Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 5, Issue 2, 2016, Pages 37-48
This paper is available online at http://www.frdint.com/
Published online April 13, 2016

PSEUDOMETRIZABLE SPACES ON A FINITE SET WITH A CHECK

REBECCA STEWARD and CHARLES DORSETT*

Department of Mathematics
Texas A&M University-Commerce
Commerce, Texas 75429
USA
e-mail: charles.dorsett@tamuc.edu

Abstract

Given a finite set, how many topologies can be defined on that set? This question has received considerable attention from the mathematics community. It is known for sets of 18 or less elements. Of these topologies how many are T_0? Of the T_0 topologies how many are topologically distinct? These things have been investigated. Of those topologies how many are pseudometrizable? Amongst those pseudometrizable topologies how many are distinct? Utilizing somewhat tedious determination the answers to the last two questions are known. However, because of the possibility of accidentally overlooking a possible topology, a check would be appropriate and needed. In this paper, a check is done to verify the correctness of the answer to the last question.

1. Introduction

For a finite set there are only finitely many topologies on the set, but how many? R. E. Stong [8] was amongst the first to pose this question in the literature. In response it is currently known the number of labeled topologies on a set with 18 or

Keywords and phrases: pseudometrizable, finite set, topologies.

2010 Mathematics Subject Classification: 54A05, 54A10.

*Corresponding author

Received March 16, 2016; Accepted April 06, 2016

less elements [7]. The set of all topologies on a set without regard to homeomorphic classes is called the set of labeled topologies on the set.

In the paper [7], the number of labeled T_0 spaces, the number of topologically distinct spaces, and the number of topologically distinct T_0 spaces were given for finite sets with 10 or less elements.

For the known number of labeled topologies on a set with 18 or less elements, a pattern was observed: If the number of elements in the finite set is $n = 3p + 2$; $p = 0, 1, ..., 5$, the number of labeled topologies on the set is even and the remaining labeled topologies are odd. Thus the question of whether the pattern continued for all finite sets naturally arose.

The key used to obtain a positive answer to the above question were R_0 spaces, strong Alexandroff spaces, and Sterling numbers of the second type [2]. Thus without knowing exactly how many, a set with 19 elements has an odd number of labeled topologies, a set with 20 elements has an even number of labeled topologies, ... and there is a check for any further calculations.

Definition 1.1. A space (X, T) is R_0 iff for each closed set C and each $x \notin C$, $Cl(\{x\}) \cap C = \phi$ [6].

Definition 1.2. A space (X, T) is strong Alexandroff iff $C(T) = T$, where $C(T)$ denotes the family of closed sets of (X, T) [1].

Sterling numbers of the second kind; $S(n, k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j}(k-j)^n$, give the number of decompositions of a set with finite elements.

With the important role of R_0 spaces in resolving the question above, the question of the number of labeled topologies and the number of topological distinct R_0 spaces on a finite set arose and resolved in a 2014 paper [4].

Since for a finite set all of pseudometrizable, completely normal R_0, normal R_0, regular, weakly Urysohn, R_1, R_0 [3], and completely regular [4] are equivalent, then much knowledge was gained from the 2014 paper [4].

2. The Calculations and Check

Within the 2014 paper [4], a one-to-one correspondence between the decompositions of a finite set and the labeled R_0 spaces on the set was established, a one-to-one correspondence between the ordered vectors for a fixed natural number n and the topologically distinct R_0 spaces was given, and formulas for determining the number of labeled topologies topologically equivalent to each topologically distinct R_0 space were given for future calculations and a check to make sure nothing has been accidentally overlooked. Within this paper, the calculations and checks are given for the sets size $n \leq 10$.

Definition 1.3. Let $n \in \mathbb{N}$ and let $k \in \{1, ..., n\}$. Then a $1 \times k$ vector $[n_{ij}]_{j=1}^{k}$ is an ordered vector for n if $n_{ij} \in \mathbb{N}$ for each j, $n_{ij} \leq n_{ij+1}$, $j = 1, ..., k-1$, and $\sum_{j=1}^{k} n_{ij} = n$ [4].

