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POTENTIAL THEORY 1: GRADIENT TENSORS

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Abstract

The electromagnetic potential A^i is a quadrivector in Minkowski spacetime x^k and its gradient $[a_k{}^i]$ is a tensor of rank two whose elements are the sixteen partial derivatives $\partial A^i / \partial x^k$. We study in this article the properties of a family of tensors resulting from $[a_k{}^i]$. We first introduce the covariant tensor $[a_{ki}]$. Four initial tensors are obtained by separating $[a_k{}^i]$ on the one hand, and $[a_{ki}]$ on the other hand into their symmetric and antisymmetric parts. These are $([s_k{}^i], [f_k{}^i], [S_{ki}], [F_{ki}])$. As the lowering-raising index operations and symmetrization-antisymmetrization operations do not commute, these four tensors are different and each describes different physical phenomena: $[F_{ki}]$ is the well-known electromagnetic tensor, $[S_{ki}]$ contains the source terms, $[s_k{}^i]$ introduces the electromagnetic

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particles and $[f_k{}^i]$ is the origin of the electrostatic field. The article is divided into two main parts. In the first, we study $[s_k{}^i]$ and $[f_k{}^i]$ which show that there is a particular coordinate system where the scalar potential obeys the Hemholtz equation. The solutions allow to describe the "electromagnetic particles", characterized by three quantum numbers n, ℓ and m. The condition of existence of these particles is related to a property of the electron described in Wheeler-Feynman's absorber theory. We give the potentials corresponding to the first five solutions. We associate a Lagrangian density $\mathcal{L}$ to the determinant of $[a_k{}^i]$ which is an invariant of this tensor in an operation of symmetry of the Poincaré group. Potential energy and electric charge distributions are included in the tensors properties. In the second part, we first check that $[F_{ki}]$ is the usual electromagnetic tensor whose components are the electric and magnetic fields. We prove that Maxwell's equations are obtained by applying the principle of least action to the 4-potential endowed with $\mathcal{L}$. The source terms (ρ and $\vec{j}$) are expressed in terms of the components of $[S_{ki}]$. The results obtained are covariant. The formulation of the potential and its derivatives being independent of scale, they unify the human and the electron scales, giving a new way to understand elementary particles.

1. Introduction

The notion of potential has been introduced by the French mathematician and physicist Simeon-Denis Poisson in the year 1813 when he was studying a modification to Laplace's equation [1]. It was then understood and extended [2] as a 4-vector A^i in Minkowski's spacetime x^k. Its time-component is the scalar potential ϕ/c, and the spatial part $\vec{A} = (A^x, A^y, A^z)$ is the vector potential. Spacetime being also 4-dimensional, there are 16 partial derivatives $a_k{}^i = \partial A^i / \partial x^k$ which form the gradient tensor $[a_k{}^i]$. The aim of this article is to describe some mathematical properties which can be deduced from $[a_k{}^i]$ and to interpret them in the context of electromagnetism and elementary particles.

We will show that $[a_k{}^i]$ is the source of four different tensors: one of them is the Faraday tensor $[F_{ki}]$, or the electromagnetic tensor, which was described by Hermann Minkowski as early as 1909 [3]. This tensor is usually deduced from Maxwell's equations and classical textbooks in electromagnetism [4, 5, 6] describe $[F_{ki}]$ as an object which neatly groups the components of the electric and magnetic fields. An essential application of $[F_{ki}]$ is to show that a pure electric field in a standing system of coordinates gives a magnetic field in a moving system through a Lorentz transformation.

$[F_{ki}]$ is only one member of a family of tensors which are obtained from $[a_k{}^i]$ at each event M in spacetime. The members of this family are generated through the use of two operations which do not commute:

(1) The raising-lowering (r.l.) index operation which is done with the metric tensor and

(2) The antisymmetric-symmetric (a.s.) splitting which makes use of the transpose of the tensors.

Four different tensors are thus obtained. One of them is $[F_{ki}]$, it results from the lowering index operation acting on $[a_k{}^i]$ followed by the a.s. operation where it corresponds to the antisymmetric part.

This preliminary remark has led us to study the properties of each of these tensors and to develop a general theory where the sole assumption is that of a 4-potential A^i in a flat spacetime. A characteristic of these tensors is that their formulation is **scale-independent**. They should thus be perfect tools to unify phenomena at ordinary and microscopic scales.

Indeed, many unsolved problems in modern physics are due to the difficulties associated with scaling. One of the most important is the structure of a "point-like" particle. In string models, such an idealized particle is represented by a one dimensional object called a string. However, despite the abundance of developments in these models the solution remains questionable [7, 8]. We show in this article that the potential theory offers a new way to solve it and especially a method to unify macroscopic and microscopic scales.

The first authors to use tensor techniques to study the properties of an elementary particle, the electron, were Max Born and Leopold Infeld in the year 1934 [9]. However, their attempt was not successful because their tensor was phenomenological: the antisymmetric part was indeed $[F_{ki}]$ but the symmetric part was chosen to be the metric tensor and both parts of the tensor were of a different nature. The tensors we use in this article were obtained from $[a_k{}^i]$ and $[a_{ki}]$ and both parts are combinations of partial derivatives.

The non-commutativity mentioned above leads to divide the theory into two parts:

The first part is based on the study of the symmetric ($[s_k{}^i]$) and antisymmetric ($[f_k{}^i]$) parts of $[a_k{}^i]$. We show that a proper time exists in which the scalar potential obeys a Helmholtz equation whose solutions describe "electromagnetic particles". Splitting $[a_k{}^i]$ into its symmetric and antisymmetric parts induced us to associate the former to the description of the mass and the latter to the field which could explain the particle-field duality.

The second part is based on the study of the symmetric ($[S_{ki}]$) and antisymmetric ($[F_{ki}]$) parts of $[a_{ki}]$. There is absolutely no reason to reduce $[a_{ki}]$ to its antisymmetric part $[F_{ki}]$: we will show that both parts contribute to Maxwell equations. If $[S_{ki}]$ is forgotten, which is the case in classical electromagnetism, it becomes necessary to replace it with phenomenological quantities. These are charges and currents: we demonstrate that they can be expressed in terms of the elements of $[S_{ki}]$.

A key point of the theory is the association of a Lagrangian density with $[a_k{}^i]$. This density is a local scalar invariant of the tensor and plays a central role: it allows the description of the structure, or the geometrical distribution of energy in the particle where it is used to compute the canonical momenta. In the second part, it is used to deduce Maxwell's equations from $[a_{ki}]$ through the least action principle.

The second section of this paper describes the few basic assumptions and the notations.

The third section develops the properties of the electromagnetic particles. Solutions of the Helmholtz equation in the proper time of $[a_k{}^i]$ describe these particles which are classified by three quantum numbers n, ℓ, m. We studied the spatial distribution of energy and the electric charge of these solutions. Here an interesting result occurs: while the distribution of field and mass energies are different, their sum, when integrated over spacetime, vanishes. A second important result concerns the electric charge: it is found that it can be different from zero for the even and odd solutions $n = 1$, $\ell = m = 0$ only, which confers the status of electron and positron to these solutions. Finally, we obtained the tensors of a spinning particle.

In the fourth section, we computed the electromagnetic inductions which are the derivatives of the Lagrangian with respect to the fields. The second set of Maxwell's equations is obtained from Euler-Lagrange equations. A new result is the expression of the sources (charge ρ and current $\vec{j}$ densities) in terms of the elements of $[S_{ki}]$.

Associated with this subject is the question of the preeminence of potentials and fields which arises in standard textbooks on classical electromagnetism: In ref. [4], electric and magnetic fields are deduced from a 4-potential. In refs. [5, 6], the 4-potential is deduced from the fields. In classical theory, fields are more fundamental than the potential because they are observable quantities and the 4-potential is a *Deus ex machina* which can be seen only through its gradient $[a_k{}^i]$. In quantum theory, it is the potential that is more fundamental as illustrated by the Aharonov-Bohm effect [10, 11]. Recently, two articles [12, 13] have been published where the authors develop Richard Feynman's idea of introducing potentials before fields [14]. The theory which is presented here brings the proof that the potential is also more fundamental than the fields in classical electromagnetism.

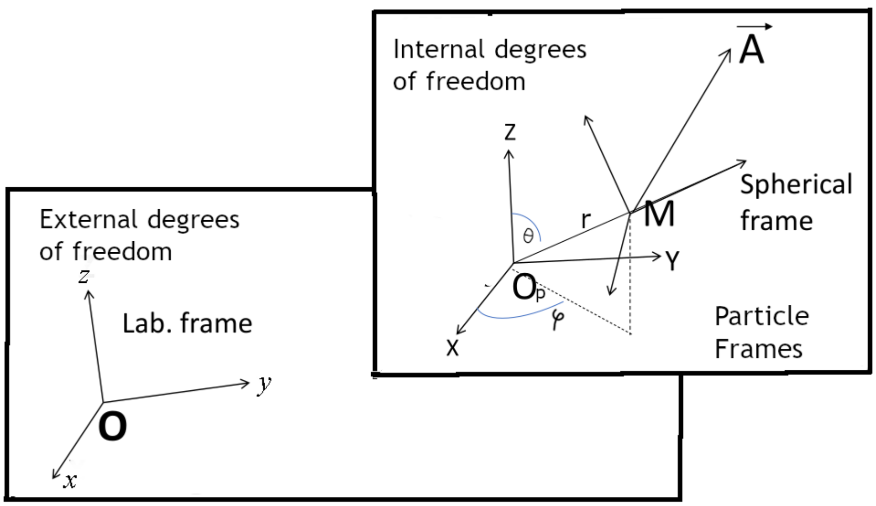

Figure 1. Schematics of the three frames of coordinates in the geometrical space. M is the point of observation. Tensors leading to Maxwell's equations are expressed in the laboratory frame. Electromagnetic particles tensors are first described at M in spherical coordinates r, θ, φ with M fixed with respect to the origin O_p. A particle spinning around the Z axis is described by the tensors obtained from a local Lorentz transformation at M. Note that M close to O_p allows the description of the structure of the particle. The long range behavior of the particle is obtained when M is far from O_p.

2. Foundations and Notations

The theory that we are developing is nothing but the study of the properties of the electromagnetic potential and its derivatives in different frames of observation. It is based on few ingredients:

1- One considers a point M (an "event") in a continuous and flat Minkowski's spacetime where the observer is located. Given a coordinates frame with an origin O, M is defined by the vector $\overrightarrow{OM}$.

The natural coordinates [17] will be used to describe the electromagnetic particles in Section 3. They happen to be the proper time and the geometrical spherical system at M. On the other hand, Maxwell's equations are covariant and can be computed in any coordinate system.

We will thus have two independent Cartesian coordinate systems with two different origins, the first is the laboratory frame with origin O. Let us name the second the particle frame with O_p as an origin. The external and internal coordinates of M are noted, respectively, as $x^k =$ (ct, x, y, z) and $X^k = (cT, X, Y, Z)$ in Cartesian coordinates. Here, t or T stands for the time, (x, y, z) or (X, Y, Z) for the geometrical coordinates and c is the speed of light. We will also use a spherical system of coordinates at M. Figure 1 shows the three coordinate frames which will be used. The two systems of coordinates (x, y, z) or (X, Y, Z) are the analog to those of a moving body, where the internal degrees of freedom allow the description of its shape, its rotations (Euler's angles) and its deformations and where the independent external coordinates are used to describe the motion of the center of gravity.

2- A four-potential $A^i = (\phi/c, A^x, A^y, A^z)$ is associated with each point M. ϕ/c is the scalar potential and (A^x, A^y, A^z) are the three contravariant components of the vector potential in the direct space. This is the standard notation in classical electromagnetism. We will make use of the direct and the inverse spaces where the potential is defined, respectively, by its contravariant and covariant components. Convention $\eta = diag\,(1, -1, -1, -1)$ is taken for the metric tensor η. Covariant components of the potential in the reciprocal space are: $A_i = (\phi/c\,,\, A_x,\, A_y,\, A_z) = (\phi/c\,,\, -A^x,\, -A^y,\, -A^z)$. We divide the potential into two parts: the first is coherent and will describe an isolated particle while the second is incoherent and describes the space in which it is embedded. This second part originates from the fundamental noise. The following study will essentially be concerned by the isolated particle.

3- The 16 partial derivatives $\partial A^i / \partial x^k$ of the potential are the components of the gradient tensor $[a_k{}^i]$. They are obtained from the total derivative of each component A^i at M. The extended form of $[a_k{}^i]$ in a Cartesian frame is:

$$[a_k{}^i] = \begin{bmatrix} (\phi/c)_{,t} & A^x_{,t} & A^y_{,t} & A^z_{,t} \\ (\phi/c)_{,x} & A^x_{,x} & A^y_{,x} & A^z_{,x} \\ (\phi/c)_{,y} & A^x_{,y} & A^y_{,y} & A^z_{,y} \\ (\phi/c)_{,z} & A^x_{,z} & A^y_{,z} & A^z_{,z} \end{bmatrix}. \tag{1}$$

We have adopted the conventional notation:

$$(\phi/c)_{,t} \equiv \frac{\partial(\phi/c)}{c\partial t}, \qquad (\phi/c)_{,x} \equiv \frac{\partial(\phi/c)}{\partial x} \;\ldots$$

$$A^x_{,t} \equiv \frac{\partial A^x}{c\partial t}, \qquad A^x_{,x} \equiv \frac{\partial A^x}{\partial x} \;\ldots$$

k and i, respectively, indicate the line and the column index for a reason which will appear in eq. (109). The above derivatives become more complicated in curvilinear coordinates where they include Christoffel's coefficients [16].

The theory is essentially local: a particle will be described by vector and tensor fields. Global properties are obtained after integrations over spacetime.

4- The fourth ingredient is the set of constraints that are imposed by nature:

a) Coordinate transformations form the Poincaré group: they include translations with respect to time and space, rotations, and Lorentz transformations (or boosts). Tensors are characterized by **invariants** in such transformations. Our attention should thus be fixed on these invariants.

b) The evolution of a system is subject to the principle of least action which is expressed by the Euler-Lagrange equations.

c) The principle of relativity couples time and the geometrical coordinates of M in two frames in relative motion.

5- Finally, transformations of vectors or tensors have to obey mathematical rules. A fundamental concept is the splitting of a tensor into its antisymmetric (or skew-symmetric) and symmetric parts (a.s. operation). This operation does not commute with other manipulations such as the raising or lowering index operations (r.l. operations) performed with the metric tensor. This non-commutativity is illustrated in Appendix A. This simple law is at the origin of the splitting of electromagnetism into two parts where the first gives the description of electromagnetic particles (Section 3) and the second describes how Maxwell's equations can be deduced from the gradient tensor (Section 4).

Table 1. Synopsis of the different electromagnetic tensors. Tensors in the upper part are obtained from the gradient $a_k{}^i$ of the 4-potential A^i at point M. $a_k{}^i$ is split into its symmetric and antisymmetric parts $s_k{}^i$ and $f_k{}^i$ in the coordinate frame where M is at rest with respect to the origin (a.s. splitting). A_i and a_{ki} in the lower part are obtained from the lowering index operation acting on A^i or $a_k{}^i$. Then a_{ki} is split into its symmetric part S_{ki} and its antisymmetric part F_{ki} which is the usual electromagnetic tensor. The non-commutativity of the r.l. and a.s. operations is fundamental.

A^i, $[a_k{}^i]$	a.s. splitting: $[a_k{}^i] = [s_k{}^i] + [f_k{}^i]$
lowering i	
A_i, $[a_{ki}]$	a.s. splitting: $[a_{ki}] = [S_{ki}] + [F_{ki}]$

Table (1) shows the different tensors a_{ki}, S_{ki}, F_{ki}, $a_k{}^i$, $s_k{}^i$, $f_k{}^i$ which will be described as we progress. One has to distinguish between those tensors which are defined in the pure direct space $(a_k{}^i, s_k{}^i, f_k{}^i)$ where contravariant components A^i and x^k only are involved and (a_{ki}, S_{ki}, F_{ki}) where the geometrical coordinates can depend on time and where we need covariant components A_i. One should note that:

1- covariant or contravariant tensors keep their symmetry or antisymmetry properties in a coordinate transformation which is not the case of a mixed tensor [15].

