

$$\times\left[\frac{\Gamma(r-k+1)}{\alpha(j+1)^2}+\beta\Gamma(r-k-1)\right]. \quad (4.1)$$

**Proof.** We start with the well known definition of the $M_X(t)$ of the random variable *X* with probability density function $f(x)$ given by

$$M_X(t) = \int_0^{\infty} e^{tx} f(x;\, a,\, b,\, \alpha,\, \beta)dx, \quad (4.2)$$

using series expansion of $e^{tx}$, we have

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x;\, a,\, b,\, \alpha,\, \beta)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'. \quad (4.3)$$

Substituting from Eq. (3.9) into Eq. (4.3), we get

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}a^{j+1}b(i+1)^j\beta^k t^r}{j!\,k!\,r!\,\alpha^{r-k-1}(j+1)^{r-2k-1}}\binom{b-1}{i}$$

$$\times\left[\frac{\Gamma(r-k+1)}{\alpha(j+1)^2}+\beta\Gamma(r-k-1)\right].$$

This completes the proof.

## 5. Order Statistics

In this section, we derive closed form expressions for the PDFs of the *r*th order statistic of the EG-FWE distribution. Let $X_{1:n},\, X_{2:n},\, \cdots,\, X_{n:n}$ denote the order statistics obtained from a random sample $x_1,\, x_2,\, \cdots,\, x_n$ which is taken from a continuous population with cumulative distribution function cdf $F(x; \varphi)$ and probability density function pdf $f(x; \varphi)$, then the probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x; \varphi) = \frac{1}{B(r, n-r+1)}[F(x; \varphi)]^{r-1}[1-F(x; \varphi)]^{n-r} f(x; \varphi), \qquad (5.1)$$

where $f(x; \varphi)$, $F(x; \varphi)$ are the pdf and cdf of EG-FWE $(\varphi)$ distribution given by Eq. (2.2) and Eq. (1.7), respectively, $\varphi = (a, b, \alpha, \beta)$ and $B(.,.)$ is the Beta function, also we define first order statistics $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$ and the last order statistics as $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$. Since $0 < F(x; \varphi) < 1$ for $x > 0$, we can use the binomial expansion of $[1-F(x; \varphi)]^{n-r}$ given as follows

$$[1-F(x; \varphi)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \varphi)]^i. \qquad (5.2)$$

Substituting from Eq. (5.2) into Eq. (5.1), we obtain

$$f_{n:n}(x; a, b, \alpha, \beta) = \sum_{i=0}^{n-r} \frac{(-1)^i n!}{i!(r-1)!(n-r-i)!} f(x; \varphi)[F(x; \varphi)]^{i+r-1}. \qquad (5.3)$$

Substituting from Eq. (2.1) and Eq. (2.2) into Eq. (5.3), we obtain the probability density function for $r$th order statistics.

Relation (5.3) shows that $f_{r:n}(x; \varphi)$ is the weighted average of the Exponentiated Generalized Flexible Weibull Extension distribution with different shape parameters.

## 6. Parameters Estimation

In this section, point and interval estimation of the unknown parameters of the EG-FWE distribution are derived by using the maximum likelihood method based on a complete sample.

**6.1. Maximum likelihood estimation**

Let $x_1, x_2, \cdots, x_n$ denote a random sample of complete data from the EG-FWE distribution. The Likelihood function is given as

$$L = \prod_{i=1}^{n} f(x_i; a, b, \alpha, \beta), \tag{6.1}$$

substituting from (2.2) into (6.1), we have

$$L = \prod_{i=1}^{n} ab\left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}} e^{-ae^{\alpha x_i - \frac{\beta}{x_i}}} \left[1 - \exp\left\{-ae^{\alpha x_i - \frac{\beta}{x_i}}\right\}\right]^{b-1}.$$

The log-likelihood function is

$$\mathcal{L} = n\ln(ab) + \sum_{i=1}^{n} \ln\left(\alpha + \frac{\beta}{x_i^2}\right) + \sum_{i=1}^{n}\left(\alpha x_i - \frac{\beta}{x_i}\right) - a\sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}}$$
$$+ (b-1)\sum_{i=1}^{n} \ln\left[1 - \exp\left\{-ae^{\alpha x_i - \frac{\beta}{x_i}}\right\}\right]. \tag{6.2}$$

The maximum likelihood estimation of the parameters are obtained by differentiating the log-likelihood function $\mathcal{L}$ with respect to the parameters $a, b, \alpha$ and $\beta$ and setting the result to zero, we have the following normal equations.

