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# SOME GLOBAL PROPERTIES ON LP-SASAKIAN MANIFOLDS

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## Abstract

We classify Lorentzian para-Sasakian manifolds admitting locally and globally $\varphi$-pseudo-quasi-conformal structure. Among others it is proved that a globally $\varphi$-pseudo-quasi-conformally symmetric LP-Sasakian manifold is globally $\varphi$-symmetric. Some results for a 3-dimensional

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locally $\varphi$-pseudo-quasi-conformally symmetric LP-Sasakian manifold are also given. The existence of a 3-dimensional locally $\varphi$-pseudo-quasi-conformally symmetric LP-Sasakian manifold is also ensured by an example.

## 1. Introduction

In 1989, Matsumoto [4] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [5] familiarized the same notion independently and obtained several results. LP-Sasakian manifolds are also studied by De et al. [7], Shaikh and Biswas [8] and so many authors.

In [3], Yano and Sawaki introduced the notion of quasi-conformal curvature tensor on an $n(n \geq 3)$-dimensional Riemannian manifold. Recently, the authors of [2] defined the notion of pseudo-quasi-conformal curvature tensor $\tilde{C}$ on a Riemannian manifold of dimension $n(n \geq 3)$ which includes the projective, quasi-conformal, Weyl conformal and concircular curvature as special cases. This tensor is defined by

$$\tilde{C}(X, Y, Z) = (p + d)R(X, Y, Z) + \left(q - \frac{d}{n-1}\right)[S(Y, Z)X - S(X, Z)Y]$$
$$+q\{g(Y, Z)QX - g(X, Z)QY\}$$
$$- \frac{r}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)X - g(X, Z)Y], \tag{1.1}$$

where $R, S, g, Q \in \chi(M)$, and $p, q, d$ are real constants such that $p^2 + q^2 + d^2 > 0.$ In particular, if

(i) $p = q = 0, d = 1;$

(ii) $p \neq 0, q \neq 0, d = 0;$

(iii) $p = 1, q = -\frac{1}{n-2}, d = 0;$

(iv) $p = 1, q = d = 0;$

then $\tilde{C}$ reduces to the projective curvature tensor, quasi-conformal curvature tensor, conformal curvature tensor and concircular curvature tensor, respectively.

In view of (1.1), we obtain

$$(\nabla_W \tilde{C})(X, Y, Z) = (p + d)(\nabla_W R)(X, Y, Z)$$
$$+ \left(q - \frac{d}{n-1}\right)[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y]$$
$$+q\{g(Y, Z)(\nabla_W Q(X) - g(X, Z)(\nabla_W Q)(Y)\}$$
$$- \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)X - g(X, Z)Y]. \quad (1.2)$$

In [1], authors introduced the notion of $\varphi$-quasi-conformal symmetric structure on a contact metric manifold. Recently, the author of [10] defined the notion of $\varphi$-pseudo-quasi-conformal structure on a paracontact metric manifold. With reference to above study, we introduced such notion on Lorentzian para-Sasakian manifold, as follows:

**Definition 1.1.** A Lorentzian para-Sasakian manifold is said to be locally $\varphi$-pseudo-quasi-conformally symmetric if the pseudo-quasi-conformal curvature tensor $\tilde{C}$ satisfies the condition

$$\varphi^2((\nabla_X \tilde{C})(Y, Z)W) = 0, \quad (1.3)$$

for all $X, Y, Z, W \in \chi(M)$ which are orthogonal to $\zeta$.

**Definition 1.2.** A Lorentzian para-Sasakian manifold is said to be globally $\varphi$-pseudo-quasi-conformally symmetric if the pseudo-quasi-conformal curvature tensor $\tilde{C}$ satisfies the condition

$$\varphi^2((\nabla_X \tilde{C})(Y, Z)W) = 0, \forall X, Y, Z, W \in \chi(M). \quad (1.4)$$

It is shown that if LP-Sasakian manifold is globally $\varphi$-pseudo-quasi-conformally symmetric, then the manifold is an Einstein provided $\{p + (n-2)q\} \neq 0.$ Also shown that an Einstein LP-Sasakian manifold admitting a globally $\varphi$-pseudo-quasi-conformally symmetric structure is globally $\varphi$-symmetric. We study 3-dimensional locally $\varphi$-pseudo-quasi-conformally symmetric LP-Sasakian manfolds and prove that it is locally $\varphi$-pseudo-quasi-conformally symmetric if and only if the scalar curvature $r$ is constant provided $(4q - 2p - 3d) \neq 0,$ that ensured by an interesting example.

