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QUANTILE-GENERATED FAMILY OF DISTRIBUTIONS: A NEW METHOD FOR GENERATING CONTINUOUS DISTRIBUTIONS

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Abstract

In this paper, we propose quantile generated family of distributions, where the generator is given by the quantile density function of a continuous random variable. Some statistical properties associated with this new family of distributions are obtained. A special case of this new family of distributions is associated to a non-linear regression problem.

1. Introduction

Eugene et al. [1] proposed the beta-generated family of distributions, where the beta distribution with PDF say b is used as the generator. The CDF of the beta generated distribution is then defined as

$$G(x) = \int_0^{F(x)} b(t)dt,$$

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where F is the CDF of any random variable. If X is continuous, the corresponding PDF of the beta generated distribution is

$$g(x) = \frac{f(x)}{B(\alpha, \beta)} F^{\alpha-1}(x)(1 - F(x))^{\beta-1}$$

$\alpha > 0, \beta > 0,$ where $B(\alpha, \beta)$ is the beta function. The PDF of the beta-generated distribution can be considered as a generalization of the distribution of order statistic [1, 2]. By applying different $F(x)$, many authors have studied variants of the beta-generated distribution and its applications, and for examples, see [3, 4, 5].

Alzaatreh et al. [6] proposed a general method by replacing the beta PDF of Eugene et al. [1] with a general PDF say r of a continuous random variable say T and replacing $F(x)$, the CDF of X, with a weighted version, $W(F(x))$, where $W(F(x))$ admits the following properties

(a) $W(F(x)) \in [a, b]$,

(b) W is a monotone increasing and differentiable function,

(c) $\lim_{x \to -\infty} W(F(x)) = a$ and $\lim_{x \to +\infty} W(F(x)) = b$,

where $[a, b]$ is the support of the random variable T for $-\infty \leq a < b \leq \infty$. The CDF of the $T - X(W)$ family is then defined as

$$G(x) = \int_a^{W(F(x))} r(t)dt.$$

If R is the CDF of T, then the CDF of the $T - X(W)$ family can be written as

$$G(x) = R(W(F(x)))$$

and the corresponding PDF (if it exists) can be written as

$$g(x) = r(W(F(x)))\frac{d}{dx}W(F(x)).$$

By applying different $F(x)$ and W, variants of the $T - X(W)$ family have been investigated, for examples, see [7, 8, 9].

Aljarrah et al. [10] proposed a generalization of the method of Alzaatreh et al. [6] by introducing a new weight function that is based on the quantile function associated with a random variable *Y*. Let Q_Y be the quantile function associated with the random variable *Y* whose CDF is continuous and strictly increasing, then the CDF of the $T - X(Y)$ family is then defined as

$$G(x) = \int_a^{Q_Y(F(x))} r(t)dt.$$

If *R* is the CDF of *T*, then the CDF of the $T - X(Y)$ family can be written as

$$G(x) = R(Q_Y(F(x)))$$

and the corresponding PDF is given by

$$g(x) = \frac{f(x)}{p(Q_Y(F(x)))} r(Q_Y(F(x))),$$

where *p* is the PDF of *Y*. Variants of the $T - X(Y)$ family have been explored, for example, see [11].

This paper is organized as follows. In Section 2, we describe a new technique which is also a "quantile approach" which differs from Aljarrah et al. [10], and in Section 3, we give some families for the new technique arising from the $T - X(W)$ family. In Section 4, we discuss some statistical properties. In Section 5, a special distribution arising from the new technique is presented, and we discuss Moors' kurtosis and Galtons' skewness based on quantile function. In Section 6, the special distribution is associated to a non-linear regression problem.

2. The New Technique

Consider the $T - X(W)$ family of Alzaatreh et al. [6] and the $T - X(Y)$ family of Aljarrah et al. [10]. Since *T* has an absolutely continuous distribution with PDF $r(t)$ and CDF $R(t)$, then the quantile function $Q(t)$ is written as $Q(t) = R^{-1}(t)$, $0 < t < 1$, and the quantile density function is written as $q(t) = \frac{dQ(t)}{dt} = \frac{1}{r(Q(t))}$, $0 < t < 1$.

To summarize, since T has an absolutely continuous distribution, we replace the integrand of $T-X(W)$ family with the quantile density function associated with T and do the same for the $T-X(Y)$ family. Thus the quantile generated distribution arising from the $T-X(W)$ family which we call the $q_T - X(W)$ family has CDF defined as

$$G(x)=\int_a^{W(F(x))}\frac{1}{r(Q(t))}dt$$

and the quantile generated distribution arising from the $T-X(Y)$ family which we call the $q_T - X(Y)$ family has CDF defined as

$$K(x)=\int_a^{Q_Y(F(x))}\frac{1}{r(Q(t))}dt.$$

Note that the CDF of the $q_T - X(W)$ family is

$$G(x)=Q(W(F(x)))$$

and the CDF of the $q_T - X(Y)$ family is

$$K(x)=Q(Q_Y(F(x))).$$

The PDF of the $q_T - X(W)$ family is given by

$$g(x)=\frac{1}{r(Q(W(F(x))))}\frac{1}{dx}W(F(x))$$

and the PDF of the $q_T - X(Y)$ family is given by

$$k(x)=\frac{1}{r(Q(Q_Y(F(x))))}\frac{f(x)}{p(Q_Y(F(x)))}.$$

3. Some Families Arising from $q_T - X(W)$

Theorem 3.1. *Assume the support of T is $[0,\infty)$, then we have the following*