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} + (b-1)\sum_{i=1}^{n} \mathcal{A}_i = 0, \tag{6.3}$$

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln\left(1 - e^{-ae^{\alpha x_i - \frac{\beta}{x_i}}}\right) = 0, \tag{6.4}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{x_i^2}{\beta + \alpha x_i^2} + \sum_{i=1}^{n} x_i - a\sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} + a(b-1)\sum_{i=1}^{n} x_i \mathcal{A}_i = 0, \tag{6.5}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{\beta + \alpha x_i^2} - \sum_{i=1}^{n} \frac{1}{x_i} + a\sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}} - a(b-1)\sum_{i=1}^{n} \frac{\mathcal{A}_i}{x_i} = 0, \quad (6.6)$$

where $\mathcal{A}_i = e^{\alpha x_i - \frac{\beta}{x_i}} \left[ e^{ae^{\alpha x_i - \frac{\beta}{x_i}}} - 1 \right]^{-1}$.

The MLEs can be obtained by solving the previous nonlinear equations, (6.3)-(6.6), numerically for $a$, $b$, $\alpha$ and $\beta$.

**6.2. Asymptotic confidence bounds**

In this section, we derive the asymptotic confidence intervals of these parameters when $a, b, \alpha > 0$ and $\beta > 0$ as the MLEs of the unknown parameters $a, b, \alpha > 0$ and $\beta > 0$ can not be obtained in closed forms, by using variance covariance matrix $I^{-1}$, see Lawless [16], where $I^{-1}$ is the inverse of the observed information matrix which is defined as follows

$$I^{-1} = \begin{pmatrix} -\frac{\partial^2 \mathcal{L}}{\partial a^2} & -\frac{\partial^2 \mathcal{L}}{\partial a \partial b} & -\frac{\partial^2 \mathcal{L}}{\partial a \partial \alpha} & -\frac{\partial^2 \mathcal{L}}{\partial a \partial \beta} \\ -\frac{\partial^2 \mathcal{L}}{\partial b \partial a} & -\frac{\partial^2 \mathcal{L}}{\partial b^2} & -\frac{\partial^2 \mathcal{L}}{\partial b \partial \alpha} & -\frac{\partial^2 \mathcal{L}}{\partial b \partial \beta} \\ -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial a} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial b} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial a} & -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial b} & -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial \alpha} & -\frac{\partial^2 \mathcal{L}}{\partial \beta^2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \mathrm{var}(\hat{a}) & \mathrm{cov}(\hat{a}, \hat{b}) & \mathrm{cov}(\hat{a}, \hat{\alpha}) & \mathrm{cov}(\hat{a}, \hat{\beta}) \\ \mathrm{cov}(\hat{b}, \hat{a}) & \mathrm{var}(\hat{b}) & \mathrm{cov}(\hat{b}, \hat{\alpha}) & \mathrm{cov}(\hat{b}, \hat{\beta}) \\ \mathrm{cov}(\hat{\alpha}, \hat{a}) & \mathrm{cov}(\hat{\alpha}, \hat{b}) & \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{cov}(\hat{\beta}, \hat{a}) & \mathrm{cov}(\hat{\beta}, \hat{b}) & \mathrm{cov}(\hat{\beta}, \hat{\alpha}) & \mathrm{var}(\hat{\beta}) \end{pmatrix}. \quad (6.7)$$