## 2. Preliminaries

A differential manifold of dimension $n$ is called Lorentzian para-Sasakian (briefly, LP-Sasakian) [4], if it admits a (1, 1)-tensor field $\varphi,$ a contravariant vector field $\zeta,$ a 1-form $\eta$ and a Lorentzian metric $g$ which satisfy

$$\eta(\zeta) = -1, \tag{2.1}$$

$$\varphi^2(X) = X + \eta(X)\zeta, \tag{2.2}$$

$$g(\varphi, X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \zeta) = \eta(X), \tag{2.4}$$

$$\nabla_X \zeta = \varphi X, \tag{2.5}$$

$$(\nabla_X \varphi)(Y) = g(X, Y)\zeta + \eta(Y)X + 2\eta(X)\eta(Y)\zeta, \tag{2.6}$$

where $\nabla$ denotes the covariant differentiation with respect to the Lorentzian metric $g$.

It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\varphi\zeta = 0, \quad \eta(\varphi X) = 0, \tag{2.7}$$

$$\text{rank}\, \varphi = n - 1. \tag{2.8}$$

If we put

$$\Phi(X, Y) = g(X, \varphi Y), \tag{2.9}$$

for any vector fields $X$ and $Y$, then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [4]. Also the 1-form $\eta$ is closed in an LP-Sasakian manifold, we have [4]

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \zeta) = 0 \tag{2.10}$$

for all $X, Y \in TM$.

Also in an LP-Sasakian manifold, the following relations hold [4]:

$$g(R(X, Y)Z, \zeta) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.11}$$

$$R(\zeta, X)Y = g(X, Y)\zeta - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\zeta = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\zeta, X)\zeta = X + \eta(X)\zeta, \tag{2.14}$$

$$S(X, \zeta) = (n-1)\eta(X), \tag{2.15}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.16}$$

for any vector fields $X, Y, Z,$ where $R$ and $S$ are the Riemannian curvature and the Ricci tensor of $M$, respectively.

An LP-Sasakian manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ of the type $(0, 2)$ is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y),$$

for any vector fields $X, Y,$ where $\alpha, \beta$ are smooth function on $M$.

**Example 1.** Let $\mathfrak{R}^5$ be the 5-dimensional real number space with a coordinate system $(x, y, z, t, s)$. Define

$$\eta = ds - ydx - tdz, \quad \zeta = \frac{\partial}{\partial s}, \quad g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2,$$

$$\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \quad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y},$$

$$\varphi\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \quad \varphi\left(\frac{\partial}{\partial s}\right) = 0.$$

The structure $(\varphi, \eta, \zeta, g)$ becomes an LP-Sasakian structure in $\mathfrak{R}^5$ [9].

## 3. Globally $\varphi$-pseudo-Quasi-Conformally Symmetric LP-Sasakian Manifolds

Let $M$ be a globally $\varphi$-pseudo-quasi-conformally symmetric LP-Sasakian manifold. Then equation (1.4) holds on $M$ and from (2.2), we have

$$(\nabla_W \tilde{C})(X, Y)Z + \eta((\nabla_W \tilde{C})(X, Y)Z)\zeta = 0. \tag{3.1}$$

In view of (1.2) and (3.1), we get

$$\begin{aligned}
0 = (p+d)(\nabla_W R)(X, Y)Z &+ \left(q - \frac{d}{n-1}\right)[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y] \\
&+ q[g(Y, Z)(\nabla_W Q)X - g(X, Y)(\nabla_W Q)Y] \\
&- \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)X - g(X, Z)Y] \\
&+ (p+d)\eta((\nabla_W R)(X, Y)Z)\zeta \\
&+ \left(q - \frac{d}{n-1}\right)[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y]\zeta \\
&+ q[g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y]\zeta \\
&- \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)X - g(X, Z)Y]\zeta.
\end{aligned}$$

