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SOME NEW EXACT SOLUTIONS OF THE SU(2) YANG-MILLS FIELD EQUATIONS

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Abstract

Some new exact solutions of the SU(2) Yang-Mills gauge field equations are obtained by solving their ansatz reduced scalar field equations, using similarity group transformations. The solutions are classified and briefly discussed.

1. Introduction

The basis of symmetry principle in Physics is that, some properties remain invariant under certain transformations. Gauge theories [1] are characterized by their invariance under a group (gauge group) that defines the symmetry transformations and gauge theories are classified into Abelian and non-Abelian types. The simplest gauge group is U(1) and that corresponding gauge theories are called Abelian gauge theories. If higher symmetry groups such as SU(2), SU(3), etc. are involved then that

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becomes non-Abelian. Yang and Mills [1], [2] (YM) generalized the principle of gauge invariance to the case where invariance is associated with a non-Abelian internal symmetry group SU(2), that are of great interest in contemporary field theory and Particle Physics, especially in the context of unified model of fundamental interactions. Even then YM field equations have not been solved in a general settings. In this study, we are reporting particular solutions of YM field equations that are reduced by ansatz.

The basic dynamical variables of SU(2) YM theory are the vector potential A_μ^a carrying, space time index μ and internal symmetry index a, which ranges over 1, 2 and 3. The YM fields $F_{\mu\nu}$ are related to the potential A_μ^a by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\nu^a + e\epsilon^{abc} A_\mu^b A_\nu^c, \tag{1.01}$$

where e is the coupling constant. The equation of motion is

$$D_\mu F^{\mu\nu} = 0. \tag{1.02}$$

A YM theory with a local gauge symmetry breaking potential is characterized by the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{1}{4}\lambda(A^2 + \mu^2 / A^2). \tag{1.03}$$

The first non-Abelian YM solution was found by Wu and Yang [3] which is point like where the gauge potential behaves like $\frac{1}{r}$ everywhere, describe a point-like, non-Abelian magnetic monopole attached to a string.

The SU(2) gauge theory with Higgs triplet is defined by the Lagrangian

$$L = -\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a - \frac{1}{2}\prod^{\mu a} \prod_\mu^a + \frac{1}{2}\mu^2 \phi^a \phi^a - \frac{1}{2}\lambda(\phi^a \phi^a)^2, \tag{1.04}$$

where

$$\Pi_\mu^a = \partial_\mu \phi^a + e\,\epsilon^{abc} A_\mu^b \phi^c. \tag{1.05}$$

The equation of motion is

$$\partial_u \Pi^{\mu a} + e\,\epsilon^{abc} A_\mu^b \Pi^{\mu c} - \frac{\partial V\phi}{\partial \phi} = 0, \tag{1.06}$$

and

$$V(\phi) = -\frac{1}{2}\mu^2 \phi^a \phi^a + \frac{1}{4}\lambda(\phi^a \phi^a)^2. \tag{1.07}$$

Wu and Yang [4] developed an ansatz in which fields are assumed to be spherically symmetric. Then Wu-Yang-'tHooft-Julia-Zee ansatz [4] further modified the equation of motion to a special case, that becomes

$$r^2\left[\frac{\partial^2 K}{\partial r^2} - \frac{\partial^2 K}{\partial t^2}\right] = K(K^2 - 1) + KH^2, \tag{1.08}$$

$$r^2\left[\frac{\partial^2 H}{\partial r^2} - \frac{\partial^2 H}{\partial t^2}\right] = 2HK^2 + \frac{\lambda}{e^2}(H^3 - C^2 r^2 H^2). \tag{1.09}$$

Prasad and Sommerfield [5] further simplified by assuming $\lambda = 0$ and reported its static version and a finite energy static point monopole solution, for which, time dependent solutions reported by this author [6] by similarity group transformations.

