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REGULARIZATION OF VECTOR FUNCTION RIEMANN BOUNDARY VALUE PROBLEM AND SOLVING

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Abstract

The present paper, at first, puts out a simple way to transfer an irregular vector function Riemann boundary value problem (R-problem) to a regular R-problem. Then using the same method, we else transfer the regular R-problem to an equivalence R-problem, in which the total index is zero. And then we provide a method to simply obtain canonical function matrix of the problem. Meanwhile, we prove a series of results to simplify the solving process and our method can avoid from partial index. Based on our results, finally, the expression of the generalized accurate solution is

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directly given here.

1. Introduction

As it is well known, *R*-problem was one of twenty three problems that were put out by great mathematician Hilbert in the very beginning of last century. In the time, the study of *R*-problem and its related problems had been coming fluently. References [1-5] were not only collected a lot of important results, but also introduced the research method and content systematically.

If we classify all of the achievements and results, we can find that the results related to *R*-problem (regular or not) seem very plenty. But, there is a little works on vector functions *R*-problem, particularly on irregular *R*-problem.

Solving vector function *R*-Problem need to use the matrix theory. Since the coefficient matrix in the problem is a function matrix (a matrix whose element consisted by function), as we know, there is no good method to copy with this kind of matrix.

One of the most important tools to deal with matrix is determinant. Since the operation between determinant and matrix is quite different, and there are many isolated zeros and singular points of the matrix in domain, the solving of the problem is seen rather difficult. As a result of solving the problem has to pass a process that reduces a basic solution matrix (or basic solution groups) to a canonical matrix (or canonical solution matrix), this means one thing should be done that is finding a method to clean up all isolated zeros and singular points from the basic solution matrix.

Written out the canonical matrix need know the partial index in before. But, it was pity to say that there was no formula to determine the partial index. For this reason, the solution of the problem could not be written out indeed (see refs. [1-3]). What the people done here was that gave the solving process a descriptive one.

The present paper gives a method to avoid from the partial index mentioned above. In fact, we prove that the conception is of no use here. To obtain the result, in our paper, we firstly introduce some diagonal matrices to help us to transfer an irregular vector function Riemann boundary value problem (*R*-problem) to a regular *R*-problem. Then using the same method, we else transfer the regular *R*-problem to an equivalence *R*-problem, in which the total index is zero. Based on this, we prove a series of results to simplify the solving process and our method can avoid from partial index. That is, in the whole solving process of problem, known the total index is enough. Partial indexes are a redundant conception.

2. Illumination

For simple reason, we suppose L is a smooth closed contour in the complex plane oriented counter-clockwise, the interior region bounded by which is denoted as D^+, exterior D^-, and assume $0 \in D^+$.

Denote

$$\varphi(z) = \begin{pmatrix} \varphi_1(z) \\ \varphi_2(z) \\ \vdots \\ \varphi_n(z) \end{pmatrix} = (\varphi_1(z)\varphi_2(z) \cdots \varphi_n(z))^T. \tag{1.1}$$

The Mention of the problem

The regular type functions *R*-problem is that finding a sectional holomorphic vector function $\Phi(z)$ as form (1.1) with jumps on *L*, satisfies

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L, \tag{1.2}$$

where $G(t) = (G_{jk}(t))_{n\times n}$ is a known $n \times n$ matrix on $L, \in H$, $g(t)$ is a known vector function $(\in H)$ as form (1.1). $\Phi(z) = (\Phi_1(z), \Phi_2(z), \cdots, \Phi_n(z))^T$ is an unknown sectional holomorphic vector function, which belongs to H on L also. Suppose the singular order of $\Phi(z)$ at ∞ is finite, and belongs to *H* on *L* too.

Here, the irregular type vector function R-problem is studied. That is, in the time, $\det G(t)$ can be equal to zero at some isolate points on L. No more than this, we can suppose the $\det G(t)$ have some isolate singular points. Furthermore, without losing the generality, we just solve the problem in the class R_0 here.