Taking inner product with $V$, we have

$$\begin{aligned}
0 = (p+d)(\nabla_W R)(X, Y, Z, V) &+ \left(q - \frac{d}{n-1}\right)[(\nabla_W S)(Y, Z)g(X, V) \\
&- (\nabla_W S)(X, Z)g(Y, V)] \\
&+ q[g(Y, Z)g((\nabla_W Q)X, V - g(X, Z)g((\nabla_W Q)Y, V)] \\
&- \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)g(X, V) - g(X, Z)g(Y, V)] \\
&+ (p+d)\eta((\nabla_W R)(X, Y)Z)\eta(V) \\
&+ \left(q - \frac{d}{n-1}\right)[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y]\eta(V) \\
&+ q[g(Y, Z)(\nabla_W Q)X - g(X, Z)(\nabla_W Q)Y]\eta(V) \\
&- \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}[g(Y, Z)X - g(X, Z)Y]\eta(V).
\end{aligned}$$

Putting $X = V = e_i$, where $\{e_i\}$, $i = 1, 2, 3, \ldots,$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, the above equation reduces to

$$0 = \left( p + q(n-2) + \frac{d}{n-1} \right)(\nabla_W S)(Y, Z)$$

$$+ \begin{bmatrix} q\eta((\nabla_W Q)e_i)\eta(e_i) + qg(\nabla_W Q)e_i, e_i) \\ -\frac{dr(W)}{n}\{p + 2(n-1)q\} + \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\} \end{bmatrix} g(Y, Z)$$

$$+ (p+d)\eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) - \left( q - \frac{d}{n-1} \right)[(\nabla_W S)(\zeta, Z)\eta(Y)]$$

$$- q\eta((\nabla_W Q)(Y)\eta(Z) + \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}\eta(Y)\eta(Z) - qg((\nabla_W Q)Y, Z).$$

Substituting $Z = \zeta$, in above equation and using (2.1) and (2.4), we get

$$0 = \left\{ p + (n-2)q + \frac{d}{n-1} \right\}(\nabla_W S)(Y, \zeta)$$

$$+ \begin{bmatrix} q\eta((\nabla_W Q)e_i)\eta(e_i) + qg(\nabla_W Q)e_i, e_i) \\ -\frac{dr(W)}{n}\{p + 2(n-1)q\} + \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\} \end{bmatrix} \eta(Y)$$

$$+ (p+d)\eta((\nabla_W R)(e_i, Y)\zeta)\eta(e_i) - \left( q - \frac{d}{n-1} \right)[(\nabla_W S)(\zeta, \zeta)\eta(Y)]$$

$$- \frac{dr(W)}{n(n-1)}\{p + 2(n-1)q\}\eta(Y).$$

Also, we have

$$g((\nabla_W Q)e_i, e_i) = (\nabla_W S)(e_i, e_i) = dr(W).$$

Hence, using the above relation, we have

$$0 = \left\{ p + (n-2)q + \frac{d}{n-1} \right\}(\nabla_W S)(Y, \zeta) + \begin{bmatrix} q\eta((\nabla_W Q)e_i)\eta(e_i) + qdr(W) \\ -\frac{dr(W)}{n}\{p + 2(n-1)q\} \end{bmatrix} \eta(Y)$$

$$+ (p+d)\eta((\nabla_W R)(e_i, Y)\zeta)\eta(e_i) - \left( q - \frac{d}{n-1} \right)[(\nabla_W S)(\zeta, \zeta)\eta(Y)]. \tag{3.2}$$

Using (2.1), (2.5) and (2.15) in equation (3.2), we get

$$\eta((\nabla_W Q)(e_i)) = g((\nabla_W Q)\zeta, \zeta) = 0. \tag{3.3}$$

Now, equation

$$g((\nabla_W R)(e_i, Y)\zeta, \zeta) = g(\nabla_W R(e_i, Y)\zeta, \zeta) - g(R(\nabla_W e_i), Y\zeta, \zeta)$$
$$-g(R(e_i, \nabla_W Y)\zeta, \zeta) - g(R(e_i, Y)\nabla_W \zeta, \zeta),$$

leads to

$$g((\nabla_W R)(e_i, Y)\zeta, \zeta) = g(\nabla_W R(e_i, Y)\zeta, \zeta) - g(R(\nabla_W e_i, Y)\zeta, \zeta)$$
$$-g(R(e_i, Y)\nabla_W \zeta, \zeta). \tag{3.4}$$

