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REVISITING WORMHOLES SUPPORTED BY TWO NON-INTERACTING FLUIDS

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Abstract

This paper extends several previous studies of wormholes supported by two non-interacting fluids beginning with a combined model of ordinary matter and phantom dark energy with an anisotropic matter distribution. After noting that such wormholes could only exist on very large scales, the two-fluid model is extended to neutron stars, previously treated only for the isotropic case. The model is completed by incorporating certain generic features proposed by the author in an earlier study.

1. Introduction

Wormholes are handles or tunnels in spacetime connecting widely separated regions of our Universe or even entirely different universes in a

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multiverse. Apart from some forerunners, macroscopic traversable wormholes were first studied in detail by Morris and Thorne [1] in 1988. They had proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $e^{2\Lambda(r)} = 1 - b(r)/r$. (We are using units in which $c = G = 1$.) In the terminology introduced in Ref. [1], $\Phi = \Phi(r)$ is called the *redshift function*, which must be everywhere finite to prevent an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [1]. The spherical surface $r = r_0$ is called the *throat* of the wormhole, where $b(r_0) = r_0$. The shape function must also meet the requirement $b'(r_0) < 1$, called the *flare-out condition*, while $b(r) < r$ for $r > r_0$. We also require that $b'(r_0) > 0$. In classical general relativity, the flare-out condition can only be met by violating the null energy condition (NEC), which states that for the energy-momentum tensor $T_{\alpha\beta}$,

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0 \text{ for all null vectors } k^\alpha. \tag{2}$$

In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation becomes

$$T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0. \tag{3}$$

Here $T^t{}_t = -\rho$ is the energy density, $T^r{}_r = p_r$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t$ is the lateral (transverse) pressure.

Matter that violates the NEC is called "exotic" in Ref. [1]. So in classical general relativity, the violation of the NEC is equivalent to the need for exotic matter, at least in the vicinity of the throat. It follows that exotic matter can only be avoided by departing from a strict classical setting. Examples are an appeal to $f(R)$ modified gravity [2] or to the existence of an extra spatial dimension [3].

The idea that a wormhole can be supported by two non-interacting fluids is not new [4, 5, 6]. Using the superscripts 1 and 2 to represent the two fluids, the energy-momentum tensor takes on the form

$$T^t{}_t = -(\rho^1 + \rho^2), \tag{4}$$

$$T^r{}_r = p_r^1 + p_r^2, \tag{5}$$

and

$$T^\theta{}_\theta = T^\phi{}_\phi = p_t^1 + p_t^2. \tag{6}$$

Referring now to line element (1), the Einstein field equations are

$$\frac{b'}{r^2} = 8\pi(\rho^1 + \rho^2), \tag{7}$$

$$-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r} = 8\pi(p_r^1 + p_r^2), \tag{8}$$

and

$$\left(1 - \frac{b}{r}\right)\left[\Phi'' - \frac{b'r - b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r-b)}\right] = 8\pi(p_t^1 + p_t^2). \tag{9}$$

Ref. [4] discusses quintom wormholes, so that the two fluids are quintessence and phantom dark energy. In Ref. [5], dealing with neutron stars, the matter sources are neutron and quark matter, respectively. In the third case, to be discussed further below, the two non-interacting fluids are ordinary matter and phantom dark energy [6].

The main purpose of this paper is to compare and contrast the last two cases.

2. Background

As noted in the Introduction, we are going to be primarily interested in the models discussed in Refs. [5] and [6]. Since these describe very different physical situations, some preliminary remarks are going to be needed.