(a) *If $W(F(x))=H(x)$, then $g(x)=\dfrac{h(x)}{r(G(x))}$, where $h(x)$ is the Hazard* *function of the random variable X with CDF* $F(X)$ and $H(x)$ *is the cumulative Hazard function of the random variable X with CDF* $F(X)$.

(b) *If* $W(F(x)) = \dfrac{F(x)}{1 - F(x)}$, *then* $g(x) = \dfrac{f(x)}{[1 - F(x)]^2 r(G(x))}$.

(c) *If* $W(F(x)) = -\log(1 - F^{\alpha}(x))$, *then* $g(x) = \dfrac{\alpha f(x) F^{\alpha - 1}(x)}{[1 - F^{\alpha}(x)] r(G(x))}$.

(d) *If* $W(F(x)) = \dfrac{F^{\alpha}(x)}{1 - F^{\alpha}(x)}$, *then* $g(x) = \dfrac{\alpha f(x) F^{\alpha - 1}(x)}{[1 - F^{\alpha}(x)]^2 r(G(x))}$.

Proof. In all cases the proof follows from the definition of $G(x)$ and $g(x)$. Moreover, in the case of (a), we note that $H(x) = -\log(1 - F(x))$ and $h(x) = \dfrac{f(x)}{1 - F(x)}$.

Theorem 3.2. *Assume the support of T is* $(-\infty, \infty)$, *then we have the following*

(a) *If* $W(F(x)) = \log(-\log(1 - F(x)))$, *then*

$$g(x) = \frac{f(x)}{[F(x) - 1][\log(1 - F(x))] r(G(x))}.$$

(b) *If* $W(F(x)) = \log\left(\dfrac{F(x)}{1 - F(x)}\right)$, *then* $g(x) = \dfrac{h(x)}{F(x) r(G(x))}$, *where* $h(x)$ *is the Hazard function of the random variable X with CDF* $F(X)$.

(c) *If* $W(F(x)) = \log(-\log(1 - F^{\alpha}(x)))$, *then*

$$g(x) = \frac{\alpha f(x) F^{\alpha - 1}(x)}{(F^{\alpha}(x) - 1)[\log(1 - F^{\alpha}(x)] r(G(x))}.$$

(d) *If* $W(F(x)) = \log\left(\dfrac{F^{\alpha}(x)}{1 - F^{\alpha}(x)}\right)$, *then* $g(x) = \dfrac{\alpha f(x)}{F(x)(1 - F^{\alpha}(x)) r(G(x))}$.

Proof. In all cases the proof follows from the definition of $G(x)$ and $g(x)$.

Moreover, in the case of (b), we note that $h(x) = \frac{f(x)}{1 - F(x)}$.

Theorem 3.3. *Assume the support of T is* $[0, \infty)$ *and let* $W(F(x)) = -\log(1 - F(x))$, *then we have the following*

(a) *If T follows the Exponential distribution with parameter* θ, *then* $g(x) = \frac{h(x)}{\theta(1 - H(x))}$, *where* $h(x)$ *is the Hazard function of the random variable X with CDF* $F(X)$ *and* $H(x)$ *is the cumulative Hazard function of the random variable X with CDF* $F(X)$.

(b) *If T follows the Weibull distribution with parameters* k, λ, *then* $g(x) = \frac{\lambda m(x)^{\frac{1-k}{k}} h(x)}{k(1 - H(x))}$, *where* $h(x)$ *is the Hazard function of the random variable X with CDF* $F(X)$, $H(x)$ *is the cumulative Hazard function of the random variable X with CDF* $F(X)$, *and* $m(x) = \ln\left(\frac{1}{1 - H(x)}\right)$.

(c) *If T follows the Rayleigh distribution with parameter* σ, *then* $g(x) = \frac{\sigma h(x)}{\sqrt{2m(x)}(1 - H(x))}$, *where* $h(x)$, $H(x)$ *and* $m(x)$ *are defined as before*.

(d) *If T follows the Lomax distribution with parameters* α, λ, *then* $g(x) = \frac{\lambda h(x)}{\alpha V(x)^{\alpha+1}}$, *where* $V(x) = (1 - H(x))^{\frac{1}{\alpha}}$, *and* $h(x)$ *and* $H(x)$ *are defined as before*.

Proof. (a) Follows from the definition of $g(x)$ and noting that $r(t) = \theta e^{-\theta t}$, $Q = R^{-1}$, where $R(t) = 1 - e^{-\theta t}$, and $H(x)$ and $h(x)$ are defined as before.