Just as doing before, introduce the concept of total index as

$$\kappa = \frac{1}{2\pi i}[\log \det G(t)]_L = \frac{1}{2\pi}[\arg \det G(t)]_L. \tag{1.3}$$

2. Regularization of Vector Function Riemann Boundary Value Problem

Suppose the zero points of $\tilde{G}(t)$ and $1/\det \tilde{G}(t)$ on L are $\alpha_1, \alpha_2, \cdots, \alpha_p$ and $\beta_1, \beta_2, \cdots, \beta_q$ respectively. Its order is $l_1, l_2, \cdots l_p$ and $m_1, m_2, \cdots, m_p$. Note

$$\prod_1(t) = \prod_{j=1}^{p}(t-\alpha_j)_j^l; \quad \prod_2(t) = \prod_{j=1}^{q}(t-\beta_j)_j^m; \quad \prod(t) = \prod_1(t)/\prod_2(t),$$

$$l = \sum_{j=1}^{p} l_j; \quad m = \sum_{j=1}^{q} m_j,$$

$E(t) = (E(t))_{n\times n}$ is a diagonal matrix too, in which the elements on diagonal are $d_{kk}(t)$, $(d_{kk}(t) = 1\ (k = 1, 2, \cdots, n-1)$, but $d_{nn} = 1/\prod(t))$; $E^*(t) = (E^*(t))_{n\times n}$ is also a diagonal matrix, in which the elements on diagonal are $(d_{kk}^*(t) = 1$ $(k = 1, 2, \cdots, n-1)$, but $d_{nn}^*(t) = \prod(t))$. Let

$$G_*(t) = E^*(t)G(t), \quad t \in L, \tag{2.1}$$

then

$$\det G_*(t) \neq 0; \quad 1/\det G_*(t) \neq 0.$$

Obviously, problem (2.2) is transferred to

$$E_1(t)\Phi^+(t) = G_*(t)E_2(t)\Phi^-(t),\ t \in L, \tag{2.2}$$

where $E_1(t) = \mathrm{diag}(1, 1, \cdots, 1, \prod_2(t)),\ E_2(t) = \mathrm{diag}(1, 1, \cdots, 1, \prod_1(t))$.

Now, introduce a new sectional holomorphic vector function:

$$\Psi(z) = \begin{cases} E_1(t)\Phi(z), & z \in D^+, \\ E_2(t)\Phi(t), & z \in D^-, \end{cases} \tag{2.3}$$

then problem (2.2) is transferred to

$$\Psi^+(t) = G_*(t)\Psi^-(t),\ t \in L. \tag{2.4}$$

It is a regular vector function *R*-problem. The problem is regularized at all.

3. Equivalence Problem

In general, the index κ is not equal to zero in problem (1.2), which would make many a trouble in solving it. We can use the same method mentioned above to simplify the solving.

Because the index is only related to the homogeneous problem, we just need to hand on the follow problem

$$\Phi^+(t) = G(t)\Phi^-(t),\ t \in L. \tag{3.1}$$

Define two matrixes as: $T(z) = (T(z))_{n\times n}$ a diagonal matrix whose elements on diagonal are denoted by $\delta_{kk}(z)$ and the $\delta_{kk}(z) = 1\ (k = 1, 2, \cdots, n-1)$, but $\delta_{nn}(z) = z^{-\kappa}$; $T^*\ (z) = (T^*\ (z))_{n\times n}$ an other diagonal matrix, in which the elements on diagonal are denoted by $\delta^*_{kk}(z)$, and the $\delta^*_{kk}(z) = 1\ (k = 1, 2, \cdots, n-1)$ but $\delta^*_{nn}(z) = z^{\kappa}$. Now, multiply $T(t)$ to both sides of (3.1), note

$$G_0(t) = T(t)G(t), \quad t \in L \tag{3.2}$$

and complement define that the functional value of $G_0(t)$ at removable isolated singular point is equal to its limiting value, then the original problem is transferred as