Theorem 1.1. *Let X be a set with n elements. For the ordered vector* $\vec{v} = [n_{ij}]_{j=1}^{k} = \begin{bmatrix} p_{11} \\ \vdots \\ p_{nk} \end{bmatrix}$, *where* $p_{ij} < p_{ij+1}$; $j = 1, ..., k-1$, *the number of* R_0 *topologies on X topologically equivalent to the topologically distinct* R_0 *space corresponding to* $\vec{v}$ *is* $\dfrac{n!}{p_{11}! \times p_{12}! \times \ldots \times p_{1k}!}$ *and for the ordered vector* $\vec{v} = [n_{ij}]_{j=1}^{k}$, *where the top* n_1 *entries are all equal to* p_1, *followed by* n_2 *entries all equal to* $p_2 > p_1, \ldots,$ *followed by the bottom* n_l *entries all equal to* $p_l > p_{l-1}$ *and* $\sum_{i=1}^{l} n_i p_i = n$, *the number of* R_0 *topologies on X topologically equivalent to the topologically distinct* R_0 *space corresponding to* $\vec{v}$ *is* $\dfrac{n!}{(p_1)!^{n_1}(p_2)!^{n_2} \ldots (p_l)!^{n_l}} \times \dfrac{1}{(n_1)!(n_2)! \ldots (n_l)!}$ [4].

In the results below DT stands for Distinct Topologies and N(ET) stands for the Number of Equivalent Topologies on X for each of the vectors in the columns $k = 1, ..., 10$.

$\lvert X \rvert = 1$	
DT	N(ET)
$k = 1$	
[1]	1

$\lvert X \rvert = 2$			
DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$	
[2]	1	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	1

$\lvert X \rvert = 3$					
DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$	
[3]	1	$\begin{bmatrix}1\\2\end{bmatrix}$	3	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	1

$\lvert X \rvert = 4$							
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$		$k = 4$	
[4]	1	$\begin{bmatrix}1\\3\end{bmatrix}$	4	$\begin{bmatrix}1\\1\\2\end{bmatrix}$	6	$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$	1
		$\begin{bmatrix}2\\2\end{bmatrix}$	3				

$\lvert X \rvert = 5$									
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$		$k = 4$		$k = 5$	
[5]	1	$\begin{bmatrix}1\\4\end{bmatrix}$	5	$\begin{bmatrix}1\\1\\3\end{bmatrix}$	10	$\begin{bmatrix}1\\1\\1\\2\end{bmatrix}$	10	$\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$	1
		$\begin{bmatrix}2\\3\end{bmatrix}$	10	$\begin{bmatrix}1\\2\\2\end{bmatrix}$	15				

$\lvert X\rvert = 6$							
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$		$k = 4$	
[6]	1	$\begin{bmatrix}1\\5\end{bmatrix}$	6	$\begin{bmatrix}1\\1\\4\end{bmatrix}$	15	$\begin{bmatrix}1\\1\\1\\3\end{bmatrix}$	20
		$\begin{bmatrix}2\\4\end{bmatrix}$	15	$\begin{bmatrix}1\\2\\3\end{bmatrix}$	60	$\begin{bmatrix}1\\1\\2\\2\end{bmatrix}$	45
		$\begin{bmatrix}3\\3\end{bmatrix}$	10	$\begin{bmatrix}2\\2\\2\end{bmatrix}$	15		
$k = 5$		$k = 6$					
$\begin{bmatrix}1\\1\\1\\1\\2\end{bmatrix}$	15	$\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$	1				

$\lvert X\rvert = 7$							
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$		$k = 4$	
[7]	1	$\begin{bmatrix}1\\6\end{bmatrix}$	7	$\begin{bmatrix}1\\1\\5\end{bmatrix}$	21	$\begin{bmatrix}1\\1\\1\\4\end{bmatrix}$	35
		$\begin{bmatrix}2\\5\end{bmatrix}$	21	$\begin{bmatrix}1\\2\\4\end{bmatrix}$	105	$\begin{bmatrix}1\\1\\2\\3\end{bmatrix}$	210

		$\begin{bmatrix}3\\4\end{bmatrix}$	35	$\begin{bmatrix}1\\3\\3\end{bmatrix}$	70	$\begin{bmatrix}1\\2\\2\\2\end{bmatrix}$	105
				$\begin{bmatrix}2\\2\\3\end{bmatrix}$	105		
$k = 5$		$k = 6$		$k = 7$			
$\begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix}$	35	$\begin{bmatrix}1\\1\\1\\1\\1\\2\end{bmatrix}$	21	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}$	1		
$\begin{bmatrix}1\\1\\1\\2\\2\end{bmatrix}$	105						