Since from (2.13), we have

$$g(R(\nabla_W e_i, Y)\zeta, \zeta) = g(\eta(Y)\nabla_W e_i - \eta(\nabla_W e_i)Y, \zeta) = 0$$

and

$$g(R(e_i, \nabla_W Y)\zeta, \zeta) = g(\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W R, \zeta) = 0.$$

Therefore, equation (3.4) reduces to

$$g((\nabla_W R)(e_i, Y)\zeta, \zeta) = g(\nabla_W R(e_i, Y)\zeta, \zeta) - g(R(e_i, Y)\nabla_W \zeta, \zeta). \tag{3.5}$$

In view of definition of the Levi-Civita connection of $g$, we have

$$(\nabla_W g)(R(e_i, Y)\zeta, \zeta) = 0,$$

and then, using (2.13), we get

$$g(\nabla_W R(e_i, Y)\zeta, \zeta) - g(R(e_i, Y)\nabla_W \zeta, \zeta) = 0.$$

From (3.5), it follows that

$$g((\nabla_W R)(e_i, Y)\zeta, \zeta) = \eta((\nabla_W R)(e_i, Y)\zeta) = 0. \tag{3.6}$$

In view of (3.3) and (3.6), we obtain from (3.2)

$$\left\{p + (n-2)q + \frac{d}{n-1}\right\}(\nabla_W S)(Y, \zeta) = \left[\frac{1}{n}\{p + 2(n-1)q\} - q\right]dr(W)\eta(Y). \tag{3.7}$$

From (3.7) it is clear that for $Y = \zeta$, we obtain $dr(W) = 0$, provided $\{p + (n-2)q\} \neq 0$, which implies $r$ is constant. Thus, we have the following:

**Theorem 3.1.** *If a Lorentzian para-Sasakian manifold is globally* $\varphi$*-pseudo-quasi-conformal structure then the scalar curvature of the manifold is constant provided* $\{p + (n-2)q\} \neq 0.$

Also from (3.7) it follows

$$(\nabla_W S)(Y, \zeta) = 0, \left\{p + (n-2)q + \frac{d}{n-1}\right\} \neq 0. \tag{3.8}$$

Using (2.10) and (2.15), equation (3.8) reduces to

$$S(Y, \varphi W) = (n-1)g(W, \varphi Y). \tag{3.9}$$

Replacing $Y$ by $Y\varphi,$ and using (2.1) and (2.16) we have,

$$S(Y, W) = \lambda g(Y, W), \quad \lambda = (n-1). \tag{3.10}$$

Thus, we have the following:

**Theorem 3.2.** *If a Lorentzian para-Sasakian manifold is globally* $\varphi$*-pseudo-quasi-cnonformally symmetric then the manifold is an Einstein manifold provided* $\left\{p + (n-2)q + \frac{d}{n-1}\right\} \neq 0.$

Also, if $p, q \neq 0$ and $d = 0,$ then pseudo-quasi-conformal curvature reduces to quasi-conformal curvature tensor, therefore from Theorem 3.2, we can state as follows.

**Corollary 3.3.** *A globally* $\varphi$*-quasi-conformal Lorentzian para-Sasakian manifold is an Einstein manifold.*

Moreover, if $p = 1, q = -\frac{1}{n-2}$ and $d = 0,$ then pseudo-quasi-conformal curvature tensor reduces to conformal curvature tensor, we can state as follows:

**Corollary 3.4.** *A globally* $\varphi$*-conformal Lorentzian para-Sasakian manifold can not be an Einstein manifold.*

We suppose the LP-Sasakian manifold to be Einstein one. Then

$$S(X, Y) = \lambda g(X, Y),$$

where $\lambda$ is constant and $X, Y \in \chi(M).$ Then, $QX = \lambda X.$ Then from (1.1), we have

$$\tilde{C}(X, Y)Z = (p + d)R(X, Y)Z + \left[\lambda\left(q - \frac{d}{n-1}\right) + q\lambda - \frac{r}{n(n-1)}\{p + 2(n-1)q\}\right]$$

$$\times\{g(Y, Z)X - g(X, Z)Y\},$$

or

$$(\nabla_W \tilde{C})(X, Y, Z) = (p + d)(\nabla_W R)(X, Y, Z)$$

$$-\left[\frac{1}{n(n-1)}\{p + 2(n-1)q\}\right][g(Y, Z)X - g(X, Z)Y]dr(W).$$

Applying $\varphi^2$ to both sides of above equation, we have

$$\varphi^2\{(\nabla_W \tilde{C})(X, Y, Z) = (p + d)\varphi^2\{(\nabla_W R)(X, Y, Z)$$

$$-\left[\frac{1}{n(n-1)}\{p + 2(n-1)q\}\right][g(Y, Z)\varphi^2 X - g(X, Z)\varphi^2 Y]dr(W).$$