(b) Follows from the definition of $g(x)$ and noting that $r(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}$

$e^{-\left(\frac{t}{\lambda}\right)^k}$, and $Q = R^{-1}$, where $R(t) = 1 - e^{-\left(\frac{t}{\lambda}\right)^k}$.

(c) Follows from the definition of $g(x)$ and noting that $r(t) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}$, and $Q = R^{-1}$, where $R(t) = 1 - e^{-\frac{t^2}{2\sigma^2}}$.

(d) Follows from the definition of $g(x)$ and noting that $r(t) = \frac{\alpha}{\lambda} \left[1 + \frac{t}{\lambda}\right]^{-(\alpha+1)}$, and $Q = R^{-1}$, where $R(t) = \frac{\lambda\left[1 - (1-t)^{\frac{1}{\lambda}}\right]}{(1-t)^{\frac{1}{\lambda}}}$.

Theorem 3.4. *Assume support of* T *is* $(-\infty, \infty)$ *and* $W(F(x)) = \log(H(x))$, *where* $H(x)$ *is defined as before, then we have the following*

(a) *Assume* T *follows Cauchy distribution with parameters* t_0 *and* γ, *then* $g(x) = \frac{\pi\gamma[\tan^2(B(x)) + 1]h(x)}{H(x)}$, *where* $B(x) = \frac{(2\log(H(x)) - 1)\pi}{2}$, *and* $H(x)$ *and* $h(x)$ *are defined as before.*

(b) *Assume* T *follows Gumbel or Type I Extreme Value distribution with parameters* σ *and* μ, *then* $g(x) = \frac{\sigma e^{Z(x)} h(x)}{Z(x)H(x)}$, *where* $Z(x) = \ln\left(\frac{1}{\log(H(x))}\right)$, $H(x)$ *and* $h(x)$ *are defined as before.*

(c) *Assume* T *follows Laplace distribution with parameters* μ *and* b, *and let* $C(x) = \frac{1}{2 - 2H(x)}$, then

$$g(x) = \begin{cases} \dfrac{2bh(x)C(x)^{\frac{1}{b}}}{H(x)} & \text{if } H(x) \geq \mu, \\ \dfrac{bh(x)}{[H(x)]^2} & \text{if } H(x) < \mu, \end{cases}$$

where $H(x)$ *and* $h(x)$ *are defined as before.*

Proof. (a) Follows from the definition of $g(x)$, and noting that $r(t) = \frac{1}{\pi\gamma}\left[\frac{\gamma^2}{(t-t_0)^2+\gamma^2}\right]$, and $Q = R^{-1}$, where $R(t) = \frac{1}{\pi}\tan^{-1}\left(\frac{t-t_0}{\gamma}\right) + \frac{1}{2}$.

(b) Follows from the definition of $g(x)$, and noting that $r(t) = \frac{1}{\sigma}n(t)e^{-n(t)}$, where $n(t) = e^{-\frac{(t-\mu)}{\sigma}}$, *and* $Q = R^{-1}$, where $R(t) = e^{-n(t)}$.

(c) Follows from the definition of $g(x)$, and noting that

$$r(t) = \begin{cases} \frac{1}{2b}e^{\frac{\mu-t}{b}} & \text{if } t \geq \mu, \\ \frac{1}{2b}e^{\frac{t-\mu}{b}} & \text{if } t < \mu \end{cases}$$

and $Q = R^{-1}$, where

$$R(t) = \begin{cases} \frac{1}{2} + \frac{1}{2}\left[1 - e^{-\frac{(t-\mu)}{b}}\right] & \text{if } t \geq \mu, \\ \frac{1}{2} - \frac{1}{2}\left[1 - e^{\frac{(t-\mu)}{b}}\right] & \text{if } t < \mu. \end{cases}$$

4. Some Statistical Properties of $q_T - X(W)$ Family

We will write Q_T for the quantile function associated with the continuous random variable *T*, and write $Q_{q_T - X(W)}$ for the quantile function arising from $q_T - X(W)$ family given by *W*.

Theorem 4.1. *Assume the support of T* is $[0, \infty)$, *then we have the following*

(a) $Q(\lambda)_{q_T - X(H(\lambda))} = F^{-1}(1 - e^{-R(\lambda)}),\ 0 < \lambda < 1.$

(b) $Q(\lambda)_{q_T - X\left(\frac{F(\lambda)}{1-F(\lambda)}\right)} = F^{-1}\left(\frac{R(\lambda)}{1+R(\lambda)}\right), \ 0 < \lambda < 1.$

Proof. (a) By definition, $Q(\lambda)_{q_T - X(H(\lambda))} = G^{-1}(\lambda),$ where $G(\lambda) = Q_T(H(\lambda))$ and $H(\lambda) = -\log(1 - F(\lambda)).$ Since $Q_T = R^{-1},$ it is enough to check that

$$G(Q(\lambda)_{q_T - X(H(\lambda))}) = \lambda \text{ or } Q(\lambda)_{q_T - X(H(\lambda))}(G(\lambda)) = \lambda.$$