$\lvert X\rvert = 8$							
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$		$k = 4$	
[8]	1	$\begin{bmatrix}1\\7\end{bmatrix}$	8	$\begin{bmatrix}1\\1\\6\end{bmatrix}$	28	$\begin{bmatrix}1\\1\\1\\5\end{bmatrix}$	56
		$\begin{bmatrix}2\\6\end{bmatrix}$	28	$\begin{bmatrix}1\\2\\5\end{bmatrix}$	168	$\begin{bmatrix}1\\1\\2\\4\end{bmatrix}$	420
		$\begin{bmatrix}3\\5\end{bmatrix}$	56	$\begin{bmatrix}1\\3\\4\end{bmatrix}$	280	$\begin{bmatrix}1\\1\\3\\3\end{bmatrix}$	280

		$\begin{bmatrix}4\\4\end{bmatrix}$	35	$\begin{bmatrix}2\\2\\4\end{bmatrix}$	210	$\begin{bmatrix}1\\2\\2\\3\end{bmatrix}$	840
				$\begin{bmatrix}2\\3\\3\end{bmatrix}$	280	$\begin{bmatrix}2\\2\\2\\2\end{bmatrix}$	105
$k=5$		$k=6$		$k=7$		$k=8$	
$\begin{bmatrix}1\\1\\1\\1\\5\end{bmatrix}$	70	$\begin{bmatrix}1\\1\\1\\1\\1\\3\end{bmatrix}$	56	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\2\end{bmatrix}$	28	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}$	1
$\begin{bmatrix}1\\1\\1\\2\\3\end{bmatrix}$	560	$\begin{bmatrix}1\\1\\1\\1\\2\\2\end{bmatrix}$	210				
$\begin{bmatrix}1\\1\\2\\2\\2\end{bmatrix}$	420						

$\lvert X\rvert = 9$									
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k=1$		$k=2$		$k=3$		$k=4$		$k=5$	
[9]	1	$\begin{bmatrix}1\\8\end{bmatrix}$	9	$\begin{bmatrix}1\\1\\7\end{bmatrix}$	36	$\begin{bmatrix}1\\1\\1\\6\end{bmatrix}$	84	$\begin{bmatrix}1\\1\\1\\1\\5\end{bmatrix}$	126

		$\begin{bmatrix}2\\7\end{bmatrix}$	36	$\begin{bmatrix}1\\2\\6\end{bmatrix}$	252	$\begin{bmatrix}1\\1\\2\\5\end{bmatrix}$	756	$\begin{bmatrix}1\\1\\1\\2\\4\end{bmatrix}$	1260
		$\begin{bmatrix}3\\6\end{bmatrix}$	84	$\begin{bmatrix}1\\3\\5\end{bmatrix}$	504	$\begin{bmatrix}1\\1\\3\\4\end{bmatrix}$	1260	$\begin{bmatrix}1\\1\\1\\3\\3\end{bmatrix}$	840
		$\begin{bmatrix}4\\5\end{bmatrix}$	126	$\begin{bmatrix}1\\4\\4\end{bmatrix}$	315	$\begin{bmatrix}1\\2\\2\\4\end{bmatrix}$	1890	$\begin{bmatrix}1\\1\\2\\2\\3\end{bmatrix}$	3780
				$\begin{bmatrix}2\\2\\5\end{bmatrix}$	378	$\begin{bmatrix}1\\2\\3\\3\end{bmatrix}$	2520	$\begin{bmatrix}1\\2\\2\\2\\2\end{bmatrix}$	945
				$\begin{bmatrix}2\\3\\4\end{bmatrix}$	1260	$\begin{bmatrix}2\\2\\2\\3\end{bmatrix}$	1260		
				$\begin{bmatrix}3\\3\\3\end{bmatrix}$	280				
$k = 6$		$k = 7$		$k = 8$		$k = 9$			
$\begin{bmatrix}1\\1\\1\\1\\1\\4\end{bmatrix}$	126	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\3\end{bmatrix}$	84	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\2\end{bmatrix}$	36	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}$	1		

$\begin{bmatrix}1\\1\\1\\1\\2\\3\end{bmatrix}$	1260	$\begin{bmatrix}1\\1\\1\\1\\1\\2\\2\end{bmatrix}$	378						
$\begin{bmatrix}1\\1\\1\\2\\2\\2\end{bmatrix}$	1260								