Our first concern is the extreme radial tension at the throat of a Morris-Thorne wormhole, discussed in Ref. [1]. We first need to recall that the radial tension $\tau(r)$ is the negative of the radial pressure $p_r(r)$. According to Ref. [1], the Einstein field equations can be rearranged to yield

$$\tau(r) = \frac{b(r)/r - 2[r - b(r)]\Phi'(r)}{8\pi G c^{-4} r^2}, \tag{10}$$

temporarily reintroducing c and G. So the radial tension at the throat becomes

$$\tau(r_0) = \frac{1}{8\pi G c^{-4} r_0^2} \approx 5 \times 10^{41} \frac{\text{dyn}}{\text{cm}^2}\left(\frac{10\text{ m}}{r_0}\right)^2. \tag{11}$$

It is noted in Ref. [1] that for $r_0 = 3\,\text{km}$, τ has the same magnitude as the pressure at the center of a massive neutron star. It follows from Eq. (11) that the widely discussed wormhole solutions based on dark matter and dark energy could only exist on very large scales, i.e., by possessing very large throat radii r_0. For further discussion of this problem, see Ref. [7].

Before continuing, we need to consider two other topics discussed in Ref. [8]. The first is the shape function

$$b(r) = r_0\left(\frac{r}{r_0}\right)^{\alpha}, \qquad 0 < \alpha < 1, \tag{12}$$

introduced in Ref. [1] and used in Ref. [6]. Ref. [8] defines a shape function to be *typical* if it meets the following requirements: $b(r_0) = r_0$, $0 < b'(r_0) < 1$, and $b(r)$ is concave down near the throat. It is shown that any such shape function can be approximated by Eq. (12). So this form of the shape function is essentially generic.

Having defined a typical shape function, Ref. [8] goes on to discuss the nature of exotic matter in a Morris-Thorne wormhole. First we need to recall that phantom dark energy is characterized by the equation of state $p = \omega\rho$, $\omega < -1$, and can therefore support traversable wormholes [9]. The use of Eq. (12) leads to the surprising conclusion that the converse is also true in the following sense: if we are dealing with a typical shape function, the equation of state of exotic matter in the vicinity of the throat is given by $p_r = \omega\rho$, $\omega < -1$, where p_r is the radial pressure.

In this paper, it turns out to be convenient to use the form

$$p_r = -\omega\rho, \qquad \omega > 1. \tag{13}$$

3. The Combined Model

In this section, we will examine the combined model in Ref. [6] comprising ordinary matter and phantom dark energy. Here ordinary matter has the standard perfect-fluid equation of state

$$p = m\rho, \qquad 0 < m < 1. \tag{14}$$

The phantom-energy density ρ^{ph} is assumed to have the form

$$\rho^{ph} = n\rho, \qquad n > 0. \tag{15}$$

This is a natural assumption but subject to the condition that n be extremely small. For the phantom-energy case, we have from Eq. (13) that

$$p_r^{ph} = -\omega\rho^{ph}, \qquad \omega > 1. \tag{16}$$

So in Eqs. (7)-(9), the superscript 1 stands for ordinary matter and the superscript 2 for phantom energy, i.e., $\rho^1 = \rho$ and $\rho^{ph} = n\rho$. Substituting Eq. (12) in Eq. (7) yields

$$\rho = \frac{\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3}, \tag{17}$$

which leads at once to

$$p = m\rho = \frac{m\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3}, \tag{18}$$

and

$$p_r^{ph} = -\omega\rho^{ph} = -\omega n\rho = -\frac{\omega n\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3}. \tag{19}$$

The expression for p_t^{ph} is given in Ref. [6]. Since $p_t^{ph} \neq p_r^{ph}$, the pressure is anisotropic.