Since the manifold is Einstein one, therefore, the scalar curvature $r$ is constant and hence, from above we can state as follows:

**Theorem 3.4.** *An Einstein globally* $\varphi$*-pseudo-quasi-conformally symmetric LP-Sasakian manifold is globally* $\varphi$ *-symmetric*.

## 4. 3-Dimensional Locally $\varphi$-pseudo-Quasi-Conformally Symmetric LP-Sasakian Manifolds

For a 3-dimensional LP-Sasakian manifold, we have

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y$$

$$+ \frac{r}{2}\{g(X, Z)Y - g(Y, Z)X\}. \quad (4.1)$$

Replacing $Z$ by $\zeta$ in (4.1) and using (2.13) and (2.15), we get

$$\left\{\frac{r}{2} - 1\right\}(\eta(Y)X - \eta(X)Y) = \eta(Y)QX - \eta(X)QY. \quad (4.2)$$

Putting $Y = \zeta$ in (4.2), we get

$$S(X, Y) = \left\{\frac{r}{2} - 1\right\} g(X, Y) + \left\{\frac{r}{2} - 3\right\} \eta(X)\eta(Y). \tag{4.3}$$

In view of (4.1) and (4.3), we obtain

$$R(X, Y)Z = \left\{\frac{r}{2} - 2\right\}[g(Y, Z)X - g(X, Z)Y]$$

$$+ \left\{\frac{r}{2} - 3\right\}\{g(Y, Z)(X)\zeta - g(X, Z)(Y)\zeta + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \tag{4.4}$$

With reference to (1.1), (4.3) and (4.4), we have

$$\tilde{C}(X, Y)Z = \left\{\frac{r}{6}(4q - 2p - 3d)\right\}[g(X, Z)Y - g(Y, Z)X]$$

$$+ \left\{\frac{r}{2} - 3\right\}(p + q + d)[g(Y, Z)\eta(X)\zeta - g(X, Z)\eta(Y)\zeta]$$

$$+ \left(q - \frac{d}{3}\right)\left(\frac{r}{2} - 3\right)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \tag{4.5}$$

Covariently differentiating both sides of (4.5) with respect to $W$, we have

$$(\nabla_W \tilde{C})(X, Y)Z = \frac{dr(W)}{6}(4q - 2p - 3d)[g(X, Z)Y - g(Y, Z)X]$$

$$+ \frac{dr(W)}{2}(p + q + d)[g(Y, Z)\eta(X)\zeta - g(X, Z)\eta(Y)\zeta]$$

$$+ \frac{dr(W)}{2}\left(q - \frac{d}{2}\right)[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]$$

$$+ \left(q - \frac{d}{2}\right)\left(\frac{r}{2} - 3\right)\begin{bmatrix}(\nabla_W\eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X \\ -(\nabla_W\eta)(X)\eta(Z)Y - \eta(X)(\nabla_W\eta)(Z)Y\end{bmatrix}$$

$$+ \left\{\frac{r}{2} - 3\right\}(p + q + d)\begin{bmatrix}g(Y, Z)(\nabla_W\eta)(X)\zeta + g(Y, Z)\eta(X)(\nabla_W\zeta) \\ -g(X, Z)(\nabla_W\eta)(Y)\zeta - g(X, Z)\eta(Y)(\nabla_W\zeta)\end{bmatrix}.$$

Assuming $X$, $Y$ and $Z$ orthogonal to $\zeta$, above equation reduces to

$$(\nabla_W \tilde{C})(X, Y)Z) = \frac{dr(W)}{6}(4q - 2p - 3d)[g(X, Z)Y - g(Y, Z)X]$$

$$+\left\{\frac{r}{2}-3\right\}(p+q+d)\begin{bmatrix} g(Y, Z)(\nabla_W\eta)(X)\zeta + g(Y, Z)\eta(X)(\nabla_W\zeta) \\ -g(X, Z)(\nabla_W\eta)(Y)\zeta - g(X, Z)\eta(Y)(\nabla_W\zeta) \end{bmatrix}. \tag{4.6}$$

Taking $\varphi^2$ on both side of (4.6),

$$\varphi^2((\nabla_W\tilde{C})(X, Y)Z) = \frac{dr(W)}{6}(4q-2p-3d)[g(X, Z)\varphi^2Y - g(Y, Z)\varphi^2X]. \tag{4.7}$$

If possible, let us assume $\varphi^2((\nabla_W\tilde{C})(X, Y)Z) = 0,$ then $dr(W) = 0$ provided $(4q-2p-3d) \neq 0.$ Hence, $dr(W) = 0$ implies *r* is constant.