(b) By definition, $Q(\lambda)_{q_T - X\left(\frac{F(\lambda)}{1-F(\lambda)}\right)} = G^{-1}(\lambda),$ where $G(\lambda) = Q_T \left(\frac{F(\lambda)}{1-F(\lambda)}\right).$ Since $Q_T = R^{-1},$ it is enough to check that

$$G\left(Q(\lambda)_{q_T - X\left(\frac{F(\lambda)}{1-F(\lambda)}\right)}\right) = \lambda \text{ or } Q(\lambda)_{q_T - X\left(\frac{F(\lambda)}{1-F(\lambda)}\right)}(G(\lambda)) = \lambda.$$

Theorem 4.2. *Assume the support of T is* $(-\infty, \infty),$ *then we have the following*

(a) $Q(\lambda)_{q_T - X(\log(H(\lambda)))} = F^{-1}(1 - e^{-e^{R(\lambda)}}), \ 0 < \lambda < 1.$

(b) $Q(\lambda)_{q_T - X\left(\log\left(\frac{F(\lambda)}{1-F(\lambda)}\right)\right)} = F^{-1}\left(\frac{e^{R}(\lambda)}{1+e^{R(\lambda)}}\right), \ 0 < \lambda < 1.$

Proof. (a) By definition, $Q(\lambda)_{q_T - X(\log(H(\lambda)))} = G^{-1}(\lambda),$ where $G(\lambda) = Q_T(\log(H(\lambda)))$ and $H(\lambda) = -\log(1 - F(\lambda)).$ Since $Q_T = R^{-1},$ it is enough to check that

$$G(Q(\lambda)_{q_T - X(\log(H(\lambda)))}) = \lambda \text{ or } Q(\lambda)_{q_T - X(\log(H(\lambda)))}(G(\lambda)) = \lambda.$$

(b) By definition, $Q(\lambda)_{q_T - X\left(\log\left(\frac{F(\lambda)}{1-F(\lambda)}\right)\right)} = G^{-1}(\lambda),$ where $G(\lambda) = Q_T \left(\log\left(\frac{F(\lambda)}{1-F(\lambda)}\right)\right).$ Since $Q_T = R^{-1},$ it is enough to check that

$$G\left(Q(\lambda)_{q_T - X\left(\log\left(\frac{F(\lambda)}{1-F(\lambda)}\right)\right)}\right) = \lambda \text{ or } Q(\lambda)_{q_T - X\left(\log\left(\frac{F(\lambda)}{1-F(\lambda)}\right)\right)} = (G(\lambda)) = \lambda.$$

Theorem 4.3. *Assume X follows the family of distributions given by Theorem 3.1(a), then the Shannon Entropy of* X, S_X *is given by*

$$S_X = -E[\log f\{F^{-1}(1-e^{-T})\}] - \mu_T + \gamma_T,$$

where μ_T *and* γ_T *are the mean and Shannon Entropy for the random variable T with pdf* $\frac{1}{r(Q_T(t))}$.

Proof. From $h(x) = \frac{f(x)}{1-F(x)}$, $G(x) = Q_T(H(x))$ and $H(x) = -\log(1-F(x))$, observe that

$$\begin{aligned} S_X &= E[-\log\{g(X)\}] \\ &= -E[\log\{f(X)\}] + E[\log\{(1-F(X))\}] + E\left[-\log\left\{\frac{1}{r(G(X))}\right\}\right]. \end{aligned}$$

Since $T = -\log(1-F(X))$ has PDF $\frac{1}{r(Q_T(t))}$, then

$$E[\log\{f(X)\}] = E[\log f\{F^{-1}(1-e^{-T})\}]$$

and

$$E[\log\{(1-F(X))\}] = E[-T] = -E[T] = -\mu_T$$

and

$$E\left[-\log\left\{\frac{1}{r(G(X))}\right\}\right] = E\left[-\log\left\{\frac{1}{r(Q_T(t))}\right\}\right] = \gamma_T.$$

Theorem 4.4. *Assume X follows the family of distributions given by Theorem 3.1(b), then the Shannon Entropy of* X, S_X *is given by*

$$S_X = -E\left[\log f\left\{F^{-1}\left(\frac{T}{1+T}\right)\right\}\right] - 2\mu_{\log(1+T)} + \gamma_T,$$

where $\mu_{\log(1+T)}$ *is the mean of the random variable* $\log(1+T)$ *with PDF*

$\frac{1}{r(Q_T(t))}$ *and* γ_T *is the Shannon Entropy of the of the random variable T with PDF* $\frac{1}{r(Q_T(t))}$.