Since we are dealing with a combined model, we need to introduce the following notations: the effective density is denoted by $\rho^{eff} = \rho + \rho^{ph}$ and the effective radial pressure by $p_r^{eff} = p_r + p_r^{ph}$. So

$$\rho^{eff} = \frac{\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3} + \frac{n\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3} = \frac{\alpha}{8\pi r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3}. \tag{20}$$

Since Eq. (14) represents a perfect fluid, we have $p_r = m\rho$ and hence

$$p_r^{eff} = \frac{m\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3} - \omega\frac{n\alpha}{8\pi(1+n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3}$$

$$= \frac{\alpha(m - n\omega)}{8\pi(1 + n)r_0^2}\left(\frac{r}{r_0}\right)^{\alpha-3}. \tag{21}$$

We saw in the Introduction that for a Morris-Thorne wormhole, meeting the flare-out condition at the throat implies that $\rho + p_r < 0$. Since we are dealing with a more complicated model, we need to show explicitly that $\rho^{eff} + p_r^{eff} < 0$. So from Eqs. (20) and (21),

$$\rho^{eff} + p_r^{eff}\Big|_{r=r_0} = \frac{1}{8\pi r^2}\left[\alpha + \frac{\alpha(m - n\omega)}{n+1}\right]\left(\frac{r}{r_0}\right)^{\alpha-3}\Bigg|_{r=r_0}$$

$$= \frac{\alpha(n + 1 + m - n\omega)}{n+1} < 0 \tag{22}$$

provided that

$$\omega > \frac{1 + n + m}{n}. \tag{23}$$

Given that n is extremely small, the phantom-energy parameter ω is extremely large, which is completely unphysical unless r_0 is also extremely large: in the latter case, we can see from Eq. (17) that Eq. (15) could hold for larger n, say $n \geq 1$.

Remark: That wormholes supported by phantom energy could only exist on large scales has already been noted after Eq. (11).

4. Neutron Stars

We have seen that the two-fluid model could only work for much larger n and may therefore be applicable to neutron stars. Here we need to recall that while neutrons are normally held together by the strong force, the extreme conditions near the center are likely to cause the neutrons to become deconfined, resulting in quark matter. So in the vicinity of the throat $r = r_0$, deep in the interior, we can assume that both neutron matter and quark matter are present, thereby preserving the two-fluid model. (We will denote the density of neutron matter by ρ and the density of quark matter by ρ^q.)

While the interior $r < r_0$ is not part of the wormhole spacetime, the highly dense quark matter near the center still contributes to the gravitational field. This can be compared to a thin-shell wormhole from a Schwarzschild black hole [10]. Here the black hole generates the gravitational field. Using the MIT bag model, it is shown in Ref. [5] that $b = b(r)$ qualifies as a typical shape function due to the presence of quark matter. So we can retain the shape function in Eq. (12).

Our two-fluid model therefore comprises neutron matter and quark matter, assumed to be essentially non-interacting and possessing an anisotropic matter distribution in order to be consistent with our earlier discussion. This assumption leads directly to the analogue of Eq. (16):

$$p_r^q = -\omega\rho^q, \qquad \omega > 1, \tag{24}$$

where the superscript q refers to quark matter. We also need to recall from the end of Section 2 that the phantom-energy equation of state has a converse: matter that violates the NEC can be assumed to have the form in Eq. (13).

The rest of this paper is devoted to showing that these assumptions are reasonable. To that end, we first note that the density of neutron matter ranges from 3.7×10^{17} kg/ m^3 to 5.9×10^{17} kg/ m^3 and that the density of quark matter is approximately 2.7×10^{18} kg/ m^3. Denoting the former by ρ and the latter by ρ^q, the analogue of Eq. (15) becomes

$$\rho^q = n\rho, \qquad n \approx 6. \tag{25}$$

Eq. (23) now reads

$$\omega > \frac{1+6+m}{6}, \tag{26}$$

which is acceptable from a physical standpoint. Furthermore, from Eq. (21), we find that at $r = r_0$ with $n = 6$

$$p_r^{eff} = \frac{\alpha(m-6\omega)}{7}\frac{1}{8\pi r_0^2}\frac{c^4}{G} = \frac{\alpha(m-7-m)}{7}\frac{1}{8\pi r_0^2}\frac{c^4}{G} \tag{27}$$

from Eq. (23). So

$$p_r^{eff} = -\alpha\frac{1}{8\pi r_0^2}\frac{c^4}{G} = \alpha\tau(r_0). \tag{28}$$

To be consistent with Eq. (11), our generic shape function (12) must be restricted in an obvious way: α must be closer to unity, say $\alpha > 0.5$.