2- the determinant of mixed tensors remain invariant in a coordinate transformation which is not the case of a co or contravariant tensor.

3 Tensors of Electromagnetic Particles

The main part of this section deals with tensors written in the pure direct space where the geometrical coordinates do not depend on time (the observation point M is fixed with respect to the origin of coordinates). We first show that the natural coordinates of the tensor $[a_k{}^i]$ are introduced by a Helmholtz equation whose solutions describe the electromagnetic particles. We first derive this equation, then illustrate some of its solutions and the associated tensors. Some local and global properties are given. The case of a spinning particle is finally studied.

The point M in Figure 1 is **motionless** with respect to the origin O_p. We will first write the derivatives in the local Cartesian frame (T, X, Y, Z): $a_k{}^i = \partial A^i / \partial X^k$.

We use the transpose $[a_i{}^k]$ of $[a_k{}^i]$ to split $[a_k{}^i]$ into its symmetric ($[s_k{}^i]$) and antisymmetric ($[f_k{}^i]$) parts:

$$[a_k{}^i] = [s_k{}^i] + [f_k{}^i] \tag{2}$$

with elements:

$$s_k{}^i = \frac{1}{2}(\partial A^i / \partial X^k + \partial A^k / \partial X^i), \tag{3}$$

$$f_k{}^i = \frac{1}{2}(\partial A^i / \partial X^k - \partial A^k / \partial X^i). \tag{4}$$

The pair of tensors $[s_k{}^i]$ and $[f_k{}^i]$ will be a good candidate to explain the matter-field duality: $[f_k{}^i]$ describes the field properties and $[s_k{}^i]$ contains matter waves. $[s_k{}^i]$ will be referred to as the matter tensor and $[f_k{}^i]$ as the field tensor. Note that $[f_k{}^i]$ is not the usual electromagnetic field tensor (compare with eq. (97)).

3.1. Helmholtz equation

A Helmholtz equation for the scalar potential is demonstrated in the following way:

1- Being symmetric, $[s_k{}^i]$ can be diagonalized provided its determinant does not vanish. The consequence is that a time coordinate $\bar{t}$ exists in which transformed terms $\bar{s}_1^2 = \bar{s}_1^3 = \bar{s}_1^4 = \bar{s}_2^1 = \bar{s}_3^1 = \bar{s}_4^1 = 0$. We referred to $\bar{t}$ the proper time of the tensor. In the proper time system, we use the barred symbols $c\bar{t}$, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and $\overline{A}^i = (\bar{\phi}/c, \overline{A^X}, \overline{A^Y}, \overline{A^Z})$. The ordinary derivatives of a Cartesian system are replaced in a general system by absolute derivatives including Christoffel coefficients [16]. One obtains the equations:

$$\overline{A^X{}_{,\bar{t}}} + \overline{(\phi/c)_{,X}} = \overline{A^Y{}_{,\bar{t}}} + \overline{(\phi/c)_{,Y}} = \overline{A^Z{}_{,\bar{t}}} + \overline{(\phi/c)_{,Z}} = 0 \qquad (5a)$$

or $$\overrightarrow{grad}\ \overline{(\phi/c)} + \frac{\partial \overline{\vec{A}}}{c\partial \bar{t}} = 0. \qquad (5b)$$

Here a symbol like $\overline{(\phi/c)_{,X}}$ generally stands for the absolute derivative of $\overline{(\phi/c)}$ with respect to $\overline{X}$. The second formulation (5b) is tensorial. Eq. (5b) means that in the proper time, the temporal derivative $\overline{A^i_{,\bar{t}}}$ is compensated by the spatial derivative of $\overline{(\phi/c)}$ in the direction $\bar{i}$.

$[s_k{}^i]$ and $[f_k{}^i]$ are written in the proper time frame:

$[s_k{}^i] = \frac{1}{2}([a_k{}^i] + [a_i{}^k]) =$

$$\frac{1}{2}\begin{bmatrix} 2\overline{(\phi/c)}_{,\bar{t}} & 0 & 0 & 0 \\ 0 & 2\overline{A^X}_{,\overline{X}} & \overline{A^Y}_{,\overline{X}} + \overline{A^X}_{,\overline{Y}} & \overline{A^Z}_{,\overline{X}} + \overline{A^X}_{,\overline{Z}} \\ 0 & \overline{A^X}_{,\overline{Y}} + \overline{A^Y}_{,\overline{X}} & 2\overline{A^Y}_{,\overline{Y}} & \overline{A^Z}_{,\overline{Y}} + \overline{A^Y}_{,\overline{Z}} \\ 0 & \overline{A^X}_{,\overline{Z}} + \overline{A^Z}_{,\overline{X}} & \overline{A^Y}_{,\overline{Z}} + \overline{A^Z}_{,\overline{Y}} & 2\overline{A^Z}_{,\overline{Z}} \end{bmatrix}, \qquad (6)$$

and

$$[f_k^{\ i}] = \frac{1}{2}([a_k^{\ i}] - [a_i^{\ k}]) =$$

$$\frac{1}{2}\begin{bmatrix} 0 & 2\overline{A^X}_{,\bar{t}} & 2\overline{A^Y}_{,\bar{t}} & 2\overline{A^Z}_{,\bar{t}} \\ 2\overline{(\phi/c)}_{,\overline{X}} & 0 & \overline{A^Y}_{,\overline{X}} - \overline{A^X}_{,\overline{Y}} & \overline{A^Z}_{,\overline{X}} - \overline{A^X}_{,\overline{Z}} \\ 2\overline{(\phi/c)}_{,\overline{Y}} & \overline{A^X}_{,\overline{Y}} - \overline{A^Y}_{,\overline{X}} & 0 & \overline{A^Z}_{,\overline{Y}} - \overline{A^Y}_{,\overline{Z}} \\ 2\overline{(\phi/c)}_{,\overline{Z}} & \overline{A^X}_{,\overline{Z}} - \overline{A^Z}_{,\overline{X}} & \overline{A^Y}_{,\overline{Z}} - \overline{A^Z}_{,\overline{Y}} & 0 \end{bmatrix}. \quad (7)$$

One should note that, in the proper time, when geometrical coordinates are independent of time (the point M is at rest with respect to the origin O), one has: $\partial\bar{x}^j/\partial(ct) = 0$ for $j = 2, 3, 4$ and $\partial\overline{ct}^j/\partial(ct) = 1$ for $j = 1$. It follows that the potential $A^i = (\phi/c, A^1, A^2, A^3)$ becomes transformed into $A^i = ((\phi/c), \overline{A^1}, \overline{A^2}, \overline{A^3})$ in a geometrical transformation in the proper time. In the same way, one finds that the elements of the first line of the transformed tensor remain $A_1^{\ i} = ((\phi/c), 0, 0, 0)$.

2- Now we will use the invariants of $[\bar{s}_k^{\ i}]$ in a time translation. There are four scalar invariants that are the coefficients of the characteristic polynomial. The most well-known are the trace and the determinant. We use the property of the trace of $[s_k^{\ i}]$ to be invariant in a time translation to obtain:

$$\frac{\partial}{c\partial\bar{t}}\left(\frac{\partial(\overline{\phi/c})}{c\partial\bar{t}} + \frac{\partial\overline{A^X}}{\partial\overline{X}} + \frac{\partial\overline{A^Y}}{\partial\overline{Y}} + \frac{\partial\overline{A^Z}}{\partial\overline{Z}}\right) = 0 \quad (8a)$$

or $$\frac{\partial^2(\overline{\phi/c})}{c^2\partial\bar{t}^2} + div\frac{\partial\overline{\vec{A}}}{c\partial\bar{t}} = 0. \quad (8b)$$

Note that the trace of $[s_k^{\ i}]$ is the term that appears in Lorenz's gauge.

Equations (5b) and (8b) are combined to give:

$$\frac{\partial^2(\overline{\phi/c})}{c^2\partial\bar{t}^2} - \Delta(\overline{\phi/c}) = 0, \tag{9}$$

where the symbol Δ stands for the Laplacian.

In the following, we will be interested in permanent oscillatory potentials which are proportional to $\cos(\omega t + \chi)$.

$$\phi/c \propto \cos(\omega t + \chi) = \begin{cases} \pm\cos\omega t & \text{even solutions,} \\ \pm\sin\omega t & \text{odd solutions.} \end{cases} \tag{10}$$

$\chi = 0$ or π and $\chi = \pm\pi/2$ correspond, respectively, to solutions which are even or odd in time. These potentials obey a Helmholtz-type equation:

$$\frac{\omega^2}{c^2}(\overline{\phi/c})_{spatial} + \Delta(\overline{\phi/c})_{spatial} = 0, \tag{11}$$

where $(\overline{\phi/c})_{spatial}$ represents the spatial part of $(\overline{\phi/c})$. This equation is a tensor equation that remains the same in any system of spatial coordinates.

One sees that eq. (9) is invariant under space and time reversal, i.e., invariant when the sign of time t is changed. One should thus consider the solutions obtained with $\tilde{t} = -t$:

$$\phi/c \propto \cos(\omega\tilde{t} + \chi) = \begin{cases} \pm\cos\omega\tilde{t}, \\ \pm\sin\omega\tilde{t}. \end{cases} \tag{12}$$

The even and odd solutions: $\phi/c \propto \cos(\omega t)$ and $\phi/c \propto \sin(\omega t)$ will be used in the following. The $\pm$ signs do not play any role in the development of the theory. However one sees that the even solutions only remain invariant in time reversal. This property will be used to study the electric charge of the particles (Section 3.3.4).

3.2. Electromagnetic particles

Solutions of eq. (11) will describe spatial distributions of the scalar potential $\overline{\phi/c}$ in geometrical space. The components of the vector potential can be determined from $\overline{\phi/c}$ thanks to eq. (5b). The knowledge of the components of the four-potential for each solution will lead to the corresponding tensor of derivatives. A particle will be described by a potential and its derivatives at each event M in spacetime (i.e., by vector and tensor fields). These solutions are studied in this section.

Helmholtz equation can be written in the spherical reference frame attached to M with the proper time $\bar{t}$ and the geometrical coordinates (r, θ, φ) such that:

$$\overline{X} = r \sin\theta \cos\varphi, \qquad \overline{Y} = r \sin\theta \sin\varphi, \qquad \overline{Z} = r \cos\theta. \tag{13}$$

The advantage of the (r, θ, φ) system is that it makes use of the spherical or cylindrical symmetry of the solutions that we are going to use. The set $c\bar{t}$, r, θ, φ is the natural set of coordinates of the particles [17].

We introduce the *normalized* distance to the origin O_p of the coordinates: $\mathrm{x} = \omega r/c$. (Note the typography which is different from that of the coordinate x). This distance will thus be measured in units of the reference length c/ω.

Eq. (11) has been studied extensively in the context of the hydrogen atom where some of its solutions describe the electronic orbitals [18].

Solutions of eq. (11) can be split into normal and coupled modes:

1- Coupled angular-radial modes describe simultaneous vibrations on the three coordinates. They are obtained from the *ansatz*: $(\overline{\phi/c})_{spatial}$ = $R(r)\,\Theta(\theta)\,\Phi(\varphi)$ where $R(r)$, $\Theta(\theta)$, and $\Phi(\varphi)$ are functions of r, θ, and

φ, respectively. One thus obtains the coupled angular-radial modes in terms of spherical Bessel functions of order ℓ: $J_\ell(\mathrm{x})$ and spherical harmonics: $Y_\ell^m(\theta, \varphi)$:

$$\phi_{\ell,m}(\mathrm{x}, \theta, \phi) = \pm\mathcal{A}\ J_\ell(\mathrm{x})\ Y_\ell^m(\theta, \varphi) \begin{cases}\cos \omega t, \\ \sin \omega t.\end{cases} \tag{14}$$

The two even and odd solutions are explicitly written. Quantities $\mathcal{A}$ and ω are the constants of integration and are not determined at this stage. Amplitude $\mathcal{A}$ has the dimension $[\mathcal{A}] = \mathrm{ML^2T^{-2}Q^{-1}}$ in the standard nomenclature.

$J_\ell(\mathrm{x})$ is a solution to the radial equation:

$$\mathrm{x}^2 R + \frac{\partial}{\partial \mathrm{x}}\left(\mathrm{x}^2 \frac{\partial R}{\partial \mathrm{x}}\right) = \ell(\ell+1)\ R. \tag{15}$$

The $Y_\ell^m(\theta, \varphi)$ describe the solutions of the angular part of eq. (11).

2- Normal modes describe independent vibrations on one of the three coordinates. They are obtained from the *ansatz*: $(\overline{\phi/c})_{spatial} = R(r) + \Theta(\theta) + \Phi(\varphi)$. There is an important difference between the equations which describe normal and coupled modes of an oscillator. Typically an oscillator receives a sustaining energy from the outside and looses energy from, for instance, mechanical friction or electromagnetic radiation. Equilibrium is attained when both energies compensate each other. Normal and coupled modes need a source term (a seed) to develop. This source is generally a noise (again mechanical vibrations or electromagnetic waves), it starts the oscillation whose amplitude increases until the equilibrium state. In the case of coupled modes, the source originates from the coupling term and the noise does not need to be explicitly written: the source term for the radial oscillation is $\ell(\ell+1)R$ in eq. (15). The source term must also appear in the equation of a normal mode where it originates from the noise in which the particle is embedded. We consider that this noise is a homogeneous background (independent of the point of observation M). It is also isotropic and acts as a source for spherical potentials only. We thus replace $\ell(\ell+1)R$ in eq. (15) by the source term $cst\ R$ where cst is a proportionality constant:

$$\mathrm{x}^2 R + \frac{\partial}{\partial \mathrm{x}}\left(\mathrm{x}^2 \frac{\partial R}{\partial \mathrm{x}}\right) = cst\ R. \tag{16}$$

Physically acceptable spatial solutions are the spherical Bessel functions J_n which are obtained when the constant is $cst = n(n+1)$.

Grouping coupled and normal modes together shows that the potential which describes a solution finally depends on three quantum numbers $n,\ \ell,\ m$: $\phi/c = \phi/c\,(n,\ \ell,\ m)$. Its general expression is:

$$\phi/c\,(\mathrm{x},\ \theta,\ \phi) = \pm\mathcal{A}\ J_n(\mathrm{x})\ Y_\ell^{\,m}(\theta,\ \varphi) \begin{cases} \cos\omega\bar{t}, \\ \sin\omega\bar{t}. \end{cases} \tag{17}$$

Explicit values of the first spherical harmonics are:

$$Y_0^{\,0} = \sqrt{\frac{1}{4\pi}}, \qquad Y_1^{\,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,$$

$$Y_1^{\,1} = -\sqrt{\frac{3}{4\pi}}\sin\theta\cos\varphi, \qquad Y_1^{\,-1} = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\varphi. \tag{18}$$

The first spherical Bessel functions are:

$$J_0 = J_0(\mathrm{x}) = \frac{\sin \mathrm{x}}{\mathrm{x}}, \tag{19a}$$

$$J_1 = J_1(\mathrm{x}) = \frac{\sin \mathrm{x}}{\mathrm{x}^2} - \frac{\cos \mathrm{x}}{\mathrm{x}}. \tag{19b}$$

Equation (5b) allows the computation of the components of the vector potential in the proper time frame from the scalar potential.

These solutions obey the physical boundary conditions: they become asymptotically null far from the origin and they are either finite or null at the origin. Each solution describes an "electromagnetic particle" (e.m. particle).