Proof. By definition,

$$S_X = E[-\log(g(X))] = -E[\log(f(X))]$$
$$+ 2E[\log(1 - F(X))] + E\left[-\log\left\{\frac{1}{r(G(X))}\right\}\right].$$

Since $T = \frac{F(X)}{1 - F(X)}$ has PDF $\frac{1}{r(Q_T(t))}$, it implies that $E[\log(f(X))] = E\left[\log f\left\{F^{-1}\left(\frac{T}{1+T}\right)\right\}\right]$. On the other hand $E\left[-\log\left\{\frac{1}{r(G(X))}\right\}\right] = E\left[-\log\left\{\frac{1}{r(Q_T(t))}\right\}\right] = \gamma_T$. Since the random variable $V = -\log(1 - F(X))$ also has PDF $\frac{1}{r(Q_T(t))}$, it follows that the relationship between the random variables T and V is given by $V = \log(1 + T)$. Thus

$$E[\log(1 - F(X))] = E[-V] = -E[V] = -\mu_V = -\mu_{\log(1+T)}.$$

5. A Special Case of $q_T - X(W)$ Family

Let the support of T be $(0, 1) \subset (0, \infty)$ and $W(F(x)) = -\log(1 - F(x))$, where $F(x)$ is the CDF of X. In particular, let T follow the (standard) Power distribution with parameter $r > 0$, and X follow the (standard) Pareto distribution with parameter $a > 0$.

Theorem 5.1. *The quantile function of the* (*standard*) *Power distribution is given by* $Q_T(t) = t^{\frac{1}{r}}$, $0 < t < 1$, *and* $r > 0$.

Proof. $Q_T(t) = F^{-1}$, where F is the CDF of (standard) Power distribution.

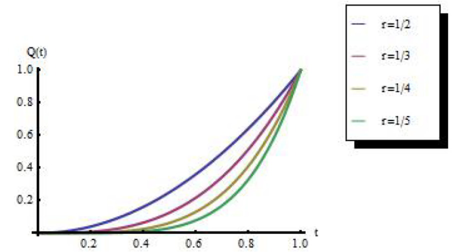

Figure 1. Quantile function of Power distribution for various values of the parameter r.

Remark 5.2. Since T follows the Power distribution and X follows the Pareto distribution, we will call this special case of the $q_T - X(W)$ family, the Quantile Power-Pareto Distribution induced by W.

Theorem 5.3. *The CDF of the Quantile Power-Pareto Distribution induced by* W, *is given by*, $G(x) = a^{\frac{1}{r}}(\log(x))^{\frac{1}{r}}$, *where* $a > 0$, $r > 0$, *and* $x > 0$.

Proof. It follows from the definition of $G(x)$, and noting that $F(x)$ is the CDF of (standard) Pareto distribution with parameter $a > 0$ and Q is given by Theorem 5.1.

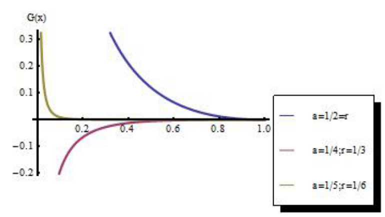

Figure 2. CDF of Quantile Power-Pareto Distribution induced by W for various values of the parameters r and a.

Theorem 5.4. *The PDF of the Quantile Power-Pareto Distribution induced by* W *is given by* $g(x) = \dfrac{a^{\frac{1}{r}}(\log(x))^{\frac{1-r}{r}}}{rx}$, *where* $a > 0$, $r > 0$, *and* $x > 0$.

Proof. One can obtain it directly from the definition of $g(x)$ or by noting that

$g(x) = \frac{d}{dx} G(x),$ where $G(x)$ is the CDF of the Quantile Power-Pareto Distribution induced by W.

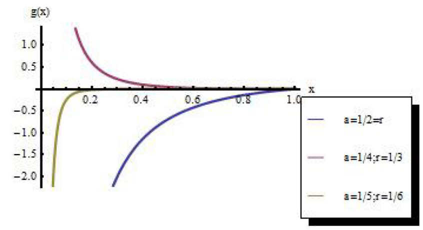

Figure 3. PDF of Quantile Power-Pareto Distribution induced by W for various values of the parameters r and a.

In order to discuss Moors' Kurtosis [12] and Galton's Skewness [13], we need the quantile function of the Quantile Power-Pareto Distribution induced by W. In particular, we have the following

Theorem 5.5. *The quantile function of the Quantile Power-Pareto Distribution induced by W is given by* $Q(x) = 10^{m(x)},\ m(x) = \frac{x^{\frac{1}{r}}}{a^{r^{\frac{1}{2}}}},\ 0 < x < 1,\ r > 0,\ a > 0.$

Proof. $Q = G^{-1},$ where G is the CDF of the Quantile Power-Pareto Distribution induced by W.

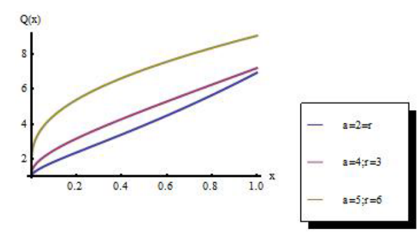

Figure 4. Quantile function of Quantile Power-Pareto Distribution induced by W for various values of the parameters r and a.

Now we have the following characterization of Galton's Skewness based on the quantile function of the Quantile Power-Pareto Distribution induced by W.