By means of a specific ansatz [7], [8], [9], the YM potential A_μ^a can be reduced and the equation of motion of pure YM theory to scalar ϕ^4 equation of motion. In Minkowski space, it is of the form

$$eA_0^a = \pm\frac{i\partial_a \phi}{\phi}, \tag{1.10}$$

$$eA_i^a = \epsilon_{ian}\partial_n \phi / \phi \pm \delta_{ai}\partial_0 \phi / \phi. \tag{1.11}$$

While in Eucledian space the above ansatz is

$$eA_0^a = \mp \frac{\partial_a \phi}{\phi}, \tag{1.12}$$

$$eA_i^a = \epsilon_{ian} \partial_n \phi / \phi \pm \delta_{ai} \partial_o \phi / \phi. \tag{1.13}$$

In both cases the equation of motion of pure SU(2) YM theory becomes scalar massless ϕ^4 equation

$$\Box \phi = -\lambda \phi^3, \tag{1.14}$$

where λ is an integration constant.

The self duality-condition

$$E_n^a = \pm B_n^a, \tag{1.15}$$

implies

$$\Box \phi / \phi = 0, \tag{1.16}$$

then both E_n^a and B_n^a are proportional to δ_{na} and the Lagrangian is

$$L = \pm D - \left(\frac{3}{2} e^2\right) (\Box \phi / \phi)^3. \tag{1.17}$$

In this study, we report new solutions of Equations (1.12) and (1.14) by using similarity group transformations.

The self duality condition implies the energy-momentum tensor $\theta_{\mu\nu} = 0$. Then the formula for the topological charge of a non-singular solution is [10]

$$q = \pm \left(\frac{1}{16} \pi^2\right) \int_{x^2 \to \infty} d\Omega x^2 x_\mu (\Box \partial_\mu \phi / \phi). \tag{1.18}$$

If $\phi \to c/(x^2)^\alpha$ as, $x^2 \to \infty$, then topological charge is $q = \pm \alpha$. Then ϕ has only two values, $\alpha = 1, 1/2$. For $\alpha = 1$ is the instanton solution and

$\alpha = 1/2$ is called meron solution with λ arbitrary.

Witten [11] introduced the following ansatz, for the Euclidean gauge potential:

$$-eW_0^a = \frac{x_o}{r} A_0, \tag{1.19}$$

$$-eW_i^a = \epsilon_{ian} \frac{x_n}{r^2}(1 + \phi_2) + \frac{x_0 x_i}{r^2} + \left(\delta_{ai} - \frac{x_a x_i}{r^2}\right)\frac{1}{r}\phi_1, \tag{1.20}$$

where $A_0(x_0, r)$, $A_1(x_0, r)$, $\phi_1(x_0, r)$ and $\phi_2(x_0, r)$ are functions of $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and x_0. This ansatz is symmetric about the time axis in E^4. In three dimensional space this corresponds to symmetry under spatial rotation. Then equations of motions are

$$\partial_\mu (r^2 F_{\mu\nu}) = 2\epsilon_{ab}\phi_a D_\nu \phi_b, \tag{1.21}$$

$$r^2 D_\mu D_\mu \phi_a = \phi_a (1 - \phi_1^2 - \phi_2^2). \tag{1.22}$$

Meron solution corresponds to $A_\mu = 0, \phi_1 = 0$, then Witten's ansatz reduced equation of motion is

$$r^2(\partial_0\partial_0 + \partial_1\partial_1)\phi_2 = \phi_2(1 - \phi_2^2). \tag{1.23}$$

This is the equation of motion of ϕ^4 theory in two-space-time dimension with curved metric $g_{\mu\nu} = r^2\delta_{\mu\nu}$. In this study, we report new exact solutions of (1.23) by similarity group transformation.

Manton [12] reported that if $\partial_\mu A_\mu = 0$ is satisfied then any solutions obtained from Witten's ansatz is gauge equivalent to ϕ^4 solution independently from assumptions such as self duality.

2. Similarity Group Transformations of Differential Equations

We shall give essential details of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a partial differential equation (PDE) so obtaining respective ordinary differential equation (ODE) [13]. Let the given PDE in two independent variables x and t and one dependent variable u be

$$P(x, t, u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \quad (2.1)$$

where $u_t, u_x, \ldots$ are partial derivatives of dependent variables $u(x, t)$ with respect to the independent variable t and x, respectively.