The general expression of the gradient tensor in the spherical system of coordinates associated with point M is:

$$[a_k{}^i] = \begin{bmatrix} (\phi/c)_{,\bar{t}} & A^r_{,\bar{t}} & A^\theta_{,\bar{t}} & A^\varphi_{,\bar{t}} \\ (\phi/c)_{,r} & A^r_{,r} & A^\theta_{,r} & A^\varphi_{,r} \\ (\phi/c)_{,\theta}/r & \frac{1}{r}(A^r_{,\theta} - A^\theta) & \frac{1}{r}(A^\theta_{,\theta} + A^r) & \frac{1}{r} A^\varphi_{,\theta} \\ \frac{(\phi/c)_{,\varphi}}{r \sin\theta} & \frac{A^r_{,\varphi}}{r\sin\theta} - \frac{A^\varphi}{r} & \frac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta\ A^\varphi) & \frac{1}{r\sin\theta} A^\varphi_{,\varphi} + \frac{1}{r} A^r + \frac{1}{r}\frac{\cos\theta}{\sin\theta} A^\theta \end{bmatrix}. \tag{20}$$

This tensor does not include the noise and will characterize the particle only. Its symmetric part (the mass part) is:

$$[s_k{}^i] = \frac{1}{2}\begin{bmatrix} 2(\phi/c)_{,\bar{t}} & A^r_{,\bar{t}} + (\phi/c)_{,\bar{t}} & A^\theta_{,\bar{t}} + (\phi/c)_{,\theta}/r & A^\varphi_{,\bar{t}} + \frac{(\phi/c)_{,\varphi}}{r\sin\theta} \\ (\phi/c)_{,r} + A^r_{,\bar{t}} & 2A^r_{,r} & A^\theta_{,r} + \frac{1}{r}(A^r_{,\theta} - A^\theta) & A^\varphi_{,r} + \frac{A^r_{,\varphi}}{r\sin\theta} - \frac{A^\varphi}{r} \\ (\phi/c)_{,\theta}/r + A^\theta_{,\bar{t}} & \frac{1}{r}(A^r_{,\theta} - A^\theta) + A^\theta_{,r} & 2\frac{1}{r}(A^\theta_{,\theta} + A^r) & \frac{1}{r}A^\varphi_{,\theta} + \frac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta\, A^\varphi) \\ \frac{(\phi/c)_{,\varphi}}{r\sin\theta} + A^\varphi_{,\bar{t}} & \frac{A^r_{,\varphi}}{r\sin\theta} - \frac{A^\varphi}{r} + A^\varphi_{,r} & \frac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta\, A^\varphi) + \frac{1}{r}A^\varphi_{,\theta} & 2\left(\frac{1}{r\sin\theta}A^\varphi_{,\varphi} + \frac{1}{r}A^r + \frac{1}{r}\frac{\cos\theta}{\sin\theta}A^\theta\right) \end{bmatrix}. \tag{21}$$

The antisymmetric part (the field part) is:

$$[f_k{}^i] = \frac{1}{2}\begin{bmatrix} 0 & A^r_{,t} - (\phi/c)_{,r} & A^\theta_{,t} - (\phi/c)_{,\theta}/r & A^\varphi_{,t} - \dfrac{(\phi/c)_{,\varphi}}{r\sin\theta} \\ (\phi/c)_{,r} - A^r_{,t} & 0 & A^\theta_{,r} - \dfrac{1}{r}(A^r_{,\theta} - A^\theta) & A^\varphi_{,r} - \dfrac{A^r_{,\varphi}}{r\sin\theta} + \dfrac{A^\varphi}{r} \\ (\phi/c)_{,\theta}/r - A^\theta_{,t} & \dfrac{1}{r}(A^r_{,\theta} - A^\theta) - A^\theta_{,r} & 0 & \dfrac{1}{r}A^\varphi_{,\theta} - \dfrac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta\, A^\varphi) \\ \dfrac{(\phi/c)_{,\varphi}}{r\sin\theta} - A^\varphi_{,t} & \dfrac{A^r_{,\varphi}}{r\sin\theta} - \dfrac{A^\varphi}{r} - A^\varphi_{,r} & \dfrac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta\, A^\varphi) - \dfrac{1}{r}A^\varphi_{,\theta} & 0 \end{bmatrix}. \quad (22)$$

In the proper time, one has:

$$A^r_{,t} + (\phi/c)_{,r} = A^\theta_{,t} + (\phi/c)_{,\theta}/r = A^\varphi_{,t} + \frac{(\phi/c)_{,\varphi}}{r\sin\theta} = 0. \quad (23)$$

The components of the field are *defined* by the relations: $\mathcal{B}^r$

$$\mathcal{E}^r/c := \frac{1}{2}(A^r_{,t} - (\phi/c)_{,r}) = A^r_{,t} = -(\phi/c)_{,r},$$

$$\mathcal{E}^\theta/c := \frac{1}{2}(A^\theta_{,t} - (\phi/c)_{,\theta}/r) = A^\theta_{,t} = -(\phi/c)_{,\theta}/r,$$

$$\mathcal{E}^\varphi/c := \frac{1}{2}\left(A^\varphi_{,t} - \frac{(\phi/c)_{,\varphi}}{r\sin\theta}\right) = A^\varphi_{,t} = -\frac{(\phi/c)_{,\varphi}}{r\sin\theta}, \quad (24)$$

$$\mathcal{B}^r := \frac{1}{2}\left(\frac{1}{r}A^\varphi_{,\theta} - \frac{1}{r\sin\theta}(A^\theta_{,\varphi} - \cos\theta\, A^\varphi)\right),$$

$$\mathcal{B}^\theta := \frac{1}{2}\left(\frac{A^r_{,\varphi}}{r\sin\theta} - \frac{A^\varphi}{r} - A^\varphi_{,r}\right),$$

$$\mathcal{B}^\varphi := \frac{1}{2}\left(A^\theta_{,r} - \frac{1}{r}(A^r_{,\theta} - A^\theta)\right). \quad (25)$$

The antisymmetric part writes in the proper time:

$$[f_k{}^i] = \begin{bmatrix} 0 & \mathcal{E}^r/c & \mathcal{E}^\theta/c & \mathcal{E}^\varphi/c \\ -\mathcal{E}^r/c & 0 & \mathcal{B}^\varphi & -\mathcal{B}^\theta \\ -\mathcal{E}^\theta/c & -\mathcal{B}^\varphi & 0 & \mathcal{B}^r \\ -\mathcal{E}^\varphi/c & \mathcal{B}^\theta & -\mathcal{B}^r & 0 \end{bmatrix}. \qquad (26)$$

Note that these components are *different* from the components of the electromagnetic field which are obtained in eq. (96) in the next section from the tensor a_{ki} which makes use of both the direct and dual spaces. The difference arises from the fact that the field is defined here from the components of the mixed tensor $[f_k{}^i]$ while they are defined from the *covariant* tensor $[F_{ki}]$ in ordinary electromagnetism.

Explicit formulas for the potential components, their derivatives and the corresponding tensors are given in Appendix B for the first 5 even and odd solutions corresponding to $n = \ell = m = 0$ (solutions g), $n = 1$, $\ell = m = 0$ (solutions e) and $n = 1$, $\ell = 1$, $m = 0, \pm 1$ (solutions q_0, q_1, q_{-1}).

3.3. Particles properties

Some properties resulting from the tensorial description of the particles are listed in this section. These properties will be illustrated with the use of the spherical "*g*" or "*e*" solution $(n = 0, 1, \ell = m = 0)$ where the potentials are written with the $+$ sign in eq. (10):

$$\phi/c = \frac{1}{\sqrt{4\pi}}\mathcal{A}\begin{cases} J\cos\omega t, \\ J\sin\omega t, \end{cases}$$

$$A^r = \frac{1}{\sqrt{4\pi}}\mathcal{A}\begin{cases} -J'\sin\omega t & \text{even solutions,} \\ \ \ J'\cos\omega t & \text{odd solutions.} \end{cases} \qquad (27)$$

As before J stands for J_0 (solutions g) or J_1 (solutions e).

Gradient tensors are obtained from the partial derivatives of the potential with respect to time $(\partial/c\partial t)$ or space $(\partial/\partial r = \omega/c\ \partial/\partial \mathrm{x})$. We will essentially use the tensor which writes for the even e solution $\phi/c \propto \cos\omega t$:

$$[a_k{}^i] = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c} \times$$

$$\begin{bmatrix} -\sin\omega t\ J & -\cos\omega t\ J' & 0 & 0 \\ \cos\omega t\ J' & -\sin\omega t\ J'' & 0 & 0 \\ 0 & 0 & -\sin\omega t\ \dfrac{J'}{\mathrm{x}} & 0 \\ 0 & 0 & 0 & -\sin\omega t\ \dfrac{J'}{\mathrm{x}} \end{bmatrix}. \qquad (28)$$

It writes for the odd solution:

$$[a_k{}^i] = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c} \times$$

$$\begin{bmatrix} \cos\omega t\ J & -\sin\omega t\ J' & 0 & 0 \\ \sin\omega t\ J' & \cos\omega t\ J'' & 0 & 0 \\ 0 & 0 & \cos\omega t\ \dfrac{J'}{\mathrm{x}} & 0 \\ 0 & 0 & 0 & \cos\omega t\ \dfrac{J'}{\mathrm{x}} \end{bmatrix}. \qquad (29)$$

The gradient tensor for the reversed-time odd solution is obtained from the odd solution $\phi/c \propto \sin\omega t$. We name this solution e^* or g^* following the value of J. One reverses the time $t \to -t$, time derivatives are taken with respect to $-t$ and one obtains $\phi/c \propto -\sin\omega t$, $A^r \propto \cos\omega t$. This tensor writes:

$$[a_k{}^i] = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c} \times$$

$$\begin{bmatrix} \cos\omega t\ J & \sin\omega t\ J' & 0 & 0 \\ -\sin\omega t\ J' & \cos\omega t\ J'' & 0 & 0 \\ 0 & 0 & \cos\omega t\ \dfrac{J'}{\mathrm{x}} & 0 \\ 0 & 0 & 0 & \cos\omega t\ \dfrac{J'}{\mathrm{x}} \end{bmatrix}. \qquad (30)$$

The symmetric part is diagonal, the antisymmetric part contains the field. Particle properties will be described with tensor (28). Tensor (30) will also be used to describe the electric charge.

3.3.1. Waves

Each element of the tensor (28) is a system of stationary waves. Their amplitudes vary radially and are maximum in the vicinity of the center. For the moment, one can recognize several kinds of waves:

1- The field is defined in eq. (24) and is expressed by the formula:

$$\mathcal{E}^r/c = -(\phi/c)_{,r} = A^r_t = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c} J'(-\cos\omega t), \qquad (31)$$

where $J' = J_0' = \cos\mathrm{x}/\mathrm{x} - (\sin\mathrm{x})/\mathrm{x}^2$ for solution g and $J' = J_1' = \sin\mathrm{x}/\mathrm{x} + 2\cos\mathrm{x}/\mathrm{x}^2 - 2\sin\mathrm{x}/\mathrm{x}^3$ for solution e.

$\mathcal{E}^r$ is a *longitudinally polarized standing wave*. When x becomes large, the long range field (for the even e solution) is proportional to:

$$\mathcal{E}^r \sim \cos\omega t \sin\mathrm{x}/\mathrm{x} = 1/2\mathrm{x}\,(\sin\omega(t + r/c) - \sin\omega(t - r/c)). \qquad (32)$$

This expression displays a travelling advanced wave $\sin\omega(t + r/c)$ and a retarded wave $\sin\omega(t - r/c)$ which corresponds to those described in Wheeler-Feynman's absorber theory [21]. Thus, we have adopted their interpretation based on causality: the outgoing wave is emitted by the particle and is absorbed by the surrounding medium which also acts as the emitter of a wave that is absorbed by the particle. Equilibrium is obtained when both incoming and outgoing waves have the same energy. This is the condition for the stability of the particle and the existence of permanent solutions. It follows that the amplitude $\mathcal{A}$ is fixed by this condition which is certainly scale-dependent.

2- The second kind of waves appears in $s_k{}^i$: these are $s_1{}^1$ and $s_2{}^2$. They can be interpreted as "matter waves" and should correspond to de Broglie's pilot wave [22]. Moreover, they could be the origin of gravitation. The physical interpretations of $s_1{}^1$ and $s_2{}^2$ are different:

- $s_1{}^1$ originates from the time derivative of the scalar potential ϕ/c. The standing wave is scalar.

- $s_2{}^2$ originates from the radial derivative of the radial potential A^r. The standing wave is radially polarized.

In the following, we will find other kinds of waves (e.g., induction waves) emitted by the particle.

3.3.2. Energy

General formulas. The Hamiltonian density describes the local density of energy at point M. We have to find the Lagrangian which is linked to the Hamiltonian through a Legendre transform. An integration over spacetime will give the total energy of the particle.

We have two hints to find the Lagrangian associated with an electromagnetic particle:

- The first hint is the invariance of a Lagrangian in a coordinate change. Among the four invariants of $[a_k{}^i]$, one is proportional to the Lagrangian density $\mathcal{L}$.

- The second hint is that the global Lagrangian of a particle should be finite. In other words, the Hamiltonian density $\mathcal{H}$, when integrated over the whole volume, should converge, giving the total energy of the particle. This integral is:

$$\int \equiv c/\omega \int_0^{2\pi} d(\omega t) \int_0^{\infty} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, \mathcal{H} \tag{33}$$

in the spherical system of coordinates. The first summation will give the mean value over a period of time. These conditions are met by the determinant of $[a_k{}^i]$ **only** and we are led to the equation:

$$\mathcal{L} = \mathcal{C} \, \| \, a_k{}^i \, \|, \tag{34}$$

where the double bar is the symbol for the determinant. The proportionality constant $\mathcal{C}$ is a physical quantity that has the dimensions $[\mathcal{C}] = \mathrm{M}^{-3}\mathrm{L}^{-2}\mathrm{T}^{2}\mathrm{Q}^{4}$.

The Hamiltonian is given by the Legendre transform:

$$\mathcal{H} = \sum_{ik} a_k{}^i \frac{\partial \mathcal{L}}{\partial a_k{}^i} - \mathcal{L} = Trace\left(a_k{}^i \left(\frac{\partial \mathcal{L}}{\partial a_k{}^i} \right) \right) - \mathcal{L}. \tag{35}$$

This equation uses the *canonical momentum* $\mathcal{L}'^k{}_i = \partial\mathcal{L}/\partial a_k{}^i$ associated with $a_k{}^i$. This term appears in the Euler-Lagrange equations (65).

Since $\mathcal{L}$ is the determinant of $[a_k{}^i]$, one sees that $\mathcal{L}'^k{}_i$ is the minor relative to the element $a_k{}^i$ (with the sign $(-1)^{i+k}$) and that $\sum_k \partial\mathcal{L}/\partial a_k{}^i \, a_k{}^i = \mathcal{L}$ (development of the determinant with respect to the elements of line k). The simple equation follows:

$$\mathcal{H} = 3\mathcal{L}. \tag{36}$$

Integration over spacetime with the 4-volume element dv gives a quantity which is proportional to the total energy of the particle:

$$W_{total} \propto \int \mathcal{L}\, dv = \int \| a_k{}^i \| \, dv. \tag{37}$$

Each term of this integral can be split into a product of 4 integrals of the form $\int_a^b \partial A^i / \partial x^m \, dx^m = A^i \Big|_{-\infty}^{+\infty}$ or $A^i \Big|_0^{2\pi}$. As the potential is periodic and vanishes at infinity, one finds that **the total energy of an electromagnetic particle is identically zero**.