Theorem 5.6. *The Quantile Power-Pareto Distribution induced by W is symmetric if the parameter r is sufficiently large*

Proof. By definition, $S = \dfrac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$. Obviously, S makes sense if $Q\left(\frac{6}{8}\right) \neq Q\left(\frac{2}{8}\right)$. Thus, the distribution is symmetric if

$$Q\left(\frac{6}{8}\right) + Q\left(\frac{2}{8}\right) = 2Q\left(\frac{4}{8}\right). \tag{7}$$

In terms of Theorem 5.5, (7) becomes

$$10^{\frac{\left(\frac{6}{8}\right)^{\frac{1}{r}}}{a^{\frac{1}{r^2}}}} + 10^{\frac{\left(\frac{2}{8}\right)^{\frac{1}{r}}}{a^{\frac{1}{r^2}}}} = 2 \cdot 10^{\frac{\left(\frac{4}{8}\right)^{\frac{1}{r}}}{a^{\frac{1}{r^2}}}}. \tag{8}$$

If there exist $r > 0$ such that $\left(\frac{6}{8}\right)^{\frac{1}{r}} = \left(\frac{2}{8}\right)^{\frac{1}{r}} = \left(\frac{4}{8}\right)^{\frac{1}{r}}$, then equality holds in (8). However, there is no such $r > 0$. Now observe that

$$\lim_{r\to\infty}\left(\frac{6}{8}\right)^{\frac{1}{r}} = \lim_{r\to\infty}\left(\frac{2}{8}\right)^{\frac{1}{r}} = \lim_{r\to\infty}\left(\frac{4}{8}\right)^{\frac{1}{r}} = 1.$$

Thus if r is sufficiently large, then equality holds in (8) and the distribution is symmetric. Clearly symmetry of distribution is independent of the parameter $a > 0$.

Theorem 5.9. *The Quantile Power-Pareto Distribution induced by W is right skewed* $\Leftrightarrow Q\left(\frac{4}{8}\right) > Q\left(\frac{2}{8}\right)$.

Proof. $(\Rightarrow)$ Suppose $S > 0$ but $Q\left(\frac{4}{8}\right) \leq Q\left(\frac{2}{8}\right)$. Since $S > 0$, we consider two cases

Case I.

$$Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)>2Q\left(\frac{4}{8}\right),$$

$$Q\left(\frac{6}{8}\right)>Q\left(\frac{2}{8}\right).$$

Case II.

$$Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)<2Q\left(\frac{4}{8}\right),$$

$$Q\left(\frac{6}{8}\right)<Q\left(\frac{2}{8}\right).$$

If we combine Case II with $Q\left(\frac{4}{8}\right)\leq Q\left(\frac{2}{8}\right)$, then there is nothing to prove. Considering Case I with $Q\left(\frac{4}{8}\right)\leq Q\left(\frac{2}{8}\right)$ yields $Q\left(\frac{2}{8}\right)=Q\left(\frac{6}{8}\right)$, which is a contradiction (the denominator of S cannot be zero).

($\Leftarrow$) Suppose $Q\left(\frac{4}{8}\right)>Q\left(\frac{2}{8}\right)$ but $S\leq 0$. Without loss of generality, we may assume $S=0$, then by Theorem 5.6

$$\lim_{r\to\infty} 10^{\,a^{\frac{\left(\frac{4}{8}\right)^{\frac{1}{r}}}{\frac{1}{r^2}}}} > \lim_{r\to\infty} 10^{\,a^{\frac{\left(\frac{2}{8}\right)^{\frac{1}{r}}}{\frac{1}{r^2}}}}$$

which is a contradiction.

Theorem 5.10. *The Quantile Power-Pareto Distribution induced by W is left skewed* $\Leftrightarrow Q\left(\frac{2}{8}\right)<Q\left(\frac{4}{8}\right)$.

Proof. ($\Rightarrow$) Suppose $S<0$ but $Q\left(\frac{2}{8}\right)\geq Q\left(\frac{4}{8}\right)$. Since $S<0$, we consider two cases

Case I.

$$Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)<2Q\left(\frac{4}{8}\right),$$

$$Q\left(\frac{6}{8}\right)>Q\left(\frac{2}{8}\right).$$

Case II.

$$Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)>2Q\left(\frac{4}{8}\right),$$

$$Q\left(\frac{6}{8}\right)<Q\left(\frac{2}{8}\right).$$

If we combine Case II with $Q\left(\frac{2}{8}\right)\geq Q\left(\frac{4}{8}\right),$ then there is nothing to prove. Considering Case I with $Q\left(\frac{2}{8}\right)\geq Q\left(\frac{4}{8}\right)$ yields $Q\left(\frac{2}{8}\right)=Q\left(\frac{6}{8}\right),$ which is a contradiction (the denominator of *S* cannot be zero).