When we apply a family of one parameter infinitesimal continuous point group transformations

$$x = x + \epsilon X(x, t, u) + O(\epsilon^2), \quad (2.2)$$

$$t = t + \epsilon T(x, t, u) + O(\epsilon^2), \quad (2.3)$$

$$u = u + \epsilon U(x, t, u) + O(\epsilon^2), \quad (2.4)$$

we get the infinitesimals of the variables u, t and x as U, T, X, respectively and ϵ is an infinitesimal parameter. The derivatives of u are also transformed as

$$u_x = u_x + \epsilon[U_x] + O(\epsilon^2), \quad (2.5)$$

$$u_{xx} = u_{xx} + \epsilon[U_{xx}] + O(\epsilon^2), \quad (2.6)$$

$$u_{tt} = u_{tt} + \epsilon[U_{tt}] + O(\epsilon^2), \quad (2.7)$$

where $[U_x], [U_{xx}], [U_{tt}]$ are the infinitesimals of the derivatives u_x, u_{xx}, u_{tt}, respectively. These are called first and second extensions and that are given by [13]

$$[U_x] = U_x + (U_u - X_x)u_x - X_u u_x^2 - T_x u_t - T_x u_x u_t, \tag{2.8}$$

$$\begin{aligned}[U_{xx}] = {} & U_{xx} + (2U_{xu} - X_{xx})u_x + (U_{uu} - 2X_{xu})u_x^2 - X_{uu}u_x^3, \\ & + (U_u - 2X_x)u_{xx} - 3X_u u_x u_{xx} - T_{xx}u_t - 2T_{xu}u_x u_t - T_{uu}u_x^2 u_t \\ & - 2T_x u_{xt} - T_u u_{xx} u_t - 2T_u u_{xt} u_x,\end{aligned} \tag{2.9}$$

$$\begin{aligned}[U_{tt}] = {} & U_{tt} + [2U_{tu} - T_{tt}]u_t - X_{tt}u_x + [U_{uu} - 2T_{uu}]u_t^2 \\ & - 2X_{tu}u_x u_t - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + [U_u - 2T_t]u_{tt} - 2X_t u_{xt}, \\ & - 3T_u u_{tt} u_t - X_u u_{tt} u_x - 3X_u u_{xt} u_t.\end{aligned} \tag{2.10}$$

The invariant requirements of given PDE (2.1) under the set of above transformations lead to the invariant surface conditions

$$T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial u} + [U_x]\frac{\partial F}{\partial u_x} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{xx}]\frac{\partial F}{\partial u_{xx}} = 0. \tag{2.11}$$

On solving above invariant surface condition (2.11), the infinitesimals X, T, U can be uniquely obtained, that give the similarity group under which the given PDE (2.1) is invariant. This gives

$$T\frac{du}{dt} + X\frac{du}{dx} - \frac{du}{dU} = 0. \tag{2.12}$$

The solution of (2.12) are obtained by Langrange's condition

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \tag{2.13}$$

This yields

$$x = x(t, c_1, c_2)$$

and

$$u = u(t, c_1, c_2), \tag{2.14}$$

where c_1 and c_2 are arbitrary integration constants and the constant c_1 plays the role of an independent variable called the similarity variable η and c_2 that of a dependent variable called the similarity solution $u(\eta)$, such that exact solution of given PDE becomes

$$u(x, t) = u(\eta). \tag{2.15}$$

On substituting (2.15) in given PDE (2.1), it is reduced to an ordinary differential equation with η as independent variable and $u(\eta)$ as dependent variable.