Now we divide $[a_k{}^i]$ into its symmetric part (elements $[s_k{}^i]$) and antisymmetric part (elements $[f_k{}^i]$). In the same way the momentum tensor can be split into its symmetric and antisymmetric parts:

$$\frac{\partial \mathcal{L}}{\partial a_k{}^i} = \frac{\partial \mathcal{L}}{\partial s_k{}^i} + \frac{\partial \mathcal{L}}{\partial f_k{}^i}. \tag{38}$$

We will use the notation:

$$\frac{\partial \mathcal{L}}{\partial s_k{}^i} = \mathcal{L}'^{\,k}_{s_i} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial f_k{}^i} = \mathcal{L}'^{\,k}_{f_i}. \tag{39}$$

Eq. (35) gives:

$$\mathcal{H} = Trace\,((s_k{}^i + f_k{}^i))\left(\mathcal{L}'^{\,k^T}_{s_i} + \mathcal{L}'^{\,k^T}_{f_i}\right) - \mathcal{L}, \tag{40}$$

where the exponent T stands for the transpose of the matrix.

As the trace of the product of a symmetric and an antisymmetric tensor nullifies, it remains:

$$\mathcal{H} = Trace\left(s_k{}^i \mathcal{L}'^{\,k^T}_{s_i}\right) + Trace\left(f_k{}^i \mathcal{L}'^{\,k^T}_{f_i}\right) - \mathcal{L}. \tag{41}$$

The energy density $\mathcal{H}$ can thus be split into two parts: $\mathcal{H} = \mathcal{H}_s + \mathcal{H}_f$

with:

$$\mathcal{H}_s = Trace\left(s_k{}^i \mathcal{L}'^{\,kT}_{s\,i}\right) - \mathcal{L}_s \tag{42}$$

and

$$\mathcal{H}_f = Trace\left(f_k{}^i \mathcal{L}'^{\,kT}_{f\,i}\right) - \mathcal{L}_f. \tag{43}$$

We have introduced the mass Lagrangian: $\mathcal{L}_s = \| s_k{}^i \|$ and the field Lagrangian: $\mathcal{L}_f = \mathcal{L} - \mathcal{L}_s = \| a_k{}^i \| - \| s_k{}^i \|$.

As the total energy nullifies, the integrals over spacetime of $\mathcal{H}_s$ and $\mathcal{H}_f$ (eq. (33)) are equal and opposite in sign.

Application to solution *e*. Distribution of energies is illustrated now in the case of solutions e and g. The Lagrangian density which corresponds to solution e is the determinant of (28):

$$\mathcal{L} = \frac{1}{(4\pi)^2}\left(\frac{\mathcal{A}\,\omega}{c}\right)^4\left(J_1\, J_1''\frac{J_1'^2}{\mathrm{x}^2}\sin^4\omega t + \frac{J_1'^4}{\mathrm{x}^2}\sin^2\omega t\cos^2\omega t\right). \tag{44}$$

The first term in the sum is the mass (or the potential energy) term:

$$\mathcal{L}_s = \frac{1}{(4\pi)^2}\left(\frac{\mathcal{A}\,\omega}{c}\right)^4 J_1\, J_1''\frac{J_1'^2}{\mathrm{x}^2}\sin^4\omega t. \tag{45}$$

It corresponds to the determinant of the symmetric part of $[a_k{}^i]$.

The second term is the field term:

$$\mathcal{L}_f = \frac{1}{(4\pi)^2}\left(\frac{\mathcal{A}\,\omega}{c}\right)^4 \frac{J_1'^4}{\mathrm{x}^2}\sin^2\omega t\cos^2\omega t. \tag{46}$$

It is the difference between $\mathcal{L}$ and $\mathcal{L}_s$.

The total energy density is:

$$W_t = \mathcal{C} \sum_{ik} a_k{}^i \frac{\partial \mathcal{L}}{\partial a_k{}^i} - \mathcal{C}\,\mathcal{L} = 3\mathcal{C}\,\mathcal{L}. \tag{47}$$

The mass energy density is:

$$W_s = \mathcal{C} \sum_{ik} s_k{}^i \frac{\partial \mathcal{L}}{\partial s_k{}^i} - \mathcal{C}\,\mathcal{L}_s = 3\mathcal{C}\,\mathcal{L}_s. \tag{48}$$

In the following, the factor 3 will be included in $\mathcal{C}$.

The integrated value of $\mathcal{L}$ over a period $\mathcal{T} = 2\pi/\omega$ is (first integral in eq. (33)):

$$\frac{c}{\omega}\int_0^{\mathcal{T}} \mathcal{L}(\omega t)\, d(\omega t) = \frac{c}{\omega}\frac{1}{(4\pi)^2}\left(\frac{\mathcal{A}\,\omega}{c}\right)^4\left(\frac{3\pi}{4} J_1\, J_1'' \frac{J_1'^2}{\mathrm{x}^2} + \frac{\pi}{4}\frac{J_1'^4}{\mathrm{x}^2}\right).$$

The total energy included in the volume element limited by two concentric spheres separated by $d\mathrm{x}$ is:

$$H_t = 4\pi\,\mathcal{C}\frac{c^4}{\omega^4} W_t\, \mathrm{x}^2\, d\mathrm{x}. \tag{49}$$

The corresponding mass energy is:

$$H_s = 4\pi\,\mathcal{C}\frac{c^4}{\omega^4} W_s\, \mathrm{x}^2\, d\mathrm{x}. \tag{50}$$

A factor 4π originates from the volume integration over the angles $(\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi)$. A factor c^3/ω^3 originates from the relation $r = c/\omega\,\mathrm{x}$. Graphical illustrations of the radial distributions of H_t, H_s and the difference $H_t - H_s$ for particle g and particle e are given in Figures 2 and 3. Formulas leading to Figure 3 are similar to those of Figure 2 if we replace J_1 by J_0.

The total energy W of the particle is the integral over x when x varies from zero to infinity. It is proportional to:

$$W \propto \int_0^\infty \left(3J_1\, J_1'' \frac{J_1'^2}{\mathrm{x}^2} + \frac{J_1'^4}{\mathrm{x}^2} \right) \mathrm{x}^2 d\mathrm{x}$$

$$= \int_0^\infty d\mathrm{x} \left(\frac{\partial (J_1\, J_1'^3)}{\partial \mathrm{x}} \right) = J_1\, J_1'^3 \Big|_0^\infty = 0. \tag{51}$$

The integral vanishes because J_1 vanishes when $\mathrm{x} \to 0$ [23] and $\mathrm{x} \to \infty$.

This result illustrates the general rule demonstrated before that the total field and mass energies which are, respectively, associated with the antisymmetric and symmetric parts of the tensor, are equal and opposite. It is a wonder that mathematics can build a world from zero total energy, the negative part of which being the mass, and the positive part the field energy.

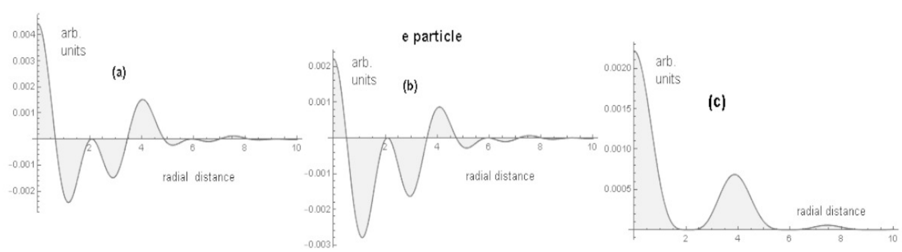

Figure 2. Radial distribution of energies for particle *e*. (a): H_t (total). (b): H_s (mass). (c): $H_t - H_s$ (field).

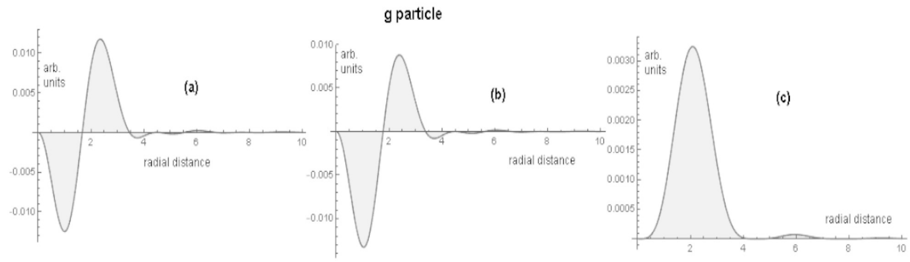

Figure 3. Radial distribution of energies for particle g. (a): H_t (total). (b): H_s (mass). (c): $H_t - H_s$ (field).

3.3.3. Far field tensors

The potential and its derivatives contain terms in $1/\mathrm{x}$, $1/\mathrm{x}^2$, $1/\mathrm{x}^3$, etc... Far from the origin O_p, only terms in $1/\mathrm{x}$ survive. This is the far field region where the potentials and the derivatives are limited to these terms. The asymptotic behavior of even spherical Bessel functions $J_0(\mathrm{x})$, $J_2(\mathrm{x})$, $J_4(\mathrm{x})$ ..., when x is large is $J_n^\infty \sim \pm \sin \mathrm{x}/\mathrm{x}$. For odd functions it is $J_n^\infty \sim \pm c\mathrm{os x}/\mathrm{x}$. The sign of successive functions is alternated. While $\Phi_{n,\ell,m}$ and A^r are functions in $1/\mathrm{x}$, A^θ and A^φ behave like $1/\mathrm{x}^2$. If we keep the $1/\mathrm{x}$ terms only, our fundamental tensor becomes:

$$[a_k{}^i]_\infty = \begin{bmatrix} (\phi/c)_{,t} & A^r_{,t} & 0 & 0 \\ (\phi/c)_{,r} & A^r_{,r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

We will use this expression later.

Asymptotic values of the potentials in the far field are for the even solutions:

$$\Phi_{n,\ell,m}/c \sim \mathcal{A}_n \frac{\sin \mathrm{x}}{\mathrm{x}} Y_\ell^m(\theta, \varphi) \cos \omega t$$

$$\sim \mathcal{A}_n \frac{1}{2\mathrm{x}} Y_\ell^m (\sin(\omega t + kr) + \sin(\omega t - kr)), \tag{53}$$

$$A^r_{n,\ell,m} \sim -\mathcal{A}_n \frac{\cos \mathrm{x}}{\mathrm{x}} Y_\ell^m \sin \omega t$$

$$\sim \mathcal{A}_n \frac{1}{2\mathrm{x}} Y_\ell^m (\sin(\omega t + kr) - \sin(\omega t - kr)), \tag{54}$$

$$A^\theta \sim 0,$$

$$A^\varphi \sim 0.$$

It follows that the field far from the particle is purely radial for any electromagnetic particles. The field becomes a superposition of an incoming and an outcoming spherical waves. As indicated above, this finding fits the conclusions published in [21].

3.3.4. Electric charge

This section describes the method to find the electric charge Q associated with an e.m. particle. The strategy is to use the fundamental equation which relates the induction $\overrightarrow{D}$ and the charge density ρ at a point M: $div\,\overrightarrow{D} = \rho$ and to integrate over the spacetime volume. For this purpose we will consider the even and the odd time-reversed spherical solutions, then we will find the relation between the field and the induction in the far field and finally we will integrate the charge density to obtain the total charge of the particle. We will find that this charge has a different sign for the two solutions which are considered.

Charge density for the even and the odd time-reversed spherical solutions. In spherical coordinates the divergence is given by the formula:

$$div\,\vec{D} = \frac{\partial D^r}{\partial r} + \frac{2D^r}{r} + \frac{\partial D^\theta}{r\,\partial\theta} + \frac{\partial D^\varphi}{r\sin\theta\,\partial\varphi} + \frac{\cos\theta}{r\sin\theta}D^\theta. \tag{55}$$

The induction is given by:

$$\vec{D} = \left(\frac{\mathcal{L}}{\mathcal{E}^r}, \frac{\mathcal{L}}{\mathcal{E}^\theta}, \frac{\mathcal{L}}{\mathcal{E}^\varphi}\right). \tag{56}$$

Let us apply these equations to the even spherical solutions (e and g) (case 1):

$$\phi/c = \frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}J\cos\omega t, \qquad A^r = -\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}J'\sin\omega t \tag{57}$$

and to the time-reversed odd solutions ($t \to -t$) (e^* and g^*) (case 2):

$$\phi/c = -\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}J\sin\omega t, \qquad A^r = \frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}J'\cos\omega t. \tag{58}$$

The Lagrangian, the field, the induction and the charge density are given by:

(case 1):

$$\mathcal{L} = \mathcal{C}\left(\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}\right)^4\left(J\,J''\,\frac{J'^2}{x^2}\sin^4\omega t + \frac{J'^4}{x^2}\cos^2\omega t\sin^2\omega t\right), \tag{59a}$$

$$\mathcal{E}^r = -\frac{1}{\sqrt{4\pi}}\mathcal{A}\,\omega\,J'\cos\omega t, \tag{59b}$$

$$\mathcal{D}^r = \frac{\partial\mathcal{L}}{\partial\mathcal{E}^r} = -\frac{\mathcal{C}}{c}\left(\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}\right)^3\frac{J'^3}{x^2}\sin^2\omega t\cos\omega t, \tag{59c}$$

$$\rho = div\,(\mathcal{D}_r) = -\frac{\omega\,\mathcal{C}}{c^2}\left(\frac{1}{\sqrt{4\pi}}\frac{\mathcal{A}\,\omega}{c}\right)^3$$

$$\times\sin^2\omega t\cos\omega t\left(3\frac{J'^2\,J''}{x^2} - 2\frac{J'^3}{x^3} + 2\frac{J'^3}{x^3}\right). \tag{59d}$$

(case 2):

$$\mathcal{L} = \mathcal{C}\left(\frac{1}{\sqrt{4\pi}}\,\frac{\mathcal{A}\,\omega}{c}\right)^4\left(J\,J''\,\frac{J'^2}{x^2}\cos^4\,\omega t + \frac{J'^4}{x^2}\sin^2\,\omega t\cos^2\,\omega t\right), \quad (60a)$$

$$\mathcal{E}^r = \frac{1}{\sqrt{4\pi}}\,\mathcal{A}\,\omega\,J'\sin\omega t, \quad (60b)$$

$$\mathcal{D}^r = \frac{\partial\mathcal{L}}{\partial\mathcal{E}^r} = \frac{\mathcal{C}}{c}\left(\frac{1}{\sqrt{4\pi}}\,\frac{\mathcal{A}\,\omega}{c}\right)^3\frac{J'^3}{x^2}\cos^2\,\omega t\sin\omega t, \quad (60c)$$

$$\rho = \frac{\omega\,\mathcal{C}}{c^2}\left(\frac{1}{\sqrt{4\pi}}\,\frac{\mathcal{A}\,\omega}{c}\right)^3\cos^2\,\omega t\sin\omega t\left(3\,\frac{J'^2\,J''}{x^2}\right). \quad (60d)$$

Long range behavior of the field and the induction: role of the noise. As a particle manifests itself at large distances by terms in $1/x$, the potential, the field or the energy stored in an element dv around a point M are very small in this region as compared to the incoherent sum of the corresponding quantities originating from the multitude of particles of the universe (we presume here that any existing particle can be described in the context of potential theory). One can thus characterize this region (the "vacuum") by a noise tensor whose components $n_k{}^i$ are incoherent and with a modulus large as compared to those originating from a single test-particle. We write the local tensor as a sum of this noise tensor and the test-particle tensor (eq. 52). The far field Lagrangian density $\mathcal{L}_{ff}$ is proportional to the determinant:

$$\mathcal{L}_{ff} = \mathcal{C}\left\| n_k{}^i + a_k{}^i{}_{ff} \right\|. \quad (61)$$

In the case of spherical solutions, the local induction due to the field $\mathcal{E}^r/c = A^r_t = -(\phi/c)_{,r}$ (see definition (24)) of the test particle is the derivative of $\mathcal{L}_{ff}$ with respect to $\mathcal{E}^r$. Here we should write the field with the $1/x$ terms only as the above expression stands for the long range. In order to keep this fact in mind we will write the field as $\mathcal{E}^r_{ff}$. This field writes for the even e solution:

$$\mathcal{E}^r_{ff} = \frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c} \cos \omega t \, \frac{\sin \mathrm{x}}{\mathrm{x}}. \tag{62}$$

Developing the determinant (61), performing the derivation with respect to $\mathcal{E}^r_{ff}$ and keeping the term proportional to the field only, give the far field radial induction:

$$D_{ff} = 2\mathcal{C}\, c^{-2} (n^3_4\, n^4_3 - n^3_3\, n^4_4)\, \mathcal{E}^r_{ff}. \tag{63}$$

This equation expresses the vacuum permittivity $\overline{\varepsilon_0}$ in terms of the noise tensor components:

$$D_{ff} = \overline{\varepsilon_0}\, \mathcal{E}^r_{ff} \qquad \text{with:} \qquad \overline{\varepsilon_0} = 2\mathcal{C}\, c^{-2} (n^3_4\, n^4_3 - n^3_3\, n^4_4). \tag{64}$$

Note that the dimensions of $\overline{\varepsilon_0}$ are: $\mathrm{M^{-1}T^2Q^2L^{-4}}$ while the usual vacuum permittivity has the dimensions $\varepsilon_0 = \mathrm{M^{-1}T^2Q^2L^{-3}}$. This is because the dimensions of $\mathcal{L}$ are that of a density of energy in four dimensional space.