($\Leftarrow$) Suppose $Q\left(\frac{2}{8}\right)>Q\left(\frac{4}{8}\right)$ but $S\geq 0.$ Without loss of generality, we may assume $S=0,$ then by Theorem 5.6

$$\lim_{r\to\infty} 10^{\frac{\left(\frac{2}{8}\right)^{\frac{1}{r}}}{a^{\frac{1}{r^2}}}} < \lim_{r\to\infty} 10^{\frac{\left(\frac{4}{8}\right)^{\frac{1}{r}}}{a^{\frac{1}{r^2}}}}$$

which is a contradiction.

By definition Moors' Kurtosis is given by

$$K=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}.$$

If we use Theorem 5.5 in the above, we get the following

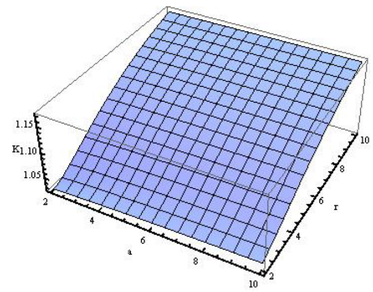

Figure 5. Kurtosis of Quantile Power-Pareto Distribution induced by W for $2 < r < 10$ and $2 < a < 10.$

From the figure, we see that for the "range" of parameter values, the Kurtosis is increasing, it must be the case the tail of the Quantile Power-Pareto Distribution induced by W is heavy.

6. A Non-linear Regression Application

We are interested in the following problem: Given n points $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$, which function $p(x)$ is closest to the given points? Recall by the method of least squares the "best" function $p(x)$ is the one whose coefficients minimize the function L, where

$$L = \sum_{i=1}^{n} [y_i - p(x_i)]^2.$$

If $p(x)$ is a linear function, say $p(x) = a + bx,$ then we have the following

Theorem 6.1 (Theorem 10.4.1 [14]). *Given n points $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$, the straight line $y = a + bx$ minimizing*

$$L = \sum_{i=1}^{n} [y_i - p(x_i)]^2$$

has slope

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

and y-intercept

$$a = \frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n}.$$

When the relationship between x and y is nonlinear it is often possible to "transform" x and y into a new set of variables - x' and y' - that are linearly related, and the previous theorem can be applied to the (x_i', y_i')'s. Once the straight line has been applied to the (x_i', y_i')'s, we can then transform back to the original data.

Among mammals, the relationship between the age at which an animal develops locomotion and the age at which it first begins to play has been widely studied. Listed below from Table 10.4.6 [14] are typical "onset" times for locomotion and for play in 11 different species (see page 617 of [14]).

Species	Locomotion begins, x(i) (days)	Play begins, y(i) (days)
Homo sapiens	360	90
Gorilla gorilla	165	105
Felis catus	21	21
Canis familiaris	23	26
Rattus norvegicus	11	14
Turdus merula	18	28
Macaca mulatta	18	21
Pan troglodytes	150	105
Saimiri sciurens	45	68
Cercocebus alb.	45	75
Tamiasciureus hud.	18	46

Figure 6. "Onset" times for locomotion and for play.

When the data was graphed in Figure 10.4.7 [14], it was observed via Theorem 6.1, that the equation $y = 5.42x^{0.56}$ was a good fit.

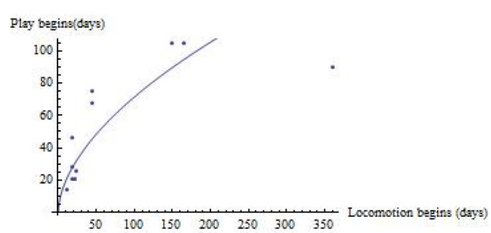

Figure 7. Regression of play onset versus locomotion onset.

In this section, we want to know if the Quantile Power-Pareto Distribution induced by W is a "best" function to play onset versus locomotion onset. For this reason let $y = g(x)$, and notice that we can write the PDF of the Quantile Power-Pareto Distribution induced by W as $y' = a' + x'$, where

$$y' = \log(y),$$

$$a' = \log\left\{\frac{a^{\frac{1}{r}}}{r}\right\}$$

and

$$x' = \frac{1-r}{r}\log[\log(x)] + \log\left[\frac{1}{x}\right].$$

Since x' is not independent of r, let us fix $r = \frac{1}{2}$, then $y' = a' + x'$ can be written as $y'' = a'' + x''$, where

$$y'' = y' = \log(y),$$

$$a'' = \log[2a^2],$$

$$x'' = \log[\log(x)] + \log\left[\frac{1}{x}\right].$$

Therefore, we transform (x_i, y_i) to (x_i'', y_i''), and apply Theorem 6.1 to find a, since $r = \frac{1}{2}$ is fixed.

When we make the transformation of the data in Figure 6, (x_i, y_i), to (x_i'', y_i''), Theorem 6.1 implies the line of best fit is given by $y'' = -0.738033x'' + 1.9697$. When the line of best fit is fitted to the transformed data, we get the following

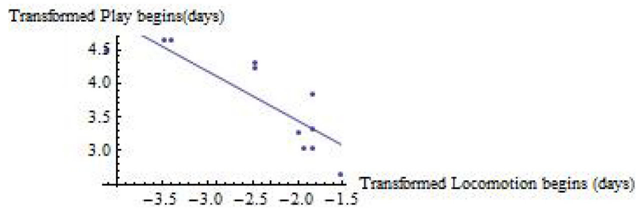

Figure 8. Regression of transformed play onset versus transformed locomotion onset.