The second partial derivatives included in $I$ are given as follows

$$\frac{\partial^2 \mathcal{L}}{\partial a^2} = -\frac{n}{a^2} - (b-1)\sum_{i=1}^{n} \mathcal{A}_i^2 e^{ae^{\alpha x_i - \frac{\beta}{x_i}}}, \tag{6.8}$$

$$\frac{\partial^2 \mathcal{L}}{\partial a \partial b} = \sum_{i=1}^{n} \mathcal{A}_i, \tag{6.9}$$

$$\frac{\partial^2 \mathcal{L}}{\partial a \partial \alpha} = -\sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} + (b-1)\sum_{i=1}^{n} x_i \mathcal{D}_i, \tag{6.10}$$

$$\frac{\partial^2 \mathcal{L}}{\partial a \partial \beta} = \sum_{i=1}^{n} \frac{e^{\alpha x_i - \frac{\beta}{x_i}}}{x_i} - (b-1)\sum_{i=1}^{n} \frac{\mathcal{D}_i}{x_i}, \tag{6.11}$$

$$\frac{\partial^2 \mathcal{L}}{\partial b^2} = -\frac{n}{b^2}, \tag{6.12}$$

$$\frac{\partial^2 \mathcal{L}}{\partial b \partial \alpha} = a\sum_{i=1}^{n} x_i \mathcal{A}_i, \tag{6.13}$$

$$\frac{\partial^2 \mathcal{L}}{\partial b \partial \beta} = -a\sum_{i=1}^{n} \frac{\mathcal{A}_i}{x_i}, \tag{6.14}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = -\sum_{i=1}^{n} \frac{x_i^4}{(\beta + \alpha x_i^2)^2} - a\sum_{i=1}^{n} x_i^2 e^{\alpha x_i - \frac{\beta}{x_i}} + a(b-1)\sum_{i=1}^{n} x_i^2 \mathcal{D}_i, \tag{6.15}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \frac{x_i^2}{(\beta + \alpha x_i^2)^2} + a\sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} - a(b-1)\sum_{i=1}^{n} \mathcal{D}_i, \tag{6.16}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \beta^2} = -\sum_{i=1}^{n} \frac{1}{(\beta + \alpha x_i^2)^2} - a\sum_{i=1}^{n} \frac{e^{\alpha x_i - \frac{\beta}{x_i}}}{x_i^2} + a(b-1)\sum_{i=1}^{n} \frac{\mathcal{D}_i}{x_i^2}, \tag{6.17}$$

where

$$\mathcal{D}_i = e^{\alpha x_i - \frac{\beta}{x_i}}\left[e^{ae^{\alpha x_i - \frac{\beta}{x_i}}}\left(1 - ae^{\alpha x_i - \frac{\beta}{x_i}}\right) - 1\right]\left[e^{ae^{\alpha x_i - \frac{\beta}{x_i}}} - 1\right]^{-2}.$$

We can derive the $(1 - \delta)100\%$ confidence intervals of the parameters $a$, $b$, $\alpha$ and $\beta$ by using variance matrix as in the following forms

$$\hat{a} \pm Z_{\frac{\delta}{2}}\sqrt{\text{var}(\hat{a})}, \quad \hat{b} \pm Z_{\frac{\delta}{2}}\sqrt{\text{var}(\hat{b})}, \quad \hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{\text{var}(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\delta}{2}}\sqrt{\text{var}(\hat{\beta})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right)$-th percentile of the standard normal distribution.

## 7. Application

In this section, we present the analysis of a real data set using the EG-FWE $(a, b, \alpha, \beta)$ model and compare it with many known distributions such as a flexible Weibull (FW), Weibull (W), modified Weibull (MW), reduced additive Weibull (RAW) and extended Weibull (EW) distributions, [2, 14, 21, 24], using Kolmogorov Smirnov (K-S) statistic, as well as Akaike information criterion (AIC), [1], Akaike Information Citerion with correction (AICC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Schwarz information criterion (SIC) [20] values. The data have been obtained from [19], it is for the time between failures (thousands of hours) of secondary reactor pumps.

**Table 1.** Time between failures of secondary reactor pumps [19]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2.160 | 0.746 | 0.402 | 0.954 | 0.491 | 6.560 | 4.992 | 0.347 |
| 0.150 | 0.358 | 0.101 | 1.359 | 3.465 | 1.060 | 0.614 | 1.921 |
| 4.082 | 0.199 | 0.605 | 0.273 | 0.070 | 0.062 | 5.320 | |

Table 2 gives MLEs of parameters of the WGFWE and K-S Statistics. The values of the log-likelihood functions, AIC, AICC, BIC, HQIC, and SIC are in Table 3.