3. Exact Solutions of Witten's Ansatz Reduced Equation

By Witten's ansatz, the equation of motion of YM theory becomes (1.23), for simplicity, we replace ϕ_2 by u, gives

$$r^2(u_{tt} + u_{rr}) = u - u^3. \tag{3.1}$$

So the PDE (3.1) can be written as

$$F(t, r, u, u_{tt}, u_{rr}) = 0. \tag{3.2}$$

Then the invariant surface condition (2.12) becomes

$$R\frac{\partial F}{\partial r} + U\frac{\partial F}{\partial u} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{rr}]\frac{\partial F}{\partial u_{rr}} = 0. \tag{3.3}$$

On substituting the expansions for $[U_{tt}]$ and $[U_{rr}]$ as second extensions (2.9) and (2.10) in the following equation:

$$2rR(u_{tt} + u_{rr}) + r^2[U_{tt}] + r^2[U_{rr}] + (3u^2 - 1)U = 0, \tag{3.4}$$

we get

$$2rR(u_{tt} + u_{rr}) + r^2\{U_{tt}[2U_{tu} - T_{tt}]u_t - R_{tt}u_r + [U_{uu} - 2T_{tu}]u_t^2$$

$$- 2R_{tu}u_r u_t - T_{uu}u_t^3 - R_{uu}u_t^2 u_r + [U_u - 2T_t]u_{tt} - 2R_t u_{rt} - 3T_u u_{tt}u_t$$

$$- R_u u_{tt}u_r - 3R_u u_{rt}u_t\} + r^2\{U_{rr} + (2U_{ru} - R_{rr})u_r$$

$$+ (U_{uu} - 2R_{ru})u_r^2 - R_{uu}u_r^3 + (U_u - 2R_r)u_{rr} - 3R_u u_r u_{rr} - T_{rr}u_t$$

$$- 2T_{ru}u_r u_t - T_{uu}u_r^2 u_t - 2T_r u_{rt} - T_u u_{rr}u_t - 2T_u u_{rt}u_r\}$$

$$+ (3u^2 - 1)U = 0. \tag{3.5}$$

On solving for $R, T, U,$ by equating coefficients of derivatives of $u(r, t)$, we get many constrained equations, out of which the essentials are the following:

$$r^2(U_{tt} + U_{rr}) + (3u^2 - 1)U = 0,$$

$2U_{tu} - T_{tt} - T_{rr} = 0,$	$U_{uu}2R_{ru} = 0,$	$U_u = 2U_r = U_t = 0,$
$R_{tt} + 2U_{ru} - R_{rr} = 0,$	$2rR + r^2(U_u - 2T_t) = 0,$	$U = 0,$
$U_{uu} - 2T_{tu} = 0,$	$2rR + r^2(U_u - 2R_r) = 0,$	$T_t + rT_{rt} - R_r = 0,$
$2R_{tu} - 2T_{ru} = 0,$	$2R_t - 2rT_r = 0,$	$rR_r - R = 0,$
$T_{uu} = 0,$	$T_u = X_u = 0,$	$rT_t - R = 0.$

(3.6)

On solving the above constrained Equation (3.6), we get

$$R = 2\lambda rt + \kappa r,$$

$$T = \lambda(r^2 + t^2) + \kappa t + \sigma,$$

$$U = 0, \tag{3.7}$$

where λ, κ, σ are integration constants, for which we get three Lie group generators [13]

$$G_1 = 2rt\frac{\partial}{\partial r} + (r^2 + t^2)\frac{\partial}{\partial t},$$

$$G_2 = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t},$$

$$G_3 = \frac{\partial}{\partial t}, \tag{3.8}$$

and they satisfy the following Lie algebra:

$$[G_1, G_2] = -G_1, [G_2, G_3] = -G_3, [G_3, G_1] = 2G_2. \tag{3.9}$$

Substitute (3.7) in the Lagrange's condition (2.13)

$$\frac{dr}{R} = \frac{dt}{T} = \frac{du}{U}. \tag{3.10}$$

From first two, we get the similarity variable χ as integration constant

$$\chi = r \Big/ \left[(t^2 + r^2) + \frac{\kappa t}{\lambda} + \frac{\kappa^2}{4\lambda^2}\right], \lambda \neq 0 \tag{3.11}$$

and the similarity solution as

$$u = u(\chi), \text{ where } \sigma = \frac{\kappa^2}{4\lambda^2}. \tag{3.12}$$

Substitute (3.12) in (3.1) that reduces to an ordinary differential equation

$$\chi^2 \frac{d^2u}{d\chi^2} = u - u^3, \tag{3.13}$$

where $u = u(\chi)$. For $\chi = \exp(z)$, (3.13) becomes

$$\frac{d^2u}{dz^2} - \frac{du}{dz} = u - u^3 \text{ where } u = u(z). \tag{3.14}$$