General relation between the gradient tensor and the charge density. Let us now compute the induction created by the test-particle only at point M. For this purpose, we will use Euler-Lagrange equations:

$$\sum_k \frac{\partial}{\partial x^k} \left(\frac{\partial \mathcal{L}}{\partial A^i_{,k}} \right) - \frac{\partial \mathcal{L}}{\partial A^i} = 0. \tag{65}$$

The term $\partial\mathcal{L}/\partial A^i$ nullifies because $\mathcal{L}$ does not depend on A^i explicitly. The remaining terms represent a 4-divergence. For the component

$A^0 = \phi/c$:

$$div\,(\mathcal{L}'^0_k) = 0, \tag{66}$$

where $(\mathcal{L}'^0_k)$ is the 4-vector:

$$(\mathcal{L}'^0_k) = \left(\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial(\phi/c)}{c\,\partial t}\right)}, \frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial(\phi/c)}{\partial r}\right)}, \frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial(\phi/c)}{r\,\partial\theta}\right)}, \frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial(\phi/c)}{r\sin\theta\,\partial\varphi}\right)}\right). \tag{67}$$

In the proper frame, components of the field $\mathcal{E}/c$ are: $(-\partial(\phi/c)/\partial r$, $-\partial(\phi/c)/(r\,\partial\theta)$, $-\partial(\phi/c)/(r\sin\theta\,\partial\varphi))$. One can thus write $(\mathcal{L}'^0_k)$ under the form $(D^r = \partial\mathcal{L}/\partial A^r_{,t} = -\partial\mathcal{L}/\partial(\phi/c)_{,r}$ etc...):

$$(\mathcal{L}'^0_k) = \left(\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial(\phi/c)}{c\,\partial t}\right)}, -c\,D^r, -c\,D^\theta, -c\,D^\varphi\right), \tag{68}$$

where D^r, D^θ, D^φ are the components of the induction $\vec{D}$ in the geometrical space.

The simple equation follows:

$$\frac{\partial}{c\,\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\dfrac{\partial(\phi/c)}{c\,\partial t}\right)}\right) = c\,div\,\vec{D}. \tag{69}$$

One sees that the term $\partial\mathcal{L}/\partial\left(\dfrac{\partial(\phi/c)}{c\,\partial t}\right)$ is simply the **minor** relative to $\dfrac{\partial(\phi/c)}{c\,\partial t}$ in the determinant $\| a_k{}^i \|$ times the constant $\mathcal{C}$ (eq. 34). This minor is the determinant which groups the spatial derivatives in $a_k{}^i$.

One gets:

$$c\, div\, \vec{D} = \mathcal{C} \frac{\partial}{c\, \partial t} \| \Delta_1^1 \|, \tag{70}$$

where:

$$\| \Delta_1^1 \| = \begin{Vmatrix} A^r_{,r} & A^\theta_{,r} & A^\varphi_{,r} \\ \frac{1}{r}(A^r_{,\theta} - A^\theta) & \frac{1}{r}(A^\theta_{,\theta} + A^r) & \frac{1}{r} A^\varphi_{,\theta} \\ \frac{A^r_{,\varphi}}{r \sin\theta} - \frac{A^\varphi}{r} & \frac{1}{r \sin\theta}(A^\theta_{,\varphi} - \cos\theta\, A^\varphi) & \frac{1}{r \sin\theta} A^\varphi_{,\varphi} + \frac{1}{r} A^r + \frac{1}{r} \frac{\cos\theta}{\sin\theta} A^\theta \end{Vmatrix} \tag{71}$$

is the minor relative to $a_1{}^1 = (\phi/c)_{,\bar{t}}$.

When applied to solutions g or e, we have:

$$\| \Delta_1^1 \| = -\left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}_e \omega_e}{c} \right)^3 J'' \frac{J'^2}{x^2} \sin^3 \omega t. \tag{72}$$

One obtains:

$$div\, \vec{D} = \frac{\mathcal{C}}{c} \frac{\partial}{c\, \partial t} \| \Delta_1^1 \|$$

$$= -\frac{3\omega\, \mathcal{C}}{c^2} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}_e \omega_e}{c^2} \right)^3 J'' \frac{J'^2}{x^2} \sin^2 \omega t \cos \omega t. \tag{73}$$

When applied to solutions g^* or e^*, we have:

$$\| \Delta_1^1 \| = \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}_e \omega_e}{c} \right)^3 J'' \frac{J'^2}{x^2} \cos^3 \omega t, \tag{74}$$

and:

$$div\, \vec{D} = \frac{\mathcal{C}}{c} \frac{\partial}{c\, \partial(-t)} \| \Delta_1^1 \|$$

$$= \frac{3\omega\, \mathcal{C}}{c^2} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}_e \omega_e}{c} \right)^3 J'' \frac{J'^2}{x^2} \cos^2 \omega t \sin \omega t. \tag{75}$$

These expressions are the same as those obtained for the density of charge eqs. (59d) and (60d) before.

Total charge. In order to find the total charge of the particle we compute now the integral of $div\,\vec{D}$ in eqs. (73) and (75) with the volume element $r^2 dr \sin\theta\, d\theta\, d\varphi$ and over the geometrical volume $\mathcal{V}$ defined by the intervals $r = [0, R]$, $\theta = [0, \pi]$, $\varphi = [0, 2\pi]$ (with $x = \omega r/c$). This volume integral is exactly computed:

$$\int_{\mathcal{V}} J'' \frac{J'^2}{\mathrm{x}^2}\, dv = \frac{c^3}{\omega^3} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^{X} \left(J'' \frac{J'^2}{\mathrm{x}^2} \right) \mathrm{x}^2 d\mathrm{x}$$

$$= \frac{4\pi\, \mathrm{c}^3}{\omega^3} \int_0^{X} J''\, J'^2 d\mathrm{x} = \frac{4\pi\, \mathrm{c}^3}{3\omega^3} \left(J'^3_{(\mathrm{x}=X)} - J'^3_{(\mathrm{x}=0)} \right). \tag{76}$$

The term $J'^3_{(\mathrm{x}=X)}$ is negligible in the far field and eqs. (73) and (75) give:

$$\int_{\mathcal{V}} div\, \vec{D}\, dv = \left(\frac{4\pi\, \mathrm{c}^3}{\omega^3} \right) \frac{3\omega\, \mathcal{C}}{c^2} \left(\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\, \omega}{c} \right)^3 J'^3_{(\mathrm{x}=0)} \begin{cases} \sin^2 \omega t \cos \omega t, \\ -\cos^2 \omega t \sin \omega t. \end{cases} \tag{77}$$

In spacetime standard electrostatics, the field does not oscillate in time, and the factor $\sin\omega t$ or $\cos\omega t$ does not appear in Gauss law expressed in its integral form:

$$\Phi_E = \int_S (\mathcal{E}^r)_{tc}\, dS = \varepsilon_0\, R^2 \sqrt{4\pi}\, \mathcal{A}\, \omega\, J' \begin{cases} -1, \\ 1, \end{cases} \tag{78}$$

where Φ_E is the flux of the electric field across the Gauss surface S which is a sphere with a radius R. dS is the surface element and $(\mathcal{E}^r)_{tc}$ is the *time component* of the electric field:

$$(\mathcal{E}^r)_{tc} = \frac{1}{\sqrt{4\pi}}\, \mathcal{A}\, \omega\, J' \begin{cases} -1, \\ 1. \end{cases}$$

The total charge Q_e for the even solution is obtained after an integration over a time-length $2\pi\, c/\omega$ of $(\int_{\mathcal{V}} div\, \vec{D}\, dv)$ in the upper line of eq. (77) divided by $\cos \omega t$:

$$Q_e = \frac{3\sqrt{\pi}\, \mathcal{C}\, \mathcal{A}^3}{2c}\, J'^3_{(\mathrm{x}=0)}. \tag{79}$$

The total charge Q_o for the time-reversed odd solution is obtained after an integration over a period of inversed time-length $-2\pi\, c/\omega$ of $(\int_{\mathcal{V}} div\, \vec{D}\, dv)$ in the lower line of eq. (77) divided by $\sin \omega t$:

$$Q_o = -\frac{3\sqrt{\pi}\, \mathcal{C}\, \mathcal{A}^3}{2c}\, J'^3_{(\mathrm{x}=0)}. \tag{80}$$

Now one sees that solution g is not charged because $J'^3_{0(\mathrm{x}=0)} = 0$ while solutions e and e^* have the charge:

$$Q = \pm \frac{\sqrt{\pi}\, \mathcal{C}\, \mathcal{A}^3}{18\, c} \tag{81}$$

because $J'^3_{1(\mathrm{x}=0)} = (1/3)^3$ [23].

This quantity depends upon the amplitude $\mathcal{A}$ but Q is not an invariant of the tensor in spacetime and should be computed again for spinning particles.

A similar calculation can be done for cylindrically symmetric solutions q_i.

For solution q_0, the determinant (71) becomes:

$$\Delta_1^1 = \left(\frac{3}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c^2}\right)^3 \sin^3 \omega t \left(-\cos^3 \theta\; J_1'' \left(\left(\frac{J_1}{\mathrm{x}}\right)'\right)^2 + \cos\theta \sin^2\theta \left(\left(\frac{J_1}{\mathrm{x}}\right)'\right)^3 \right). \tag{82}$$

Integrations over the angle θ which appear in the calculation of the electric charge Q lead to the integrals:

$$\int_0^\pi \cos^3\theta \sin\theta \, d\theta = \int_0^\pi \cos\theta \sin^3\theta \, d\theta = 0 \tag{83}$$

which implies that the charge associated to solution q_0 is identically 0. The same result occurs for solutions q_1 and q_{-1} where the determinant includes terms proportional to $\cos^3\varphi$ or $\sin^2\varphi\cos\varphi$ whose integrals between 0 and 2π vanish. One can verify that all other solutions have the same property. The **only charged electromagnetic particles** are those described by solutions e and e^*. This result leads us to name solutions e and e^* the electron and the positron.

We note that the total charge (81) is not invariant in a coordinate change. It is the spinning electron that will give the global, observable, electron charge.

3.4. Rotating tensors

Up to now, the electromagnetic tensors have been written for points M at rest with respect to the local (or internal) center of coordinates O_p, i.e., in the inertial system where the geometrical coordinates are time-independent. We next studied the situation where M is subject to a rotation around the z axis. We have computed the expressions for the components of $[a_k{}^i]$ in the frame where M is at rest. We consider now the local infinitesimal length elements $c\,dt$, dr, $r\,d\theta$, $r\sin\theta\,d\varphi$ which define the volume element dv around M and which are measured by a rotating observer attached to M. These coordinates become $c\,dt'$, dr', $r'\,d\theta'$, $r'\sin\theta'\,d\varphi'$ for a fixed observer at M in the laboratory frame. Both sets of coordinates are linked by a local, tangential, Lorentz transformation. The elementary motion is a translation along the φ' axis and the two other local axes r' and θ' are perpendicular to it. It follows that the coordinates dr and $r\,d\theta$ are not affected by the rotation. Only the length element $r \sin\theta\, d\varphi$ and the time element $c\,dt$ at event M are subject to the Lorentz transformation:

$$c\,dt' = \gamma\, c\,dt + \gamma\beta\, r\sin\theta\, d\varphi, \quad r' = r, \quad \theta' = \theta,$$

$$r\sin\theta\, d\varphi' = \gamma\, r\sin\theta\, d\varphi + \gamma\beta\, c\,dt. \tag{84}$$

We use the standard notation β for the relative tangential velocity, and γ for the Lorentz factor:

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{85}$$

where β and γ could depend upon the coordinates.

Note that the temporal phase ωt (which is a true scalar) is invariant in a Lorentz transformation: $\omega t = \overline{\omega}\bar{t}$. Note also that the factor c/ω which appears in the equation of the potential or its derivatives is the normalization parameter that transforms the invariant radial coordinate r into the non-dimensioned quantity x. This parameter is not modified here.

The expression of the potential A'^i, as seen by a non-rotating observer at M, is obtained with the Jacobian (the Lorentz matrix):

$$\mathbb{J}_L = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix}. \tag{86}$$

One gets:

$$\begin{bmatrix} \phi'/c \\ A'^r \\ A'^\theta \\ A'^\varphi \end{bmatrix} = \mathbb{J}_L \begin{bmatrix} \phi/c \\ A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = \begin{bmatrix} \gamma\,\phi/c + \gamma\,\beta\,A^\varphi \\ A^r \\ A^\theta \\ \gamma\,\beta\,\phi/c + \gamma\,A^\varphi \end{bmatrix}.$$

The angle φ is now a function of time and the gradient tensor has supplementary terms originating from $A'^{\varphi}_{,t'} = \partial\,A'^{\varphi} / c\,\partial t' + \beta\,c\,\partial A'^{\varphi} / \partial\varphi'$ and from $(\,\phi'/c\,)_{,t'} = \partial(\,\phi'/c\,)/c\,\partial t' + \beta\,c\,\partial(\,\phi'/c\,)/\partial\varphi'$. Equation (5b) no longer applies, the observer is not in the proper time.