$$\Phi^{+}(t) = T^{*}(t)\, G_0(t)\Phi^{-}(t), \quad t \in L. \tag{3.3}$$

After introducing function

$$\Omega(z) = \begin{cases} \Phi(z), & z \in D^{+}, \\ T^{*}(t)\, \Phi^{-}(t), & z \in D^{-}, \end{cases} \tag{3.4}$$

the problem is varied as a equivalence problem

$$\Omega^{+}(t) = G_0(t)\Omega^{-}(t), \quad t \in L. \tag{3.5}$$

But, in this time,

$$\kappa = \frac{1}{2\pi i}[\log \det G_0(t)]_L = \frac{1}{2\pi}[\arg \det G_0(t)]_L = 0. \tag{3.6}$$

Theorem 1. *The solving problem of the regular functions R-problem* (1.2) *in* R_0 *can be transformed to an equivalent regular functions R-problem* (2.5) *in* R_κ, *in which*, *the total index of the coefficient matrix* $G_0(t)$ *is equal to zero*.

4. Basic Solution Matrix

As the definition in ref. [1], named $\Phi(z) = (\Phi_j^k(z))(k,\ j = 1, 2, \cdots, n)$ as basic solution matrix of problem (2.1). In other words, the each column of function $\Phi(z)$ satisfies

$$\Phi_j^{k+}(t) = T^{*}(t)\, G(t)\Phi_j^{k-}(t),\ (t \in L) \text{ or}$$

$$(\Phi_j^k(t))^{+} = T^{*}(t)\, G(t)\Phi_j^k(t))^{-},\ (t \in L) \tag{3.1}$$

and $\det \Phi(t) \neq 0.$

Ref. [1] provides a method to obtain the basic solution matrix, we use the results directly but do not repeat the process here.

Since the basic solution matrix existed some isolated zero points on complex plane when the total index $\kappa \neq 0$, in past, one had have to remove all those isolated zero points so that the determinant of $\Phi(z)$ was not equal to zero. In other words, one must construct a solution matrix that would be full rank on whole complex plane, which is called normal solution matrix. Using our method showed just before, we can copy with the problem quickly.

In the time, if suppose that the basic solution matrix is $\Omega(z) = (\Phi_1(z), \Phi_2(z), \cdots, \Phi_n(z))$ and its spectrum or isolated zero points of $\det \Omega(z)$ in D^+ and D^- are $a_1, a_2, \cdots, a_p$ and $b_1, b_2, \cdots, b_q$, the order of those isolated zero points is $\lambda_1, \lambda_2, \cdots, \lambda_m$, and $\nu_1, \nu_2, \cdots, \nu_m$, respectively, now, we can introduce a couple of new diagonal matrixes $D(z)$ and $D^*(z)$:

$$D(z) = \text{diagonal}\left(1, 1, \cdots, 1, \prod_{k=1}^{p} (z - a_k)^{\lambda_k} \prod_{j=1}^{q} (z - a_j)^{\nu_j}\right),$$

and multiply them to the next equation

$$\Omega^+(t) = G(t)\Omega^-(t),\ t \in L \tag{3.2}$$

then

$$D(z)\Omega^+(t) = G(t)D(z)\Omega^-(t),\ t \in L. \tag{3.3}$$

If let $\Theta(z) = D(z)\Omega(z)$, obviously, $\det \Theta(z) \neq 0$ on whole complex plane. In other words, the solution matrix $\Theta(z)$ is full rank on plane at all.