That neutron stars can support traversable wormholes had already been proposed in Ref. [5] under the assumption of isotropic pressure. Neutron stars connected by wormholes are also discussed in Ref. [11].

The above discussion suggests that wormholes with moderate throat sizes are actually compact stellar objects, made explicit in Ref. [12].

The results in this section depend on Eq. (26), an assumption that has proved to be physically reasonable. Further confirmation comes from the gradient of the redshift function, discussed in the next section.

5. The Gradient of the Redshift Function

Combining Eqs. (7) and (8), we obtain

$$\frac{rb'-b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r} = 8\pi(\rho + \rho^q + p_r + p_r^q)$$

$$= 8\pi(\rho + n\rho + m\rho - \omega n\rho) = 8\pi\frac{p}{m}(1+n+m-\omega n) \tag{29}$$

since $p = m\rho$. Solving for Φ', we get

$$\Phi' = \frac{1}{r[r - b(r)]}\left[-\frac{rb'(r) - b(r)}{2} + 4\pi\frac{p}{m}(1 + n + m - \omega n)r^3\right]. \tag{30}$$

As another check on the plausibility of our model, let us return to line element (1): if $e^{2\Lambda(r)} = 1 - 2m(r)/r$ with $m(0) = 0$, the line element represents a stellar model [13]. Here the parameter Φ is usually viewed as the relativistic version of the Newtonian potential $\Phi = -M/r$ [13]:

$$\frac{d\Phi}{dr} = \frac{M + 4\pi r^3 p}{r(r - 2M)}, \tag{31}$$

where M is the mass of the star. So $d\Phi/dr = M/r^2$ in the Newtonian limit. If we let $b(r) = 2M$ in Eq. (30), we get

$$\Phi' = \frac{1}{r(r - 2M)}\left[M + 4\pi\frac{p}{m}(1 + n + m - \omega n)r^3\right]. \tag{32}$$

To complete the comparison, we must also have

$$\frac{1 + n + m - \omega n}{m} = 1, \tag{33}$$

which implies that $\omega = 1 + 1/n$; thus $\omega \approx 7/6$, since $n \approx 6$. So ω is only slightly larger than unity.

6. Conclusion

Wormholes supported by two non-interacting fluids are discussed in Refs. [4, 5, 6]. This paper begins with the combined model of ordinary matter and phantom dark energy with an anisotropic matter distribution, discussed in Ref. [6]. While leading to a valid wormhole solution, such wormholes could only exist on very large scales, i.e., with very large throat sizes, shown previously in Ref. [7].

Extending this model to neutron stars calls for two non-interacting (or at least weakly interacting) fluids, neutron matter and quark matter. Both are assumed to be present near the throat, being close enough to the center of the neutron star. The two-fluid model carries over directly to neutron stars provided that the equation of state $p_r^{ph} = -\omega\rho^{ph}$, $\omega > 1$, for phantom dark energy can be replaced by the equation of state $p_r^q = -\omega\rho^q$, $\omega > 1$, for quark matter. This replacement can be justified as follows: it was noted in Section 2 that our shape function can be approximated by $b(r) = r_0(r/r_0)^{\alpha}$, $0 < \alpha < 1$ [8], thereby meeting the flare-out condition. Since we are going beyond a basic Morris-Thorne wormhole, the violation of the NEC needs to be checked separately; this is verified in Eqs. (22) and (23). Moreover, it is shown in Ref. [8] that matter that violates the NEC can be assumed to have the form $p_r = -\omega\rho$, $\omega > 1$, enough to suggest that the analogous equation of state for quark matter is $p_r^q = -\omega\rho^q$, $\omega > 1$. It is shown in Section 4 that this is indeed is a reasonable assumption, thereby preserving the two-fluid model. It is interesting to note that the resulting radial pressure turns out to be $p_r^{eff} = \alpha\tau(r_0)$. To be consistent with Eq. (11), α must be sufficiently large.