$\lvert X\rvert = 10$									
DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)	DT	N(ET)
$k = 1$		$k = 2$		$k = 3$		$k = 4$		$k = 5$	
[10]	1	$\begin{bmatrix}1\\9\end{bmatrix}$	10	$\begin{bmatrix}1\\1\\8\end{bmatrix}$	45	$\begin{bmatrix}1\\1\\1\\7\end{bmatrix}$	120	$\begin{bmatrix}1\\1\\1\\1\\6\end{bmatrix}$	210
		$\begin{bmatrix}2\\8\end{bmatrix}$	45	$\begin{bmatrix}1\\2\\7\end{bmatrix}$	360	$\begin{bmatrix}1\\1\\2\\6\end{bmatrix}$	1260	$\begin{bmatrix}1\\1\\1\\2\\5\end{bmatrix}$	2520
		$\begin{bmatrix}3\\7\end{bmatrix}$	120	$\begin{bmatrix}1\\3\\6\end{bmatrix}$	840	$\begin{bmatrix}1\\1\\3\\5\end{bmatrix}$	2520	$\begin{bmatrix}1\\1\\1\\3\\4\end{bmatrix}$	4200
		$\begin{bmatrix}4\\6\end{bmatrix}$	210	$\begin{bmatrix}1\\4\\5\end{bmatrix}$	1260	$\begin{bmatrix}1\\1\\4\\4\end{bmatrix}$	1575	$\begin{bmatrix}1\\1\\2\\2\\4\end{bmatrix}$	9450

		$\begin{bmatrix}5\\5\end{bmatrix}$	126	$\begin{bmatrix}2\\2\\6\end{bmatrix}$	630	$\begin{bmatrix}1\\2\\2\\5\end{bmatrix}$	3780	$\begin{bmatrix}1\\1\\2\\3\\3\end{bmatrix}$	12600
				$\begin{bmatrix}2\\3\\5\end{bmatrix}$	2520	$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$	12600	$\begin{bmatrix}1\\2\\2\\2\\3\end{bmatrix}$	12600
				$\begin{bmatrix}2\\4\\4\end{bmatrix}$	1575	$\begin{bmatrix}1\\3\\3\\3\end{bmatrix}$	2800	$\begin{bmatrix}2\\2\\2\\2\\2\end{bmatrix}$	945
				$\begin{bmatrix}3\\3\\4\end{bmatrix}$	2100	$\begin{bmatrix}2\\2\\2\\4\end{bmatrix}$	3150		
						$\begin{bmatrix}2\\2\\3\\3\end{bmatrix}$	6300		
$k = 6$		$k = 7$		$k = 8$		$k = 9$		$k = 10$	
$\begin{bmatrix}1\\1\\1\\1\\1\\5\end{bmatrix}$	252	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\4\end{bmatrix}$	210	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\3\end{bmatrix}$	120	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\2\end{bmatrix}$	45	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}$	1

$\begin{bmatrix}1\\1\\1\\1\\2\\4\end{bmatrix}$	3150	$\begin{bmatrix}1\\1\\1\\1\\1\\2\\3\end{bmatrix}$	2520	$\begin{bmatrix}1\\1\\1\\1\\1\\1\\2\\2\end{bmatrix}$	630				
$\begin{bmatrix}1\\1\\1\\1\\3\\3\end{bmatrix}$	2100	$\begin{bmatrix}1\\1\\1\\1\\2\\2\\2\end{bmatrix}$	3150						
$\begin{bmatrix}1\\1\\1\\2\\2\\3\end{bmatrix}$	12600								
$\begin{bmatrix}1\\1\\2\\2\\2\\2\end{bmatrix}$	4725								

References

[1] C. Dorsett, From observations and questions in introductory topology to some answers, Questions and Answers in General Topology 28 (2010), 55-64.

[2] C. Dorsett, The number of R_0 spaces on a finite set and a check for the total number of topologies, Far East J. Math. Sci. 46(2) (2010), 133-141.

[3] C. Dorsett, Characterizations of finite and countable sets using equivalences of non-T_0 separation axioms and Alexandroff and strong Alexandroff spaces, Pioneer J. Math. Math. Sci. 9(2) (2013), 73-82.

[4] C. Dorsett, The number of pseudometrizable and equivalent topologies on a finite set, Questions and Answers in General Topology 32 (2014), 105-112.

[5] P. Johnston, Stone Spaces, Cambridge Univ. Press (1982), 1986 edition.

[6] N. Shanin, On separations in topological spaces, Akademia Nauk SSSR Comptes Rendus (Doklady) 48 (1943), 110-113.

[7] N. Sloane, Finite Topological Space, Wikipedia, The on-Line Free Encyclopedia, 2012, pp. 1-7.

[8] R. Stong, Finite topological spaces, Transactions of the A.M.S. 123(2) (1966), 325-340.