Let us apply the Lorentz transformation to solution e. The rotating potential is:

$$\begin{bmatrix} \phi'/c \\ A'^r \\ A'^\theta \\ A'^\varphi \end{bmatrix} = \begin{bmatrix} \gamma\,\phi/c \\ A^r \\ 0 \\ \gamma\,\beta\,\phi/c \end{bmatrix}.$$

The components of the gradient tensor are the derivatives of A'^i with respect to the coordinates eqs. (84). The general expression (20) gives:

$$[a_k{}^i] = \begin{bmatrix} (\phi'/c)_{,t'} & A^r_{,t'} & 0 & A'^{\varphi}_{,t'} \\ (\phi'/c)_{,r} & A^r_{,r} & 0 & A'^{\varphi}_{,r} \\ 0 & 0 & A^r\,/r & 0 \\ 0 & -\,A'^{\varphi}\,/r & -\,\dfrac{\cos\theta}{r\sin\theta}\,A'^{\varphi} & A^r\,/r \end{bmatrix}$$

$$= \begin{bmatrix} (\phi/c)_{,t} & 1/\gamma\,A^r_{,t} & 0 & \beta\,(\phi/c)_{,t} \\ \gamma\,(\phi/c)_{,r} & A^r_{,r} & 0 & \gamma\,\beta\,(\phi/c)_{,r} \\ 0 & 0 & A^r\,/r & 0 \\ 0 & -\,\dfrac{\gamma\,\beta\,(\phi/c)}{r} & -\,\dfrac{\cos\theta}{r\sin\theta}\,\gamma\,\beta\,(\phi/c) & A^r\,/r \end{bmatrix}. \quad (87)$$

The covariant tensor is:

$$[a_{ki}] = \begin{bmatrix} (\phi/c)_{,t} & -1/\gamma\, A^r_{,t} & 0 & -\beta\,(\phi/c)_{,t} \\ \gamma\,(\phi/c)_{,r} & -A^r_{,r} & 0 & -\gamma\beta\,(\phi/c)_{,r} \\ 0 & 0 & -A^r/r & 0 \\ 0 & \dfrac{\gamma\beta\,(\phi/c)}{r} & \dfrac{\cos\theta}{r\sin\theta}\gamma\beta\,(\phi/c) & -A^r/r \end{bmatrix}. \quad (88)$$

The electromagnetic field is the antisymmetric part:

$$[F_{ki}] = 0.5 \begin{bmatrix} 0 & -1/\gamma\, A^r_{,t} - \gamma\,(\phi/c)_{,r} & 0 & -\beta\,(\phi/c)_{,t} \\ \gamma\,(\phi/c)_{,r} + 1/\gamma\, A^r_{,t} & 0 & 0 & -\gamma\beta\,(\phi/c)_{,r} - \dfrac{\gamma\beta\,(\phi/c)}{r} \\ 0 & 0 & 0 & -\dfrac{\cos\theta}{r\sin\theta}\gamma\beta\,(\phi/c) \\ \beta\,(\phi/c)_{,t} & \gamma\beta\,(\phi/c)_{,r} + \dfrac{\gamma\beta\,(\phi/c)}{r} & \dfrac{\cos\theta}{r\sin\theta}\gamma\beta\,(\phi/c) & 0 \end{bmatrix}. \quad (89)$$

Using the relation: $(\phi/c)_{,r} = -A^r_{,t}$, one obtains:

$$[F_{ki}] = 0.5 \begin{bmatrix} 0 & -1/\gamma\, A^r_{,t} + \gamma\, A^r_{,t} & 0 & -\beta\,(\phi/c)_{,t} \\ -\gamma\, A^r_{,t} + 1/\gamma\, A^r_{,t} & 0 & 0 & \gamma\beta\,(A^r_{,t} - (\phi/c)/r) \\ 0 & 0 & 0 & -\dfrac{\cos\theta}{r\sin\theta}\gamma\beta\,(\phi/c) \\ \beta\,(\phi/c)_{,t} & -\gamma\beta\,(A^r_{,t} - (\phi/c)/r) & \dfrac{\cos\theta}{r\sin\theta}\gamma\beta\,(\phi/c) & 0 \end{bmatrix}.$$

Components of the electric field are:

$$E^r = -1/\gamma\, A^r_{,t} + \gamma\, A^r_{,t} = A^r_{,t}\,(\gamma - 1/\gamma), \tag{90a}$$

$$E^\theta = 0, \tag{90b}$$

$$E^\varphi = -\beta\,(\phi/c)_{,t}. \tag{90c}$$

Components of the magnetic field are:

$$B^r = \frac{\cos\theta}{r\sin\theta}\,\gamma\beta\,(\phi/c), \tag{91a}$$

$$B^\theta = \gamma\beta\,(A^r_{,t} + (\phi/c)/r), \tag{91b}$$

$$B^\varphi = 0. \tag{91c}$$

The factor 0.5 is left aside. As expected one notes the appearance of a magnetic field perpendicular to the electric field. In the far field range, $B^r \to 0$ and $B^\theta \to \gamma\beta\, A^r_{,t}$. The magnetic field becomes oriented along the local axis $\vec{\theta}$.

These fields are computed in the coordinate frame where the observer M is motionless with respect to the origin O_p. They are the **electrostatic fields** of the electron.

4. Tensors in the Laboratory Frame

This section shows that the standard electromagnetic tensor F_{ki} is the antisymmetric part of a_{ki} expressed in a general coordinates frame. The Lagrangian density which is associated with ${a_k}^i$ allows the calculation of inductions. Then Euler-Lagrange equations are applied to find Maxwell's equations.

4.1. Splitting the covariant derivative

One starts with the contravariant components of a general electromagnetic four-potential vector A^i ($i = 0, 1, 2, 3$). This potential can be related to those studied in the preceding section in each particular problem. The scalar potential is noted again $A^0 = \phi/c$ and the set $\vec{A} = (A^x, A^y, A^z)$ represents also the vector potential. An event M in real Minkowski's spacetime is defined by its coordinates $x^k = (ct, x, y, z)$ and a four-potential corresponds to each event: $A^i = A^i(M)$. The coordinates are defined in the Cartesian frame spanned by the normalized basis vectors $(\vec{e}_t, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ with origin O (see Figure 1). All the theory described here is *local*: the point M is surrounded by an arbitrarily small volume. There are quantities, like fields, which are defined at M and densities which are defined around M.

To obtain the corresponding covariant components A_i in the dual space, we use again the $(+, -, -, -)$ convention for the metric tensor $[\eta_{mn}]$ and, therefore, one has the relation: $A_i = \eta_{im} A^m$ written with Einstein's summation convention.

The 16 partial derivatives $a_k{}^i = \partial A^i / \partial x^k$ at M are the components of the tensor $[a_k{}^i]$ which are given in the expression (1).

The covariant form $[a_{ki}]$ is written explicitly using co or contravariant components of the potential:

$$[a_{ki}] = \begin{bmatrix} (\phi/c)_t & A_{x,t} & A_{y,t} & A_{z,t} \\ (\phi/c)_{,x} & A_{x,x} & A_{y,x} & A_{z,x} \\ (\phi/c)_{,y} & A_{x,y} & A_{y,y} & A_{z,y} \\ (\phi/c)_{,z} & A_{x,z} & A_{y,z} & A_{z,z} \end{bmatrix}$$

$$= \left[\sum_m a_k{}^m \eta_{mi} \right] = \begin{bmatrix} (\phi/c)_t & -A_t^x & -A_t^y & -A_t^z \\ (\phi/c)_{,x} & -A_{,x}^x & -A_{,x}^y & -A_{,x}^z \\ (\phi/c)_{,y} & -A_{,y}^x & -A_{,y}^y & -A_{,y}^z \\ (\phi/c)_{,z} & -A_{,z}^x & -A_{,z}^y & -A_{,z}^z \end{bmatrix}. \tag{92}$$

This tensor is divided into its symmetric and antisymmetric parts:

$$[S_{ki}] = \frac{1}{2}([a_{ki}] + [a_{ik}]) = \frac{1}{2}(\partial A_i/\partial x^k + \partial A_k/\partial x^i), \tag{93}$$

$$[F_{ki}] = \frac{1}{2}(\partial A_i/\partial x^k - \partial A_k/\partial x^i). \tag{94}$$

The antisymmetric part of $[a_{ki}]$ is:

$$[F_{ki}] = \frac{1}{2}\begin{bmatrix} 0 & -A_t^x - (\phi/c)_{,x} & -A_t^y - (\phi/c)_{,y} & -A_t^z - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A_t^x & 0 & A_{,y}^x - A_{,x}^y & A_{,z}^x - A_{,x}^z \\ (\phi/c)_{,y} + A_t^y & A_{,x}^y - A_{,y}^x & 0 & A_{,z}^y - A_{,y}^z \\ (\phi/c)_{,z} + A_t^z & A_{,x}^z - A_{,z}^x & A_{,y}^z - A_{,z}^y & 0 \end{bmatrix}. \tag{95}$$

The electromagnetic field is defined from the components of $[F_{ki}]$. Below are the usual equations which condense these definitions:

$$\vec{E} := -\frac{\partial \vec{A}}{\partial t} - \overrightarrow{grad}\,\phi, \qquad \vec{B} := \overrightarrow{curl}\,\vec{A}. \tag{96}$$

In these equations, the vector potential $\vec{A}$ is expressed with its contravariant components.

The fields which are used in Maxwell's equations are pseudovectors.

We will use the special notation E^X, E^Y, E^Z, B^X, B^Y, B^Z to distinguish them from usual vectors.

The electromagnetic tensor writes:

$$[F_{ki}] = \frac{1}{2}\begin{bmatrix} 0 & E^X/c & E^Y/c & E^Z/c \\ -E^X/c & 0 & -B^Z & B^Y \\ -E^Y/c & B^Z & 0 & -B^X \\ -E^Z/c & -B^Y & B^X & 0 \end{bmatrix}. \tag{97}$$

The preceding formulas are not new: they belong to the basic knowledge of electromagnetism (apart from the factor 1/2 in $[F_{ki}]$). This is not the case for the symmetric part of $[a_{ki}]$:

$$[S_{ki}] = \frac{1}{2}\begin{bmatrix} 2(\phi/c)_t & -A_t^x + (\phi/c)_{,x} & -A_t^y + (\phi/c)_{,y} & -A_t^z + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A_t^x & -2A_{,x}^x & -A_{,y}^x - A_{,x}^y & -A_{,z}^x - A_{,x}^z \\ (\phi/c)_{,y} - A_t^y & -A_{,x}^y - A_{,y}^x & -2A_{,y}^y & -A_{,z}^y - A_{,y}^z \\ (\phi/c)_{,z} - A_t^z & -A_{,x}^z - A_{,z}^x & -A_{,y}^z - A_{,z}^y & -2A_{,z}^z \end{bmatrix}. \tag{98}$$

$[S_{ki}]$ has been ignored in textbooks [4, 5] and in the specialized literature. This neglect leads to its replacement by charge and current densities, which are phenomenological quantities. We name this tensor the source part because it is responsible for the source terms which, as seen below, will appear in Maxwell's equations.

4.2. The Lagrangian

The remaining of this section is devoted to the demonstration of Maxwell's equations written in the standard form:

$$div\,\vec{B} = 0, \qquad \overrightarrow{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$div\,\vec{D} = \rho, \qquad \overrightarrow{curl}\,\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}. \tag{99}$$

We will show that these equations result from the principle of least action applied to the electromagnetic potential and its derivatives.

The first pair of eqs. (99) shows the relation between electric $\vec{E}$ and magnetic $\vec{B}$ fields. The second couple links the inductions $\vec{D}$ and $\vec{H}$ to the sources ρ (charge density) and $\vec{j}$ (current density). Inductions are defined from the Lagrangian $\mathcal{L}$ which is proportional to the determinant (eq. 34):

$$\mathcal{L} = \mathcal{C} \left\| a_k{}^i \right\| = \mathcal{C} \begin{Vmatrix} (\phi/c)_t & A^x_t & A^y_t & A^z_t \\ (\phi/c)_{,x} & A^x_{,x} & A^y_{,x} & A^z_{,x} \\ (\phi/c)_{,y} & A^x_{,y} & A^y_{,y} & A^z_{,y} \\ (\phi/c)_{,z} & A^x_{,z} & A^y_{,z} & A^z_{,z} \end{Vmatrix}.$$

This expression will not be explicitly used in the following. However, it shows that the derivative of $\mathcal{L}$ with respect to the term $A_k{}^i$ is the determinant of the minor relative to $A_k{}^i$ (accompanied by the proper sign). For instance:

$$\frac{\partial \mathcal{L}}{\partial((\phi/c)_t)} = \mathcal{C} \begin{Vmatrix} A^x_{,x} & A^y_{,x} & A^z_{,x} \\ A^x_{,y} & A^y_{,y} & A^z_{,y} \\ A^x_{,z} & A^y_{,z} & A^z_{,z} \end{Vmatrix}.$$

4.3. Maxwell's equations

First pair. The first pair of Maxwell's equations are identities which are nicely expressed [9] by the equation:

$$\frac{\partial F_{k\ell}}{\partial x^m} + \frac{\partial F_{\ell m}}{\partial x^k} + \frac{\partial F_{mk}}{\partial x^\ell} = 0. \tag{100}$$

Second pair. Let us now show that the source terms ρ and $\vec{j}$ in the second pair are related to the induction tensor $[\mathcal{L}'^k{}_i]$.

An element $\mathcal{L}'^{k}{}_{i}$ of $[\mathcal{L}'^{k}{}_{i}]$ is obtained from the derivative of the Lagrangian with respect to the element $a_k{}^i$. The developed form is:

$$[\mathcal{L}'^{k}{}_{i}] = \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial(\phi/c)_t} & \frac{\partial\mathcal{L}}{\partial A_t^x} & \frac{\partial\mathcal{L}}{\partial A_t^y} & \frac{\partial\mathcal{L}}{\partial A_t^z} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial\mathcal{L}}{\partial A_{,x}^x} & \frac{\partial\mathcal{L}}{\partial A_{,x}^y} & \frac{\partial\mathcal{L}}{\partial A_{,x}^z} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial\mathcal{L}}{\partial A_{,y}^x} & \frac{\partial\mathcal{L}}{\partial A_{,y}^y} & \frac{\partial\mathcal{L}}{\partial A_{,y}^z} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} & \frac{\partial\mathcal{L}}{\partial A_{,z}^x} & \frac{\partial\mathcal{L}}{\partial A_{,z}^y} & \frac{\partial\mathcal{L}}{\partial A_{,z}^z} \end{bmatrix}. \tag{101}$$

An element $\mathcal{L}'^{k}{}_{i}$ is the *canonical momentum* corresponding to $a_k{}^i$.

Now the corresponding covariant tensor $[\mathcal{L}'_{ki}]$ is split into its symmetric and antisymmetric parts which are then transformed back into mixed tensors. The first operation ensures that the symmetry (or antisymmetry) of the tensors is independent of the coordinates system.

One obtains the separation of $[\mathcal{L}'^{k}{}_{i}]$ into two parts: $[\mathcal{L}'^{k}{}_{i}] = [\mathbb{D}^{k}{}_{i}] + [\mathbb{S}^{k}{}_{i}]$.

The first part is directly linked to the usual induction tensor, it corresponds to the antisymmetric part of $[\mathcal{L}'_{ki}]$. The second part corresponds to the symmetric part of $[\mathcal{L}'_{ki}]$ and will be referred to as the source tensor. The expressions of these tensors are:

$$[\mathbb{D}^{k}{}_{i}] = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial\mathcal{L}}{\partial A_t^x} + \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial\mathcal{L}}{\partial A_t^y} + \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial\mathcal{L}}{\partial A_t^z} + \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} \\ \frac{\partial\mathcal{L}}{\partial A_t^x} + \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & 0 & \frac{\partial\mathcal{L}}{\partial A_{,x}^y} - \frac{\partial\mathcal{L}}{\partial A_{,y}^x} & \frac{\partial\mathcal{L}}{\partial A_{,x}^z} - \frac{\partial\mathcal{L}}{\partial A_{,z}^x} \\ \frac{\partial\mathcal{L}}{\partial A_t^y} + \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial\mathcal{L}}{\partial A_{,y}^x} - \frac{\partial\mathcal{L}}{\partial A_{,x}^y} & 0 & \frac{\partial\mathcal{L}}{\partial A_{,y}^z} - \frac{\partial\mathcal{L}}{\partial A_{,z}^y} \\ \frac{\partial\mathcal{L}}{\partial A_t^z} + \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} & \frac{\partial\mathcal{L}}{\partial A_{,z}^x} - \frac{\partial\mathcal{L}}{\partial A_{,x}^z} & \frac{\partial\mathcal{L}}{\partial A_{,z}^y} - \frac{\partial\mathcal{L}}{\partial A_{,y}^z} & 0 \end{bmatrix} \tag{102}$$

and:

$$[\mathbb{S}^k{}_i] = \frac{1}{2}\begin{bmatrix} 2\dfrac{\partial\mathcal{L}}{\partial(\phi/c)_t} & \dfrac{\partial\mathcal{L}}{\partial A_t^x} - \dfrac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & \dfrac{\partial\mathcal{L}}{\partial A_{,t}^y} - \dfrac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & \dfrac{\partial\mathcal{L}}{\partial A_{,t}^z} - \dfrac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} \\ \dfrac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} - \dfrac{\partial\mathcal{L}}{\partial A_{,t}^x} & 2\dfrac{\partial\mathcal{L}}{\partial A_{,x}^x} & \dfrac{\partial\mathcal{L}}{\partial A_{,x}^y} + \dfrac{\partial\mathcal{L}}{\partial A_{,y}^x} & \dfrac{\partial\mathcal{L}}{\partial A_{,z}^y} + \dfrac{\partial\mathcal{L}}{\partial A_{,x}^z} \\ \dfrac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} - \dfrac{\partial\mathcal{L}}{\partial A_{,t}^y} & \dfrac{\partial\mathcal{L}}{\partial A_{,x}^y} + \dfrac{\partial\mathcal{L}}{\partial A_{,y}^x} & 2\dfrac{\partial\mathcal{L}}{\partial A_{,y}^y} & \dfrac{\partial\mathcal{L}}{\partial A_{,y}^z} + \dfrac{\partial\mathcal{L}}{\partial A_{,z}^y} \\ \dfrac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} - \dfrac{\partial\mathcal{L}}{\partial A_{,t}^z} & \dfrac{\partial\mathcal{L}}{\partial A_{,z}^x} + \dfrac{\partial\mathcal{L}}{\partial A_{,x}^z} & \dfrac{\partial\mathcal{L}}{\partial A_{,z}^y} + \dfrac{\partial\mathcal{L}}{\partial A_{,y}^z} & 2\dfrac{\partial\mathcal{L}}{\partial A_{,z}^z} \end{bmatrix}. \quad (103)$$

Electric and magnetic inductions are given by the derivatives of the Lagrangian with respect to the components of the fields $\vec{E}$ and $\vec{B}$. One applies the chain rule and equations (96) to obtain:

$$D_X = \frac{\partial\mathcal{L}}{\partial(E^X)} = -c\,\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} - c\,\frac{\partial\mathcal{L}}{\partial A_{,t}^x}, \quad (104)$$

$$H_X = \frac{\partial\mathcal{L}}{\partial B^X} = \left(\frac{\partial\mathcal{L}}{\partial A_{,y}^z} - \frac{\partial\mathcal{L}}{\partial A_{,z}^y}\right). \quad (105)$$

Other components D_Y, D_Z, H_Y and H_Z are obtained from circular permutations of x, y, z.