Returning now to the parameter ω, we saw in Section 3 that the combined model comprising ordinary matter and phantom dark energy leads to a large physically unacceptable value of the parameter ω. By contrast, for neutron stars, an examination of the gradient of the redshift function, discussed in Section 5, shows that ω is only slightly larger than unity.

The result is a viable extension of the two-fluid model in Ref. [6] to neutron stars. This outcome suggests that wormholes with moderately-sized throat sizes are actually compact stellar objects, made explicit in Ref. [12].

Neutron stars connected by wormholes are also discussed in Ref. [5], assuming an isotropic matter distribution. It is noted in Ref. [11] that an observable tell-tale sign of such a wormhole would be any observed variation in the mass of the neutron star. The existence of compact stellar objects with an anisotropic matter distribution is also discussed in Refs. [14, 15] in $f(R, T)$ modified gravity. The detection of phantom-energy wormholes, as well as phantom-energy black holes, by means of gravitational lensing is discussed in Ref. [16].

References

[1] M. S. Morris and K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, Am. J. Phys. 56 (1988), 395-412.

[2] F. S. N. Lobo and M. A. Oliveira, Wormhole geometries in $f(R)$ modified theories of gravity, Phys. Rev. D 80 (2009), ID: 104012.

[3] P. K .F. Kuhfittig, Traversable wormholes sustained by an extra spatial dimension, Phys. Rev. D 98 (2018), ID: 064041.

[4] P. K. F. Kuhfittig, F. Rahaman and A. Ghosh, Quintom wormholes, Int. J. Theo. Phys. 49 (2010), 1222-1231.

[5] P. K. F. Kuhfittig, Neutron star interiors and topology change, Adv. Math. Phys. 2013 (2013), ID: 630196.

[6] F. Rahaman, S. Ray and S. Islam, Wormholes supported by two non-interacting fluids, Astrophys. Space Sci. 346 (2013), 245-252.

[7] P. K. F. Kuhfittig, A survey of recent studies concerning the extreme properties of Morris-Thorne wormholes, arXiv: 2202.07431 [gr-qc].

[8] P. K. F. Kuhfittig, On the nature of exotic matter in Morris-Thorne wormholes, New Horizons in Mathematical Physics 4 (2020), 29-32.

[9] O. B. Zaslavskii, Exactly solvable model of a wormhole supported by phantom energy, Phys. Rev. D 72 (2005), ID: 061303.

[10] E. Poisson and M. Visser, Thin-shell wormholes: Linearized stability, Phys. Rev. D 52 (1995), ID: 7318.

[11] V. Dzhunushaliev, V. Folomeev, B. Kleihaus and J. Kunz, Mixed neutron-star-plus-

wormhole systems: Equilibrium configurations, Phys. Rev. D 85 (2012), ID:124028.

[12] P. K. F. Kuhfittig, A note on wormholes as compact stellar objects, Fundamental J. Modern Physics 17(1) (2022), 63-70.

[13] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation, W. H. Freeman, New York, 1973, Ch. 23.

[14] Z. Yousaf, M. Zaeem-ul-Haq Bhatti and M. Ilyas, Existence of compact structures in $f(R, T)$ gravity, Eur. Phys. J. C 78 (2018), ID: 307.

[15] Z. Yousaf, Kazuharu Bamba, M. Z. Bhatti and U. Farwa, Quasi static evolution of compact objects in modified gravity, Gen. Relativ. Gravit. 54 (2022), ID: 7.

[16] A. Ovgun, G. Gyulchev and K. Jusufi, Weak gravitational lensing by phantom black holes and phantom wormholes using the Gauss-Bonnet theorem, Ann. Phys. 406 (2019), 152-172.