All the solutions of (3.13) and (3.14) are exact solutions of the Witten's ansatz reduced PDE (3.1), through the similarity variable χ of Equation

(3.11).

Since $U = 0$, in the set of infinitesimals (3.7), the exact form of the right hand side of Equation (3.1) is not depending in the above similarity reduction to ODE.

We found

$$u(\chi) = \frac{\pm \alpha\chi^{1/3}}{1 \pm \alpha\chi^{1/3}} \tag{3.15}$$

as an exact solution of the following modified form of (3.13) and (3.14) as

$$\chi^2 \frac{d^2u}{d\chi^2} = \frac{-2}{9}(u - u^3), \tag{3.16}$$

and

$$\frac{d^2u}{dz^2} - \frac{du}{dz} = \frac{-2}{9}(u - u^3), \tag{3.17}$$

respectively.

The α is a nonzero arbitrary integration constant. Hence

$$u(r, t) = \frac{\pm \alpha\chi^{1/3}}{1 \pm \alpha\chi^{1/3}}, \tag{3.18}$$

where χ given by Equation (3.11) is an exact solution of the following modified Witten's ansatz reduced equation:

$$r^2(u_{tt} + u_{rr}) = \frac{-2}{9}(u - u^3). \tag{3.19}$$

If we put $\kappa = 0$ in the similarity variable (3.11), we get another exact solution of Equations (3.16), (3.17) and so of (3.19) where new similarity variable is χ_1,

$$\chi_1 = \frac{r}{(t^2 + r^2)}, \tag{3.20}$$

and

$$u(\chi_1) = \frac{\pm \alpha\chi_1^{1/3}}{1 \pm \alpha\chi_1^{1/3}}. \tag{3.21}$$

4. Similarity Transformatin of Klein-Gordon Equation

Here we apply the similarity method to find exact solutions of the Klein-Gordon equations

$$\Box\phi = g(\phi), \tag{4.1}$$

where $g(\phi)$ can be zero, $-\lambda\phi^3$, and $\lambda\phi^5$ or $\lambda\phi^{n+1}$.

For simplicity, we replace ϕ by u and apply the similarity method for two independent variables X and t, then extent to $(3+1)$ dimensions. So (4.1) becomes

$$u_{tt} + u_{xx} = g(u). \tag{4.2}$$

So general form of (4.1) is

$$F(u, u_{xx}, u_{tt}, x, t) = 0. \tag{4.3}$$

The invariant surface condition (2.11) gives

$$[U_{xx}]2\frac{\partial F}{\partial u_{xx}} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + U\frac{\partial g(u)}{\partial u} = 0. \tag{4.4}$$

On substituting the expansions of $[U_{xx}], [U_{tt}]$, we get the constrained equations as

$$\begin{array}{lll} U_{xu} + U_{tt} = 0, & X_x = X_u = 0, & T_t = T_u = U = 0, \\ X_{xx} - 2U_{xu} - X_{tt} = 0, & 2U_{tu} - T_{tt} + T_{xt} = 0, & X_x = X_u = 0. \\ T_x - X_t = 0, & T_{xu} - X_{tu} = 0, & \end{array}$$

$$\tag{4.5}$$

On solving above set of constraints, we get [14, 15]

$$X = ct + \omega,$$

$$T = cx + k,$$

$$U = 0. \tag{4.6}$$

The Lagrange's condition (2.13) is

$$\frac{dt}{(k + cx)} = \frac{dx}{(\omega + ct)} = \frac{du}{0}. \tag{4.7}$$

This gives the similarity variable η as an integration constant, when solve first two factors of (4.7), as we get

$$\eta = \left[\frac{c}{2}(x^2 - t^2) + (kx - \omega t) + \left(\frac{k^2 - \omega^2}{2c}\right)\right], \tag{4.8}$$

where c, k, and ω are arbitrary constants and $c \neq 0$.