For the converse part, if the scalar curvature *r* is constant, then from (4.7) we can say that the LP-Sasakian manifold is locally $\varphi$-pseudo-quasi-conformally symmetric. Thus, we have the following:

**Theorem 4.1.** *A* 3-*dimensional Lorentzian para-Sasakian manifold is locally* $\varphi$*-pseudo-quasi-conformally symmetric if and only if the scalar curvature r is constant provided* $(4q-2p-3d) \neq 0.$

## 5. Example of a $\varphi$-pseudo-Quasi-Conformally Symmetric LP-Sasakian Structure

We consider a 3-dimensional Riemannian manifold $M = \{(x, y, z) \in \Re^3 : z > 0\},$ where $x, y, z$ are the standard coordinates in $\Re^3$. Let $\{e_1, e_2, e_3\}$ be a linearly independent global frame on *M* given by

$$e_1 = e^z\frac{\partial}{\partial x},\ e_2 = e^{z-ax}\frac{\partial}{\partial y},\ e_3 = -\frac{\partial}{\partial z}, \tag{5.1}$$

where $a$ is a non-zero constant such that a $a \neq 1.$ Let $g$ be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -1. \tag{5.2}$$

Let $\eta$ be the 1-form defined by $\eta(V) = g(V, e_3)$ for any $V \in TM$. Let $\varphi$ be the (1, 1) tensor field defined by $\varphi e_1 = -e_1,\ \varphi e_2 = -e_2$ and $\varphi e_3 = 0.$ Then, using the linearity of $\varphi$ and $g$, we have $\varphi(e_3) = -1$, $\varphi^2 V = V + \eta(V)e_3$, and $g(\varphi U, \varphi V) = g(U, V) + \eta(U)\eta(V)$ for any $U, V \in TM$. Thus for $e_3 = \zeta$, $(\varphi, \zeta, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

Let $r$ be the Levi-Civita connection with respect to the Lorentzian metric $g$. Then we have

$$[e_1, e_2] = -ae^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2. \tag{5.3}$$

Using Koszul's formulae [9] for the Lorentzian metric $g$, we can easily calculate

$$\nabla_{e_1} e_1 = -e_3, \ \nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_3 = -e_1,$$

$$\nabla_{e_2} e_1 = ae^z e_2, \ \nabla_{e_2} e_2 = -ae^z e_1 - e_3, \ \nabla_{e_2} e_3 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_3 = 0.$$

Also, the Riemannian curvature tensor $R$ is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Then

$$R(e_1, e_2)e_2 = e_1, \ R(e_1, e_3)e_3 = -e_1, \ R(e_2, e_1)e_1 = e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \ R(e_3, e_1)e_1 = e_3, \ R(e_3, e_2)e_2 = e_3,$$

$$R(e_1, e_2)e_3 = 0, \ R(e_2, e_3)e_2 = 0, \ R(e_3, e_1)e_2 = 0.$$

Then, the Ricci tensor $S$ is given by

$$S(e_1, e_1) = 2, \ S(e_2, e_2) = 2, \ S(e_3, e_3) = -2,$$

$$S(e_1, e_2) = 0, \ S(e_1, e_3) = 0, \ S(e_2, e_3) = 0.$$

Thus, the scalar curvature $r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = 2$ is constant. Thus conditions (2.5) and (2.6) for any vector fields $X$ and $Y$ in $M$ holds. It can be shown that all the properties of an LP-Sasakian manifold hold for any vector fields $X, Y$ in $M$. Since the given 3-dimensional LP-Sasakian manifold is of constant scalar curvature $r = 2$, therefore, by virtue of Theorem 4.1, it implies that it is locally $\varphi$- pseudo-quasi-conformally symmetric in nature.

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