**Table 2.** MLEs and K-S of parameters for secondary reactor pumps

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{a}$ | $\hat{b}$ | K-S |
|---|---|---|---|---|---|---|
| EG-FWE | 0.667 | 1.9410 | - | 0.033 | 0.114 | 0.0639 |
| FW | 0.0207 | 2.5875 | - | - | - | 0.1342 |
| W | 0.8077 | 13.9148 | - | - | - | 0.1173 |
| MW | 0.1213 | 0.7924 | 0.0009 | - | - | 0.1188 |
| RAW | 0.0070 | 1.7292 | 0.0452 | - | - | 0.1619 |
| EW | 0.4189 | 1.0212 | 10.2778 | - | - | 0.1057 |

**Table 3.** Log-likelihood, AIC, AICC, BIC, HQIC and SIC values of models fitted

| Model | L | AIC | AICC | BIC | HQIC | SIC |
|---|---|---|---|---|---|---|
| EG-FEW | –29.52 | 67.0400 | 69.2622 | 71.5820 | 10.5731 | 71.5820 |
| FW | –83.3424 | 170.6848 | 171.2848 | 172.9558 | 12.5416 | 172.9558 |
| W | –85.4734 | 174.9468 | 175.5468 | 177.2178 | 12.5915 | 177.2178 |
| MW | –85.4677 | 176.9354 | 178.1986 | 180.3419 | 12.6029 | 180.3419 |
| RAW | –86.0728 | 178.1456 | 179.4088 | 181.5521 | 12.6168 | 181.5521 |
| EW | –86.6343 | 179.2686 | 180.5318 | 182.6751 | 12.6296 | 182.6751 |

Substituting the MLE's of the unknown parameters $a, b, \alpha, \beta$ into (6.7), we get estimation of the variance covariance matrix as the following

$$I_0^{-1} = \begin{pmatrix} 3.548\times10^{-3} & 1.058\times10^{-3} & -0.016 & -1.658\times10^{-3} \\ 1.058\times10^{-3} & 1.274\times10^{-3} & -4.651\times10^{-3} & -9.769\times10^{-3} \\ -0.016 & -4.651\times10^{-3} & 0.082 & 7.613\times10^{-3} \\ -1.658\times10^{-3} & -9.769\times10^{-3} & 7.613\times10^{-3} & 0.221 \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters $a, b, \alpha$ and $\beta$ are $[0, 0.15], [0.044, 0.184], [0.106, 1.228]$ and $[1.02, 2.862]$, respectively.

To show that the likelihood equation has unique solution, we plot the profiles of the log-likelihood function of $a$, $b$, $\alpha$ and $\beta$ in Figures 7 and 8.

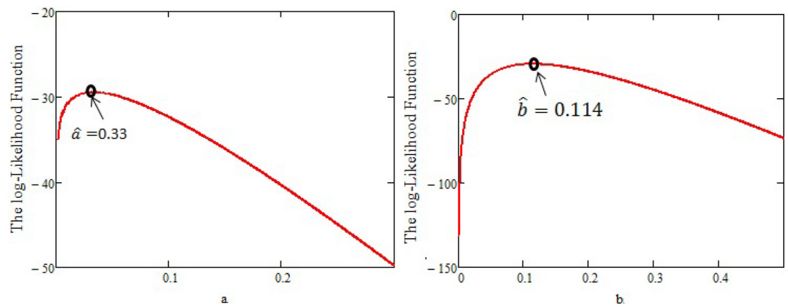


**Figure 7.** The profile of the log-likelihood function of $a$, $b$.

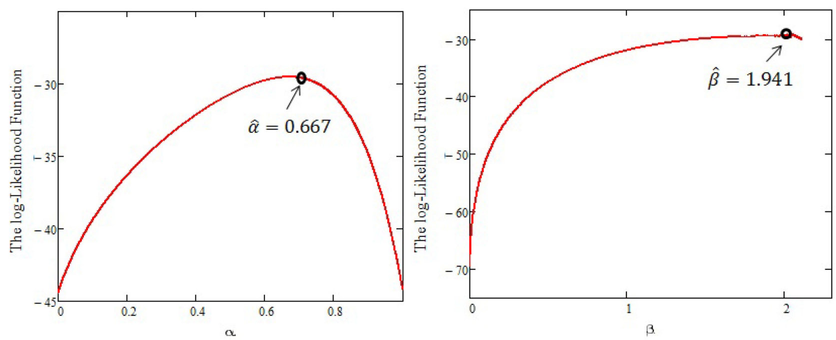


**Figure 8.** The profile of the log-likelihood function of $\alpha$, $\beta$.

The nonparametric estimate of the survival function using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be EG-FWE, FW, W, MW, RAW and EW are computed and plotted in Figure 9.