Theorem 2. *If the isolated zero points of the basic solution matrix are known, it can simple get the normal solution matrix by multiplying a special diagonal matirx.*

Note that $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_p$ and $\nu = \nu_1 + \nu_2 + \cdots + \mu_q$, meanwhile, we can obtain the following theorem

Theorem 3. *If* λ *is the total order of isolated zero points of the basic solution matrix in* D^+ *and* ν *is the total order of isolated zero points of the basic solution matrix in* D^-, *the index* κ *is*

$$\kappa = \lambda - \nu. \tag{3.4}$$

Proof. Since the $\Omega(z)$ is holomorphism on $D^{\pm}$, respectively and satisfies (3.2), we can infer

$$\begin{aligned}\kappa &= \frac{1}{2\pi}[\arg \det G_0(t)]_L \\ &= \frac{1}{2\pi}[\arg \det \Omega^+(t)]_L - \frac{1}{2\pi}[\arg \det \Omega_-(t)]_L \\ &= \lambda - \nu.\end{aligned}$$

4. Discussion of Index

It is well known that, in the past, writing out the solution expression of the vector function of *R*-problem relies on the partial indexes. And that the partial indexes are some quantities which could not be calculated by a determinate formula. Therefore, the best choice is to avoid the partial indexes out of in solving process. The next theorem is quite use of to solve the partial indexes problem.

Theorem 4. *If the total index* κ *of a vector function R-problem is zero*, *then every corresponding partial index of the problem is equal to zero too*.

Proof. Note that $X(t)$ as the canonical matrix. Suppose the partial index of the matrix is κ_j $(j = 1, 2, \cdots, n)$, we now prove that all κ_j $(j = 1, 2, \cdots, n)$ are zero.

In fact, the relation of the total and partial index is

$$\kappa = \kappa_1 + \kappa_2 + \cdots + \kappa_n.$$

Let

$$Y(t) = (z^{\kappa_j})X(t),$$

where (z^{κ_j}) is a diagonal matrix whose diagonal elements are z^{κ_j}, then we can prove that $Y(t)$ is a canonical matrix too. Actually, complement defines $0^0 = 1$, then $Y(t)$ satisfies all the demand of the canonical matrix, because

I. $Y^+(t) = G_0(t)Y^-(t)$ is obvious fact;

II. $\det Y(z) = z^{\kappa} \det X(z)$, then $Y(z)$ is holomorphism on whole complex plane;

III. It is envious that the index of $Y(z)$ at ∞ is zero.

Moreover, since the every partial index of canonical matrix $Y(t)$ is zero, and the partial index owns invariant property (seeing ref. [1]), then the partial index is zero for the canonical matrix.

Theorem 5. *If the total index* κ *of a vector function R-problem is zero*, *then the normal solution matrix is bounded at* ∞, *that is*, *it is canonical solution matrix*.

Proof. If Φ(z) is the normal solution matrix, then $\det \Phi(z) \neq 0$ on whole complex plane. Theorem 4 tells that all partial indexes are zero too. Thereby, $\Phi(z)$ should be finite at ∞. According to the definition of canonical solution matrix, it is a canonical solution matrix at all.

Corollary 2.1. *Solving problem of a vector function R-problem can be induced to an equivalence problem that total index and all partial indexes are equal to zero by use of introducing a diagonal to realize*.

Corollary 2.1 tells us we need not define the partial indexes when solving a vector function *R*-problem.

5. Expression of Solution

Based on the discussion in Section 4, we can obtain the canonical solution matrix of the equivalent problem (2.5), it have

$$X^{+}(t) = G_0(t)X^{-}(t), \text{ or } G_0(t) = X^{+}(t)[X^{-}(t)]^{-1}. \tag{5.1}$$

Therefore

$$[X^{+}(t)]^{-1}\Omega^{+}(t) = [X^{-}(t)]^{-1}\Omega^{-}(t). \tag{5.2}$$

Let $F(z) = [X(z)]^{-1}\Omega(z)$, then it is a holomorphic vector on whole complex plane. Take care of that the solving for the equivalent problem is process in class R_κ, the Liouville theorem shows that it is only a polynomial vector, denote it by $P(z)$,

$$P(z) = (P_1(z), P_2(z), \cdots, P_n(z))^T \tag{5.3),}$$

here the order of $P(z)$ is no more than κ.