Then the similarity solution of the PDE (4.2) is

$$u(x, t) = u(\eta). \tag{4.9}$$

On substituting (4.9) in the given PDE (4.2) through the similarity variable (4.8), we get

$$\eta \frac{d^2 u(\eta)}{d\eta^2} + \frac{du(\eta)}{d\eta} = -g(u). \tag{4.10}$$

Any solution of (4.10) is also exact solution of (4.2) through the similarity variable (4.8).

Case 1.

When $g(u) = 0$, then Klein-Gordon (KG) equation is

$$u_{tt} - u_{xx} = 0. \tag{4.11}$$

From (4.10) we get an exact solution of (4.11) as

$$u(x, t) = \ln\left[\frac{c}{2}(x^2 - t^2) + kx - \omega t + \left(\frac{k^2 - \omega^2}{2c}\right)\right]. \tag{4.12}$$

Case 2.

When $g(u) = -\lambda u^3$, then we have the massless $\lambda\phi^4$ equation

$$u_{tt} - u_{xx} = -\lambda u^3 \text{ where } \lambda = \frac{-5c}{2}. \tag{4.13}$$

For (4.13), we get an exact new solution

$$u(x, t) = \left[\frac{c}{2}(x^2 - t^2) + (kx - \omega t) + \left(\frac{k^2 - \omega^2}{2c}\right)\right]^{-1/2}. \tag{4.14}$$

Case 3.

When $g(u) = \frac{-5c}{2}u^5$, then the KG equation is

$$u_{tt} - u_{xx} = \frac{-5c}{2}u^5, \tag{4.15}$$

and its exact solution becomes

$$u(x, t) = \left[\frac{c}{2}(x^2 - t^2) + (kx - \omega t) + \left(\frac{k^2 - \omega^2}{2c}\right)\right]. \tag{4.16}$$

Case 4.

In general when $g(u) = u^{n+1}$, for any positive integer n the KG equation is

$$u_{tt}u_{xx} = u^{n+1}, \tag{4.17}$$

and its exact solution is

$$u(x, t) = \eta^{-1/n}. \tag{4.18}$$

Obviously, above similarity variable η fails to produce the exact solution when $g(u) = u$ as n becomes zero.

Case 5.

All of the above exact solutions of KG equations can be extended to $(3+1)$ dimensions with the following modified similarity variable as η_1,

$$\eta_1 = \frac{c}{2}(x^2 + y^2 + z^2 - t^2) + [(k_1 x - \omega_1 t) + (k_2 y - \omega_2 t) + (k_3 z - \omega_3 t)]$$

$$+ \left[\frac{(\omega_1 + \omega_2 + \omega_3)^2 - (k_1^2 + k_2^2 + k_3^2)}{2c}\right], \tag{4.19}$$

that satisfies the KG equation

$$\Box(\phi) = f(\phi), \tag{4.20}$$

where $\phi = \phi(\eta_1)$ and $f(\phi)$ takes the values, $0, -\lambda\phi^3, -\lambda\phi^5$.