From the above figure, we have a negative relationship between play onset and locomotion onset which is contradictory to Figure 7, which shows there is a positive relationship, therefore if we fix $r = \frac{1}{2}$, the Quantile Power-Pareto Distribution induced by *W* would not be a good fit or a "best function". By solving $\log[2a^2] = 1.9697$, we get *a*, and then by back-transforming and fitting to the original data, we get the following

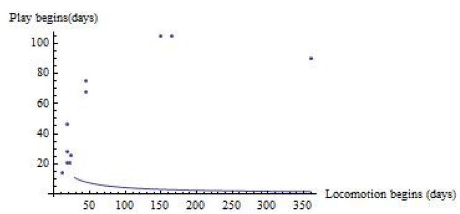

Figure 9. Regression of play onset versus locomotion onset.

However, we can still show that play onset versus locomotion onset is Quantile Power-Pareto Distributed with respect to *W*. We begin with the following

Theorem 6.2. *The Quantile Power-Pareto Distribution induced by W can be written as*

$$y = \frac{a^{\frac{1}{r}}}{r} x^{\frac{1-r}{r}}$$

provided x' *is its upper bound.*

Proof. Notice that $\frac{1-r}{r} < \frac{1}{r}$ for $r > 0$ and $\log(x) \leq x$ for $x > 0$. Now observe that

$$x' = \frac{1-r}{r}\log[\log(x)] + \log\left[\frac{1}{x}\right]$$

$$\leq \frac{1}{r}\log[\log(x)] + \log\left(\frac{1}{x}\right)$$

$$\leq \frac{1}{r}\log(x) + \log\left(\frac{1}{x}\right)$$

$$= \frac{1}{r}\log(x) - \log(x)$$

$$= \frac{1-r}{r}\log(x).$$

Therefore if x' is its upper bound, then $y' = a' + x'$ can be written as

$$\log(y) = \log\left(\frac{a^{\frac{1}{r}}}{r}\right) + \frac{1-r}{r}\log(x)$$

and it follows that

$$y = \frac{a^{\frac{1}{r}}}{r}x^{\frac{1-r}{r}}.$$

Corollary 6.3. *Onset play versus locomotion play is Quantile Power-Pareto Distributed with respect to W if* $a = \left(\frac{5.42}{1.56}\right)^{\frac{1}{1.56}}$ *and* $r = \frac{1}{1.56}$.

Proof. From the previous theorem, we get $y = 5.42x^{0.56}$.

Remark 6.4. The function in Corollary 6.3 has been graphed in Figure 7 with the original data.

References

[1] N. Eugene, C. Lee and F. Famoye, The beta-normal distribution and its applications, Comm. Stat.: Theory and Methods 31(4) (2002), 497-512.

[2] M. C. Jones, Families of distributions arising from distributions of order statistics, Test 13 (2004), 1-43.

[3] A. Akinsete, F. Famoye and C. Lee, The beta-Pareto distribution, Statistics 42 (2008), 547-563.

[4] G. M. Cordeiro and A. J. Lemonte, The β-Birnbaum-Saunder distribution: an improved distribution for fatigue life modeling, Comput. Stat. Data Anal. 55(3) (2011), 1445-1461.

[5] A. Alshawarbeh, C. Lee and F. Famoye, The beta-Cauchy distribution, J. Prob. Stat. Sci. 10 (2012), 41-58.

[6] A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, METRON 71(1) (2013b), 63-79.

[7] A. Alzaatreh, F. Famoye and C. Lee, Gamma-Pareto distribution and its applications, J. Modern Appl. Statistical Methods 11(1) (2012a), 78-94.

[8] A. Alzaatreh, C. Lee and F. Famoye, On the discrete analogues of continuous distributions, Stat. Methodol. 9 (2012b), 589-603.

[9] A. Alzaatreh, F. Famoye and C. Lee, Weibull-Pareto distribution and its applications, Comm. Stat.: Theory and Methods 42 (2013a), 1673-1691.

[10] Mohammad A. Aljarrah, Carl Lee and Felix Famoye, On generating T - X family of distributions using quantile functions, J. Stat. Distribut. Appl. (2014), 1-2.

[11] Mohammad A. Aljarrah, Felix Famoye and Carl Lee, A New Weibull-Pareto distribution, Comm. Stat.: Theory and Methods 44(19) (2015).

[12] J. J. Moors, A quantile alternative for kurtosis, Statistician 37 (1988), 25-32.

[13] F. Galton, Enquiries into Human Faculty and its Development, Macmillan and Company, London, 1883.

[14] Richard J. Larsen and Morris L. Marx, An Introduction to Mathematical Statistics and its Applications, 2nd ed., Prentice-Hall, 1986.