Moreover, polynomial vector $P(z)$ could be taken as a vector polynomial that which all the front of $n-1$ component $P_k(z)$ $(k = 1, \cdots, n-1)$ are constant, but $P_n(z) = Cz^{\kappa}$.

Take it to the original problem, we can get the expression of the general solution of (2.5) in R_0, which is

$$\Phi(z) = \begin{cases} P(z)X(z) \\ T(z)P(z)X(z) \end{cases} = \begin{cases} \sum_{k=1}^{n} P_k(z)X^k(z), & z \in D^+, \\ T(z)\sum_{k=1}^{n} P_k(z)X^k(z), & z \in D^-, \end{cases} \tag{5.4}$$

where $X(z) = (X^1(z), X^2(z), \cdots, X^n(z))$, and $P_k(z)$ $(k = 1, \cdots, n-1)$ is polynomial whose order is no more than κ.

Of course, for keeping the $\Phi(z)$ a sectional holomorphism vector function on whole complex plane, one thing is need us to do, that is, we must wipe off the ability of $z = 0$ becoming an isolated singular point. This is the reason why we take $P_n(z) = Cz^{\kappa}$.

Summary up the conclusion about, we have

Theorem 6. *Solving homogeneous problem* (2.1) *in* R_0, *if* $\kappa > 0$, *then the generalized solution is given by* (5.4), *in which the* $P(z)$ *is a polynomial vector that all the front of* $n-1$ *component* $P_k(z)$ $(k = 1, \cdots, n-1)$ *are constant, but* $P_n(z)$ *is a arbitrary polynomial whose order is no more than* κ; *if* $\kappa < 0$, *then the* $P(z)$ *must be taken to* 0.

The discussion mentioned about could be extended to the class in R_m, particularly, in R_{-1}. We could get some similar conclusions with Theorem 4. The difference is only the order of the polynomial $P(z)$.

For the inhomogeneous problem, using the result in Section 2, the original problem (1.2) can be transformed to the jump problem of

$$[X^+(t)]^{-1}\Psi^+(t) = [X^-(t)]^{-1}\Psi^-(t) + [X^+(t)]^{-1}g(t), \tag{5.5}$$

its solution is

$$\Phi(z) = \begin{cases} X(z)[\Psi(z) + P(z)], \\ \\ T(z)X(z)[\Psi(z) + P(z)] \end{cases} \tag{5.6}$$

and

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{[X^+(t)]^{-1}g(t)}{t-z}dt, \quad z \notin L. \tag{5.7}$$

Theorem 7. *The general solution of vector R-problem of* (1.2) *in* R_0 *is given by the form of* (5.6) *when* $\kappa > 0$, *in where* $X(z)$ *is a canonical solution matrix of the equivalent problem* (2.5), $\Psi(z)$ *is given by expression* (5.7), *and* $P(z)$ *is a special polynomial whose order is no more than* κ *and when* $\kappa < 0$, $P(z)$ *takes as* 0 *vector.*

If we change R_0 to R_m in the theorem, the solving process and results are almost same to all the before. But, for keeping the $\Phi(z)$ a sectional holomorphism vector function on whole complex plane, now, some interpolation method should be used. (See ref. [8]), here we do not write it all.

6. Conclusion and Discussion

The aim of this paper is to introduce some special diagonal matrix to change the original vector function of *R*-problem to a kind of regular functions of *R*-problem. Meanwhile, the method that introduces some special diagonal matrix is helpful to remove the isolate points.

In our paper, some diagonal matrices are introduced. Though the matrices are defined as a fixed form, in whith the last element of diagonal matrix is not equal to 1 but takes the pointed function, the rule is not obeyed fully. Actually, the diagonal matrix could be ordered according to the requirement of the researchers. One can range the fixed function term at each proper position on diagonal in the matrix, even can distribute the fixed function to the terms more than one but keeping the value of the determinant unchanged.

Though the solving is just for $G_0(t)$ but $G(t)$, it would not increase difficulty. In fact, the solving process and difficulty for both matrices are the same.

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