We have used a lower index notation D_X, D_Y, ... to stress the fact that the components of the induction pseudovectors are those of a type $[D^k{}_i]$ tensor:

$$[\mathbb{D}^k{}_i] = \frac{1}{2}\begin{bmatrix} 0 & -cD_X & -cD_Y & -cD_Z \\ -cD_X & 0 & H_Z & -H_Y \\ -cD_Y & -H_Z & 0 & H_X \\ -cD_Z & H_Y & -H_X & 0 \end{bmatrix}. \quad (106)$$

The relations [24] between the pseudovectors D^X, ..., H^X... and D_X, ..., H_X... are:

$$D^X = -D_X, \quad D^Y = -D_Y, \quad D^Z = -D_Z, \tag{107a}$$

$$H^X = H_X, \quad H^Y = H_Y, \quad H^Z = H_Z. \tag{107b}$$

In the following we will use this notation to write Maxwell's equations in both direct and inverse spaces. In passing, one should note that the splitting of $[a_{ki}]$ into its symmetric and antisymmetric parts allows the study of special cases where one of the tensor can be nullified in some regions of space while the other still exists. An illustration is the Aharonov-Bohm effect [10] which shows that a potential can exist in a region of space even in the absence of any field $(F_{ki} = 0)$. One sees that in such a situation it is the source tensor $[S_{ki}]$ which can change the phase of the electron when it crosses this region.

Euler-Lagrange equations. Equation (65) expresses the principle of least action and introduces the conjugate momenta $\partial\mathcal{L}/\partial A^i_{,k}$ which are the elements of the induction tensor (101). We will show that Maxwell's equations are a consequence of this principle.

The Lagrangian density $\mathcal{L}$ does not depend explicitly on the potentials (only on its derivatives) and equation (65) reduces to the first term. It introduces the tensor $[\mathcal{L}'^k{}_i]$ whose elements have been written before:

$$\mathcal{L}'^k{}_i = \frac{\partial\mathcal{L}}{\partial a_k{}^i}. \tag{108}$$

When $\partial\mathcal{L}/\partial A^i = 0$, equation (65) can be written in matrix form:

$$\left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)[\mathcal{L}'^k{}_i] = (0,\ 0,\ 0,\ 0). \tag{109}$$

This expression groups four equations and we show now that it leads to the second pair of Maxwell's equations in reciprocal space.

We use the separation of $[\mathcal{L}'^k{}_i]$ into its two parts $[\mathbb{D}^k{}_i]$ and $[\mathbb{S}^k{}_i]$ and write eq. (109) in compressed notation:

$$(\partial)[\mathcal{L}'^k{}_i] = (0) \qquad \text{or} \qquad (\partial)([\mathbb{D}^k{}_i]) = -(\partial)([\mathbb{S}^k{}_i]). \tag{110}$$

Expressions for $\mathbb{D}^k{}_i$ and $\mathbb{S}^k{}_i$ are given by eqs. (102) and (103). In the following we skip the factor $1/2$ before $\mathbb{D}^k{}_i$ and $\mathbb{S}^k{}_i$ which simplifies eq. (110).

The first term $(\partial)([\mathbb{D}^k{}_i])$ is computed first:

$$(\partial)([\mathbb{D}^k{}_i]) = \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{bmatrix} 0 & -cD_X & -cD_Y & -cD_Z \\ -cD_X & 0 & H_Z & -H_Y \\ -cD_Y & -H_Z & 0 & H_X \\ -cD_Z & H_Y & -H_X & 0 \end{bmatrix}$$

$$= \left(-c\,div\,\overrightarrow{D_I}, \left(-\overrightarrow{curl}\,\vec{H} - \frac{\partial\overrightarrow{D_I}}{\partial t}\right)\right). \tag{111}$$

$\overrightarrow{D_I} = (D_X, D_Y, D_Z)$ is the symbol for the induction in the reciprocal space.

The 4-vector $\overrightarrow{D_I}$ in equation (111) has a time component $-c\,div\,\overrightarrow{D_I}$ et 3 space components $\left(-\overrightarrow{curl}\,\vec{H} - \frac{\overrightarrow{D_I}}{c\,\partial t}\right)$. These are the induction components in Maxwell's equations.

The right hand side term $(\partial)([\mathbb{S}^k{}_i])$ in eq. (110) is computed now:

$$(\partial)([S^k{}_i]) = \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times$$

$$\begin{bmatrix} 2\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,t}} & \frac{\partial\mathcal{L}}{\partial A^x_{,t}} - \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & \frac{\partial\mathcal{L}}{\partial A^y_{,t}} - \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & \frac{\partial\mathcal{L}}{\partial A^z_{,t}} - \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} - \frac{\partial\mathcal{L}}{\partial A^x_{,t}} & 2\frac{\partial\mathcal{L}}{\partial A^x_{,x}} & \frac{\partial\mathcal{L}}{\partial A^y_{,x}} + \frac{\partial\mathcal{L}}{\partial A^x_{,y}} & \frac{\partial\mathcal{L}}{\partial A^y_{,z}} + \frac{\partial\mathcal{L}}{\partial A^z_{,x}} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} - \frac{\partial\mathcal{L}}{\partial A^y_{,t}} & \frac{\partial\mathcal{L}}{\partial A^y_{,x}} + \frac{\partial\mathcal{L}}{\partial A^x_{,y}} & 2\frac{\partial\mathcal{L}}{\partial A^y_{,y}} & \frac{\partial\mathcal{L}}{\partial A^z_{,y}} + \frac{\partial\mathcal{L}}{\partial A^y_{,z}} \\ \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} - \frac{\partial\mathcal{L}}{\partial A^z_{,t}} & \frac{\partial\mathcal{L}}{\partial A^x_{,z}} + \frac{\partial\mathcal{L}}{\partial A^z_{,x}} & \frac{\partial\mathcal{L}}{\partial A^y_{,z}} + \frac{\partial\mathcal{L}}{\partial A^z_{,y}} & 2\frac{\partial\mathcal{L}}{\partial A^z_{,z}} \end{bmatrix}$$

$$= \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{bmatrix} \frac{\partial\mathcal{L}}{\partial(\phi/c)_{,t}} & -\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,x}} & -\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,y}} & -\frac{\partial\mathcal{L}}{\partial(\phi/c)_{,z}} \\ -\frac{\partial\mathcal{L}}{\partial A^x_{,t}} & \frac{\partial\mathcal{L}}{\partial A^x_{,x}} & \frac{\partial\mathcal{L}}{\partial A^x_{,y}} & \frac{\partial\mathcal{L}}{\partial A^x_{,z}} \\ -\frac{\partial\mathcal{L}}{\partial A^y_{,t}} & \frac{\partial\mathcal{L}}{\partial A^y_{,x}} & \frac{\partial\mathcal{L}}{\partial A^y_{,y}} & \frac{\partial\mathcal{L}}{\partial A^y_{,z}} \\ -\frac{\partial\mathcal{L}}{\partial A^z_{,t}} & \frac{\partial\mathcal{L}}{\partial A^z_{,x}} & \frac{\partial\mathcal{L}}{\partial A^z_{,y}} & \frac{\partial\mathcal{L}}{\partial A^z_{,z}} \end{bmatrix}. \tag{112}$$

The second expression is obtained after simplification by eq. (109). Equating each component of the 4-vector of eq. (110) gives:

$$c\,div\,\overrightarrow{D_I} = \frac{\partial}{c\,\partial t}\frac{\partial\mathcal{L}}{(\phi/c)_{,t}} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{A^x_{,t}} - \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{A^y_{,t}} - \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{A^z_{,t}}, \tag{113a}$$

$$\frac{\partial D_X}{\partial t} + [\overrightarrow{curl}\,\vec{H}]_x = -\frac{\partial}{c\,\partial t}\frac{\partial\mathcal{L}}{(\phi/c)_{,x}} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{A^x_{,x}} + \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{A^y_{,x}} + \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{A^z_{,x}}. \tag{113b}$$

The two remaining equations along the y and z axis are obtained from circular permutations of $x,\ y,\ z$ and $X,\ Y,\ Z$.

We use these equations to introduce the following new 4-vectors in spacetime:

$$\overrightarrow{\mathcal{L}'_k} = \left(\frac{\partial\mathcal{L}}{(\phi/c)_{,k}}, -\frac{\partial\mathcal{L}}{A^x_{,k}}, -\frac{\partial\mathcal{L}}{A^y_{,k}}, -\frac{\partial\mathcal{L}}{A^z_{,k}}\right), \tag{114}$$

where k stands for $t,\ x,\ y,$ or z.

One sees that the r.h.s. of eqs. (113a, 113b) are all 4-divergences of these vectors:

$$c\, div\, \overrightarrow{D_I} = div\, \overrightarrow{\mathcal{L}'_t}, \tag{115a}$$

$$\frac{\partial D_X}{\partial t} + [\overrightarrow{curl}\, \vec{H}]_x = -\, div\, \overrightarrow{\mathcal{L}'_x}. \tag{115b}$$

These divergences define the source terms:

$$\rho := \frac{1}{c}\, div\, \overrightarrow{\mathcal{L}'_t}, \qquad j_x := -\, div\, \overrightarrow{\mathcal{L}'_x},$$

$$j_y := -\, div\, \overrightarrow{\mathcal{L}'_y}, \qquad j_z := -\, div\, \overrightarrow{\mathcal{L}'_z}. \tag{116}$$

The lower indices x, y, z label the components of the covector $\overrightarrow{j_i} = (j_x,\, j_y,\, j_z)$. One thus obtains Maxwell's equations in matrix form in the reciprocal space:

$$(c\,\rho,\, j_x,\, j_y,\, j_z)$$

$$= \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{bmatrix} 0 & -cD_X & -cD_Y & -cD_Z \\ -cD_X & 0 & H_Z & -H_Y \\ -cD_Y & -H_Z & 0 & H_X \\ -cD_Z & H_Y & -H_X & 0 \end{bmatrix}. \tag{117}$$

Finally, Maxwell's equations in the direct space are obtained after transforming the covariant quadrivector $(c\,\rho,\, j_x,\, j_y,\, j_z)$ into its contravariant counterpart $(c\,\rho,\, -j^x,\, -j^y,\, -j^z)$ and the pseudovector $\overrightarrow{D_I}$ into $\vec{D}$. These operations give the desired result:

$$div\, \vec{D} = \rho,$$

$$[\overrightarrow{curl}\, \vec{H}]^X = \frac{\partial D^X}{\partial t} + \overrightarrow{j^x} \quad (x \text{ component})$$

or in matrix form:

$$(c\,\rho, -j^x, -j^y, -j^z)$$

$$= \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{bmatrix} 0 & cD^X & cD^Y & cD^Z \\ cD^X & 0 & H^Z & -H^Y \\ cD^Y & -H^Z & 0 & H^X \\ cD^Z & H^Y & -H^X & 0 \end{bmatrix}. \tag{118}$$

One can use the above formulas to verify the continuity equation:

$$div\,\vec{j} = -\frac{\partial\rho}{\partial t}. \tag{119}$$

We have demonstrated in this section that Maxwell's equations can be deduced from a few basic operations:

1- The covariant tensor of derivatives $[a_{ki}]$ has been split into its symmetric and antisymmetric parts. This symmetry is independent of the coordinate system.

2- The Lagrangian density has been associated with the determinant of $[a_k{}^i]$. This determinant is independent of the coordinate system.

3- Induction tensors have been computed.

4- The principle of least action has been applied.

The study which is presented in this section is very general and does not need any particular form of potential. It connects the well-known electromagnetic tensor to the antisymmetric part of $[a_{ki}]$ and the source terms to the symmetric part. It shows that the fundamental quantity is the potential and that Maxwell's equations are a consequence of the least action principle. The potential is very general and can be related to the potential which describes a particle in particular situations.

5. Conclusion

We have described in this article the properties of a family of tensors whose elements are obtained from partial derivatives of a potential A^i at each point M in Minkowski's spacetime. The primary tensor is the gradient $[a_k{}^i]$ at M.

In the first main part of the study, $[a_k{}^i]$ is divided into its symmetric ($[s_k{}^i]$) and antisymmetric ($[f_k{}^i]$) parts. We use the properties of these tensors to show that there is a proper time where the scalar potential obeys the Helmholtz equation. The solutions of this equation describe electromagnetic particles. These are characterized by an accumulation of energy around the origin of the coordinates and by far fields in $1/\mathrm{x}$. The condition of existence of these particles corresponds to Wheeler-Feyman's theory where an equilibrium must exist between the incoming and outgoing waves of the electron. Each solution can be even or odd and is characterized by three quantum numbers. We have given the tensors corresponding to the first five solutions. The essential result of this part is the union in a single expression of field and matter properties. In the second main part of the study, $[a_k{}^i]$ is first transformed into the covariant tensor $[a_{ki}]$ which is then divided into its symmetric ($[S_{ki}]$) and antisymmetric ($[F_{ki}]$) parts. These two tensors are different from ($[s_{ki}]$) and ($[f_{ki}]$) because the lowering-index operation does not commute with the symmetric-antisymmetric splitting. We find that F_{ki} is the well-known tensor of classical electromagnetism. By applying the Euler-Lagrange equations, we find Maxwell's equations. We prove that the source terms are expressed as functions of the derivatives of the potential.

We summarize below the **key points** of the study.

1- There are two assumptions:

- A continuous and flat Minkowski's spacetime: $x^k = (c\,t,\ x,\ y,\ z,\ t)$.

- A 4-dimensional potential A^i: with the scalar potentiel ϕ/c and the vector potentiel $A^x,\ A^y,\ A^z$.

2- The study of the gradient of the potential: $[a^i_k] = [\partial A^i / \partial x^k]$ (rank 2 tensor, 16 components) and its splitting into its symmetric and antisymmetric parts: $[a^i_k] = [s^i_k] + [f^i_k]$ leads to the concept of "electromagnetic particles".