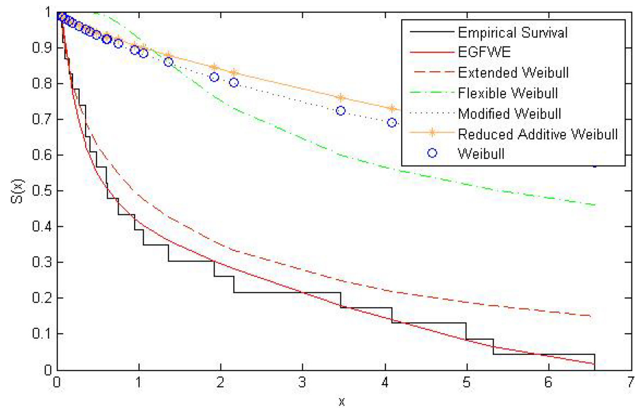


**Figure 9.** The Kaplan-Meier estimate of the survival function for the data.

Figures 10 and 11 give the form of the hazard rate and cdf for the EG-FWE, FW, W, MW, RAW and EW which are used to fit the data after replacing the unknown parameters included in each distribution by their MLE.

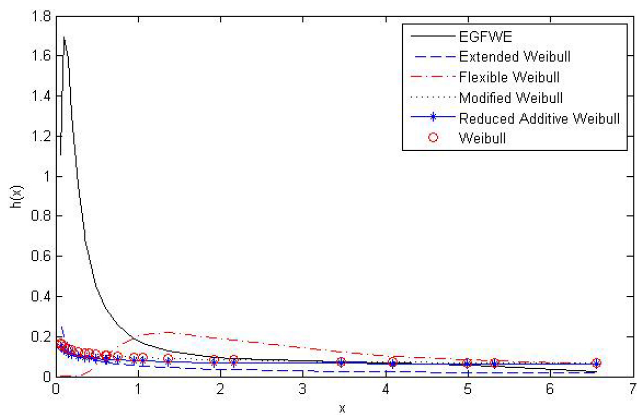


**Figure 10.** The fitted hazard rate function for the data.

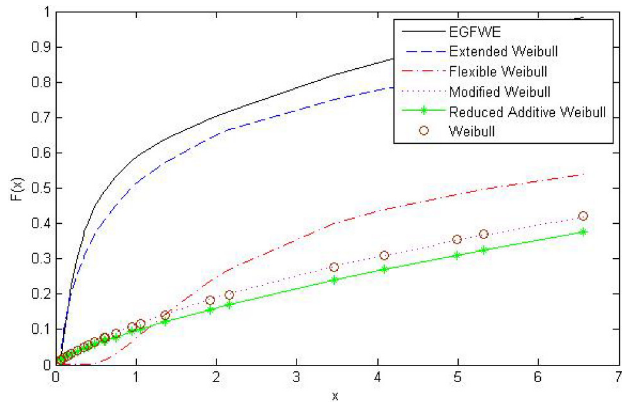


**Figure 11.** The fitted cumulative distribution function for the data.

We find that the EG-FWE distribution with the four-number of parameters provides a better fit than the previous new modified flexible Weibull extension distribution (FWE) which was the best in Bebbington et al. [2]. It has the largest likelihood, and the smallest AIC, AICC, BIC, HQIC and SIC values among those considered in this paper.

## 8. Conclusions

A new distribution, based on exponentiated generalized method distribution, has been proposed and its properties are studied. The idea is to add two parameters to flexible Weibull extension distribution, so that the hazard function is either increasing or more importantly, bathtub shaped. Using exponentiated generalized method, the distribution has flexibility to model the second peak in a distribution. We have shown that the exponentiated generalized flexible Weibull extension distribution fits certain well-known data sets better than existing modifications of the exponentiated generalized method of probability distribution.

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