From the infinitesimals (4.6), we have three Lie group generators [13], [14], [15], X_i, $i = 1, 2, 3$, corresponding to three constants c, ω and k,

$$X_1 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

$$X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{\partial}{\partial t}. \tag{4.21}$$

They obey the Lie algebra

$$[X_1, X_2] = -X_3,$$

$$[X_2, X_1] = X_2,$$

$$[X_2, X_3] = 0. \tag{4.22}$$

These three Lie group generators produce three different types of exact solutions to KG-family of Equations (4.20) [15]. The generator X_1 represents hyperbolic rotation invariant solutions with respect to the infinitesimals

$$X = ct,$$

$$T = cx,$$

$$U = 0. \tag{4.23}$$

And respective similarity variable η is

$$\eta_r = \frac{c}{2}(x^2 - t^2). \tag{4.24}$$

Then similarity reduced ODE corresponds to KG Equation (4.1) is

$$\eta_r \frac{d^2 f}{d\eta_r^2} + \frac{df}{d\eta_r} = f(u) \tag{4.25}$$

for which all the above solutions are valid with $k = 0$, $\omega = 0$. That is very rarely mentioned in other studies.

For the generator X_2, we get translation invariant solutions of KG equations corresponding to the infinitesimals

$$T = k,$$

$$X = \omega,$$

$$U = 0. \tag{4.26}$$

For which the similarity variable is η_T,

$$\eta_T = (kx - \omega t). \tag{4.27}$$

That class represents the all travelling wave type of solutions. When we take the full similarity variable (4.8) or (4.19), we get third class of solutions mentioned in above study. This method of splitting of similarity variables for various types of solutions, this author reported in the context of exact solutions of SU(2) Yang Mills-Higgs monopole solutions [6] and in the case of variable coefficient KdV equations [16].

5. Discussion

It is found that similarity Lie point group transformation method is a powerful tool for solving nonlinear PDE by converting to ODE. But method works only when given PDE is invariant under some similarity group of transformation, that need not happen always. One meron solution of the YM theory was first reported by de Alfaro et al. [17] and that was developed from the ϕ^4 ansatz, it is

$$\phi = 1 / \sqrt{\lambda x^2}. \tag{5.1}$$

They also constructed the two meron solution as

$$\phi = [(a - b)^2 / \lambda (x - a)^2 (x - b)^2]. \tag{5.2}$$

Clearly solution (5.1) is a special case of (5.2) when $a \to 0, b \to \infty, \lambda \to \infty$ and b^2 / λ is finite [10]. Our exact solutions of massless ϕ^4 equation

$$\phi(x, t) = \left[\frac{c}{2}(x^2 - t^2) + (kx - \omega t) + \frac{k^2 - \omega^2}{2c}\right]^{-1/2}, \tag{5.3}$$

is very much similar to above solution (5.1) when hyperbolic rotation invariance alone is considered when $k = \omega = 0$.

Our solution (5.3)

$$u(x, t) \to c / (x^2)^{1/2} \text{ as } x \to \infty. \tag{5.4}$$

Implies represents a meron with topological charge $q = \frac{1}{2}$, [10].

In addition to two types of exact solutions (3.11) and (3.16) there exists a possibility of another type of exact solution using the similarity variable

$$\tau = \left(\frac{r}{t + \alpha}\right), \tag{5.5}$$

which corresponds to the infinitesimals of (3.7)

$$R = kr,$$

$$T = kt + \frac{k^2}{4\lambda^2},$$

$$U = 0,\ \sigma = \frac{k^2}{4\lambda^2} \tag{5.6}$$

for which Witten's ansatz reduced Equation (3.1) becomes following form:

$$(\tau^4 + \tau^2)\frac{d^2u}{d\tau^2} + 2\sigma^3\frac{du}{d\tau} = u - u^3, \tag{5.7}$$

where $u = u(\tau)$ and τ is (5.5). This is found to be very difficult to solve.

Witten's ansatz reduced equation in terms of variable z is (3.14), its solution can be written as

$$u(z) = \frac{\pm\,\alpha \exp\left(\frac{1}{3}z\right)}{1 + \alpha \exp\left(\frac{1}{3}z\right)}, \tag{5.8}$$

where $z = \ln(\chi)$. This may be more easy to extend for multi meron solutions and it is under investigation.

Three types of solutions we suggested for KG family of equations, are very rarely mentioned in other studies. This may be due to the difficulties to solve their similarity reduced equations, compared to the translation invariant case and it is associated to travelling wave solutions of soliton types and so that type is getting more attentions.

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