The main invariants of $[a^i_k]$ in symmetry operations of the Poincaré group are its determinant $\| a^i_k \|$ and its trace $\sum_i a^i_i$. They are used to obtain the fundamental Helmholtz equation in the **proper time** and to obtain the local Lagrangian density $\mathcal{L}$.

3- The study of the covariant tensor $[a_{ki}]$ and its splitting into its symmetric and antisymmetric parts: $[a_{ki}] = [S_{ki}] + [F_{ki}]$ leads directly to $[F_{ki}]$; the usual electromagnetic tensor.

Applying the **principle of least action**, Maxwell equations are obtained from $[a_{ki}]$.

The theory that has been described is simple, synthetic and powerful.

It is **simple** because it is based (1) on a single assumption, that of a 4-potential in Minkowski's spacetime, (2) on two non-commuting mathematical manipulations of tensors, i.e., raising-lowering operation and antisymmetric-symmetric splitting, (3) on fundamental physical principles which are the principle of least action, the principle of symmetry and the principle of relativity. It is **synthetic** because it groups in a family of tensors classical electromagnetism and a description of new fundamental particles. It is **powerful** because it contains a wealth of developments. The mathematical existence of the electromagnetic particles is thought-provoking: Associating these particles together and studying their interactions will change our perception of the microscopic and cosmological universe.

6. Appendix A

The non-commutativity of the antisymmetrization-symmetrization operation (a-s operation) and the lowering-raising operation (l-r operation) is illustrated in this appendix with the use of 2×2 matrices to shorten the notation (or in a one dimensional geometrical space).

The starting tensor is written as a mixed tensor in the real space:

$$[\mathbf{a}_k{}^i] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{120}$$

The metric tensor is:

$$[\mathbf{g}_m{}^n] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{121}$$

Let us consider first the lowering index operation (1) acting on $[\mathbf{a}_k{}^i]$:

$$[\mathbf{a}_{ki}] = [\mathbf{a}_k{}^i] \cdot [\mathbf{g}_m{}^n] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$$

and followed by the a-s operation (2):

$$[f_{ki}] = \frac{1}{2}([\mathbf{a}_{ki}] - [\mathbf{a}_{ik}]) = \frac{1}{2}\begin{bmatrix} 0 & -b-c \\ c+b & 0 \end{bmatrix},$$

$$[s_{ki}] = \frac{1}{2}([\mathbf{a}_{ki}] + [\mathbf{a}_{ik}]) = \frac{1}{2}\begin{bmatrix} 2a & -b+c \\ c-b & -2d \end{bmatrix}.$$

Consider now the a-s operation (2) acting on $[\mathbf{a}_k{}^i]$:

$$[\bar{s}_k{}^i] = \frac{1}{2}([\mathbf{a}_k{}^i] + [\mathbf{a}_i{}^k]) = \frac{1}{2}\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}, \tag{122}$$

$$[\bar{f}_k{}^i] = \frac{1}{2}([\mathbf{a}_k{}^i] - [\mathbf{a}_i{}^k]) = \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}. \tag{123}$$

It is followed by the lowering index operation (1):

$$[\bar{s}_{ki}] = \frac{1}{2}\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a & -(b+c) \\ c+b & -2d \end{bmatrix},$$

$$[\bar{f}_{ki}] = \frac{1}{2}\begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & c-b \\ c-b & 0 \end{bmatrix}.$$

One sees that $[s_{ki}] \neq [\bar{s}_{ki}]$ and $[f_{ki}] \neq [\bar{f}_{ki}]$. It is this non-commutativity which leads to the two branches of electromagnetism:

The first branch is based on the tensors (1) (expressed in the Cartesian frame of coordinates):

$$[a_k{}^i] = \begin{bmatrix} (\phi/c)_{,t} & A^x_{,t} & A^y_{,t} & A^z_{,t} \\ (\phi/c)_{,x} & A^x_{,x} & A^y_{,x} & A^z_{,x} \\ (\phi/c)_{,y} & A^x_{,y} & A^y_{,y} & A^z_{,y} \\ (\phi/c)_{,z} & A^x_{,z} & A^y_{,z} & A^z_{,z} \end{bmatrix}, \tag{124}$$

$$[s_k{}^i] = \frac{1}{2}\begin{bmatrix} 2(\phi/c)_{,t} & A^x_{,t} + (\phi/c)_{,x} & A^y_{,t} + (\phi/c)_{,y} & A^z_{,t} + (\phi/c)_{,z} \\ (\phi/c)_{,x} + A^x_{,t} & 2A^x_{,x} & A^y_{,x} + A^x_{,y} & A^z_{,x} + A^x_{,z} \\ (\phi/c)_{,y} + A^y_{,t} & A^x_{,y} + A^y_{,x} & 2A^y_{,y} & A^z_{,y} + A^y_{,z} \\ (\phi/c)_{,z} + A^z_{,t} & A^x_{,z} + A^z_{,x} & A^y_{,z} + A^z_{,y} & 2A^z_{,z} \end{bmatrix},$$

(125)

$$[f_k{}^i] = \frac{1}{2}\begin{bmatrix} 0 & A^x_{,t} - (\phi/c)_{,x} & A^y_{,t} - (\phi/c)_{,y} & A^z_{,t} - (\phi/c)_{,z} \\ (\phi/c)_{,x} - A^x_{,t} & 0 & A^y_{,x} - A^x_{,y} & A^z_{,x} - A^x_{,z} \\ (\phi/c)_{,y} - A^y_{,t} & A^x_{,y} - A^y_{,x} & 0 & A^z_{,y} - A^y_{,z} \\ (\phi/c)_{,z} - A^z_{,t} & A^x_{,z} - A^z_{,x} & A^y_{,z} - A^z_{,y} & 0 \end{bmatrix}, \tag{126}$$

$$[s_{ki}] = \frac{1}{2}\begin{bmatrix} 2(\phi/c)_{,t} & -A^x_{,t} - (\phi/c)_{,x} & -A^y_{,t} - (\phi/c)_{,y} & -A^z_{,t} - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A^x_{,t} & -2A^x_{,x} & -A^y_{,x} - A^x_{,y} & -A^z_{,x} - A^x_{,z} \\ (\phi/c)_{,y} + A^y_{,t} & -A^x_{,y} - A^y_{,x} & -2A^y_{,y} & -A^z_{,y} - A^y_{,z} \\ (\phi/c)_{,z} + A^z_{,t} & -A^x_{,z} - A^z_{,x} & -A^y_{,z} - A^z_{,y} & -2A^z_{,z} \end{bmatrix}, \tag{127}$$

$$[f_{ki}] = \frac{1}{2}\begin{bmatrix} 0 & -A^x_{,t} + (\phi/c)_{,x} & -A^y_{,t} + (\phi/c)_{,y} & -A^z_{,t} + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A^x_{,t} & 0 & -A^y_{,x} + A^x_{,y} & -A^z_{,x} + A^x_{,z} \\ (\phi/c)_{,y} - A^y_{,t} & -A^x_{,y} + A^y_{,x} & 0 & -A^z_{,y} + A^y_{,z} \\ (\phi/c)_{,z} - A^z_{,t} & -A^x_{,z} + A^z_{,x} & -A^y_{,z} + A^z_{,y} & 0 \end{bmatrix}. \tag{128}$$

The second branch is based on the tensors starting from (92):

$$[a_{ki}] = \begin{bmatrix} (\phi/c)_{,t} & -A^x_{,t} & -A^y_{,t} & -A^z_{,t} \\ (\phi/c)_{,x} & -A^x_{,x} & -A^y_{,x} & -A^z_{,x} \\ (\phi/c)_{,y} & -A^x_{,y} & -A^y_{,y} & -A^z_{,y} \\ (\phi/c)_{,z} & -A^x_{,z} & -A^y_{,z} & -A^z_{,z} \end{bmatrix}, \tag{129}$$

$$[S_{ki}] = \frac{1}{2}\begin{bmatrix} 2(\phi/c)_{,t} & -A^x_{,t} + (\phi/c)_{,x} & -A^y_{,t} + (\phi/c)_{,y} & -A^z_{,t} + (\phi/c)_{,z} \\ (\phi/c)_{,x} - A^x_{,t} & -2A^x_{,x} & -A^y_{,x} - A^x_{,y} & -A^z_{,x} - A^x_{,z} \\ (\phi/c)_{,y} - A^y_{,t} & -A^x_{,y} - A^y_{,x} & -2A^y_{,y} & -A^z_{,y} - A^y_{,z} \\ (\phi/c)_{,z} - A^z_{,t} & -A^x_{,z} - A^z_{,x} & -A^y_{,z} - A^z_{,y} & -2A^z_{,z} \end{bmatrix}, \tag{130}$$

$$[F_{ki}] = \frac{1}{2}\begin{bmatrix} 0 & -A^x_{,t} - (\phi/c)_{,x} & -A^y_{,t} - (\phi/c)_{,y} & -A^z_{,t} - (\phi/c)_{,z} \\ (\phi/c)_{,x} + A^x_{,t} & 0 & -A^y_{,x} + A^x_{,y} & -A^z_{,x} + A^x_{,z} \\ (\phi/c)_{,y} + A^y_{,t} & -A^x_{,y} + A^y_{,x} & 0 & -A^z_{,y} + A^y_{,z} \\ (\phi/c)_{,z} + A^z_{,t} & -A^x_{,z} + A^z_{,x} & -A^y_{,z} + A^z_{,y} & 0 \end{bmatrix}. \tag{131}$$

$[F_{ki}]$ is the standard electromagnetic tensor (apart from the factor $1/2$). Fields are defined from the usual equations (96).

7. Appendix B

This appendix gives explicit formulas for the potentials and the corresponding tensors for the first 5 even electromagnetic particles. Odd solutions are obtained by changing ωt into $\omega t \pm \pi/2$.

g **solution** $(n = \ell = m = 0)$:

Potential:

$$\phi/c = \frac{1}{\sqrt{4\pi}} \mathcal{A}\, J_0 \cos \omega t,$$

$$A^r = -\frac{1}{\sqrt{4\pi}} \mathcal{A}\, J_0' \sin \omega t, \qquad A_r = -A^r,$$

$$A^\theta = 0, \qquad A^\varphi = 0. \tag{132}$$

Mixed tensor:

$[a_k{}^i] =$

$$\frac{1}{\sqrt{4\pi}} \frac{\mathcal{A}\,\omega}{c} \begin{bmatrix} -\sin \omega t\, J_0 & -\cos \omega t\, J_0' & 0 & 0 \\ \cos \omega t\, J_0' & -\sin \omega t\, J_0'' & 0 & 0 \\ 0 & 0 & -\sin \omega t\, \dfrac{J_0'}{\mathrm{x}} & 0 \\ 0 & 0 & 0 & -\sin \omega t\, \dfrac{J_0'}{\mathrm{x}} \end{bmatrix}. \tag{133}$$

e **solution** $(n = 1,\ \ell = m = 0)$:

The same formulas are obtained by replacing J_0 by J_1.

q_0 **solution** $(n = 1,\ \ell = 1,\ m = 0)$:

Potential:

$$\phi/c = \sqrt{\frac{3}{4\pi}}\, \mathcal{A}\, J_1 \cos\theta \cos \omega t,$$

$$A^r = -\sqrt{\frac{3}{4\pi}}\,\mathcal{A}\,J_1'\cos\theta\sin\omega t, \qquad A_r = -A^r,$$

$$A^\theta = \sqrt{\frac{3}{4\pi}}\,\mathcal{A}\,\frac{J_1}{\mathrm{x}}\sin\theta\sin\omega t, \qquad A_\theta = -A^\theta,$$

$$A^\varphi = 0. \tag{134}$$

Mixed tensor:

$$[a_k{}^i] = \sqrt{\frac{3}{4\pi}}\,\frac{\mathcal{A}\,\omega}{c}\begin{bmatrix} -\sin\omega t\,J_1\cos\theta & -\cos\omega t\,J_1'\cos\theta & \cos\omega t\,\dfrac{J_1}{\mathrm{x}}\sin\theta & 0 \\ \cos\omega t\,J_1'\cos\theta & -\sin\omega t\,J_1''\cos\theta & \sin\omega t\sin\theta\left(\dfrac{J_1'}{\mathrm{x}} - \dfrac{J_1}{\mathrm{x}^2}\right) & 0 \\ -\cos\omega t\,\dfrac{J_1}{\mathrm{x}}\sin\theta & \sin\omega t\left(\dfrac{J_1'}{\mathrm{x}} - \dfrac{J_1}{\mathrm{x}^2}\right)\sin\theta & -\sin\omega t\cos\theta\left(\dfrac{J_1'}{\mathrm{x}} - \dfrac{J_1}{\mathrm{x}^2}\right) & 0 \\ 0 & 0 & 0 & -\sin\omega t\cos\theta\left(\dfrac{J_1'}{\mathrm{x}} - \dfrac{J_1}{\mathrm{x}^2}\right) \end{bmatrix}. \tag{135}$$

q_1 **solution** ($n = 1,\ \ell = 1,\ m = 1$):

Potential:

$$\phi/c = \sqrt{\frac{3}{4\pi}}\,\mathcal{A}\,J_n\sin\theta\cos\varphi\cos\omega t,$$

$$A^r = -\sqrt{\frac{3}{4\pi}}\,\mathcal{A}\,J_n'\sin\theta\cos\varphi\sin\omega t, \qquad A_r = -A^r,$$

$$A^\theta = -\sqrt{\frac{3}{4\pi}}\,\mathcal{A}\,\frac{J_n}{\mathrm{x}}\cos\theta\cos\varphi\sin\omega t, \qquad A_\theta = -A^\theta,$$

$$A^\varphi = \sqrt{\frac{3}{4\pi}}\,\mathcal{A}\,\frac{J_n}{\mathrm{x}}\sin\varphi\sin\omega t, \qquad A_\varphi = -A^\varphi. \tag{136}$$

Mixed tensor:

$$[a_k{}^i] = \frac{\mathcal{A}\,\omega}{c}\sqrt{\frac{3}{4\pi}}\begin{bmatrix} -\sin\omega t\, J_1 \sin\theta\cos\varphi & -\cos\omega t\, J_1' \sin\theta\cos\varphi & -\cos\omega t\,\frac{J_1}{\mathrm{x}}\cos\theta\cos\varphi & \cos\omega t\,\frac{J_1}{\mathrm{x}}\sin\varphi \\ \cos\omega t\, J_1' \sin\theta\cos\varphi & -\sin\omega t\, J_1'' \sin\theta\cos\varphi & -\sin\omega t\cos\theta\cos\varphi\left(\frac{J_1'}{\mathrm{x}} - \frac{J_1}{\mathrm{x}^2}\right) & \sin\omega t\sin\varphi\left(\frac{J_1'}{\mathrm{x}} - \frac{J_1}{\mathrm{x}^2}\right) \\ \cos\omega t\,\frac{J_1}{\mathrm{x}}\cos\theta\cos\varphi & -\sin\omega t\left(\frac{J_1'}{\mathrm{x}} - \frac{J_1}{\mathrm{x}^2}\right)\cos\theta\cos\varphi & -\sin\omega t\sin\theta\cos\varphi\left(\frac{J_1'}{\mathrm{x}} - \frac{J_1}{\mathrm{x}^2}\right) & 0 \\ -\cos\omega t\,\frac{J_1}{\mathrm{x}}\sin\varphi & \sin\omega t\left(\frac{J_1'}{\mathrm{x}} - \frac{J_1}{\mathrm{x}^2}\right)\sin\varphi & 0 & -\sin\omega t\sin\theta\cos\varphi\left(\frac{J_1'}{\mathrm{x}} - \frac{J_1}{\mathrm{x}^2}\right) \end{bmatrix}. \quad (137)$$

q_{-1} **solution** $(n = 1,\ \ell = 1,\ m = -1)$.

Formulas for the q_{-1} solution are obtained by changing φ into $\varphi - \pi/2$ in the preceding expressions.

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