POSITIVE-NEGATIVE ENERGY PARTNER STATES AND CONJUGATE QUANTUM POLYNOMIALS FOR A LINEAR HARMONIC OSCILLATOR

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Abstract

Factorization in normal order form provides positive energy solutions, while the alternative anti-normal order form provides negative energy solutions of the time-independent Schroedinger equation for a linear harmonic oscillator. The positive and negative energy state spaces are related by quantum conjugation effected through sign-reversal of Planck's quantization constant $\hbar$. Photons occupying negative energy states have the same displacement $\hat{x}$ and Hamiltonian H, but opposite momentum $-\hat{p}$ compared to photons occupying the corresponding positive energy partner states. Emission of positive energy photons from a positive energy quantum state is equivalent to absorption of negative energy photons into a negative energy quantum state, leading to state lowering, while absorption of positive energy photons into a positive energy quantum state is equivalent to emission of negative energy photons from a negative energy quantum state, leading to state raising. The complete positive-negative energy spectrum of the quantized oscillator may then be

interpreted as a photon-antiphoton system. We have discovered new $\hbar$-dependent quantized oscillator polynomials and their quantum conjugates defining positive and negative energy state eigenfunctions. These polynomials satisfy corresponding second order ordinary differential equations.

1. Introduction

The one-dimensional time-independent Schroedinger equation for a linear harmonic oscillator of mass m, angular frequency ω, total energy E and displacement x is obtained as

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \right)\psi(x) = E\psi(x) \tag{1a}$$

which on introducing parameter s defined by

$$s = \sqrt{m\omega}x \tag{1b}$$

for ease of physical interpretation takes the form

$$\left(-\hbar^2\frac{d^2}{ds^2} + s^2 \right)\psi(s) = \frac{2E}{\omega}\psi(s). \tag{1c}$$

It has always been assumed that the Schroedinger equation (1a) applies only to non-negative energy, $E \geq 0$, states $\psi(x)$. Indeed, standard methods generally applied to solve equation (1a) or its alternative form (1c) yield only the expected positive energy spectrum.

We observe that operator ordering in an effective factorization procedure well developed in the present author's book [1] provides for both positive and negative energy solutions, even though only the familiar positive energy solutions based on the normal order form have been comprehensively presented in the book. We develop the solution procedure for the anti-normal order form yielding the largely ignored negative energy spectrum in the present paper. Negative energy states are generally known to exist and are associated with anti-matter within the general framework of relativistic quantum mechanics and quantum field theory [3-7]. What has remained unknown is that negative energy states also exist in non-relativistic quantum mechanics, which we establish here through solutions of the Schroedinger equation (1a).

1.1. Factorization

To develop the factorization procedure, we consider the Schroedinger equation in the form (1c). Noting that the operator $-\hbar^2 \dfrac{d^2}{ds^2} + s^2$, we apply an effective factorization procedure [1] to express equation (1c) in two alternative factorized forms

$$\left(-\hbar\frac{d}{ds} + s \right)\left(\hbar\frac{d}{ds} + s \right)\psi = \left(\frac{2E}{\omega} - \hbar \right)\psi, \tag{2a}$$

$$\left(\hbar\frac{d}{ds} + s \right)\left(-\hbar\frac{d}{ds} + s \right)\psi = \left(\frac{2E}{\omega} + \hbar \right)\psi \tag{2b}$$

which differ by operator ordering.

To determine the nature of the operator orderings, we divide equations (2a)-(2b) by 2 and introduce operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hbar\frac{d}{ds} + s \right); \quad \hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left(-\hbar\frac{d}{ds} + s \right) \tag{2c}$$

to rewrite equations (2a)-(2b) in the corresponding forms

$$\hat{a}^{\dagger}\hat{a}\psi = \left(\frac{E}{\hbar\omega} - \frac{1}{2}\hbar \right)\psi, \tag{2d}$$

$$\hat{a}\hat{a}^{\dagger}\psi = \left(\frac{E}{\hbar\omega} + \frac{1}{2}\hbar \right)\psi. \tag{2e}$$

Subtracting equation (2d) from equation (2e) and dropping the arbitrary ψ gives the commutation bracket

$$[\hat{a},\, \hat{a}^{\dagger}] = \hbar. \tag{2f}$$

It will become clear later that the operators $\hat{a}$ and $\hat{a}^{\dagger}$ defined in equation (2c) are annihilation and creation operators, respectively, of the quantized oscillator. We have deliberately excluded Planck's quantization constant $\hbar$ from the definitions of $\hat{a}$, $\hat{a}^{\dagger}$ for consistency in the application of conjugation procedure developed in Section 4. According to standard operator ordering definitions [2], equation (2d) and its original expression (2a) are in normal order form, while equation (2e) and its original

expression (2b) are in anti-normal order form. Since both forms arise from factorization of the same equation (1c), they represent alternative solutions of the original Schroedinger equation (1a). We obtain these solutions below, starting with the normal order form, which provides the familiar positive energy states in Section 2, followed by the anti-normal order form, which provides the generally ignored negative energy states in Section 3. A quantum conjugation theory for transforming the positive energy states into their negative energy partner states, leading to the discovery of quantum conjugate oscillator polynomials, is developed in Section 4.

2. Normal Order Form: Quantized Oscillator Polynomials and Positive Energy Spectrum

The normal order form in equation (2a) is an eigenvalue equation with eigenvalue $\dfrac{2E}{\omega} - \hbar.$ It has a lower bound of zero eigenvalue obtained as

$$\frac{2E}{\omega} - \hbar \equiv \frac{2E_0}{\omega} - \hbar = 0 \Rightarrow E_0 = \frac{1}{2}\hbar\omega, \tag{3a}$$

where E_0 denotes the lowest value of E obtained at zero eigenvalue. The corresponding lowest order eigenfunction $\psi_0(s)$ at zero eigenvalue $\left(\dfrac{2E_0}{\omega} - \hbar = 0\right)$ is determined through equation (2a) under the condition (3a) according to

$$E = E_0 = \frac{1}{2}\hbar\omega; \quad \psi(s) = \psi_0(s) \Rightarrow \left(-\hbar\frac{d}{ds} + s\right)\left(\hbar\frac{d}{ds} + s\right)\psi_0(s) = 0. \tag{3b}$$

Applying Hermitian conjugation of the operators $\left(-\hbar\dfrac{d}{ds} + s\right)$ and $\left(\hbar\dfrac{d}{ds} + s\right)$ by reversing the sign of $\hbar$ according to (see Section 4)

$$\hbar \to -\hbar: \quad \left(-\hbar\frac{d}{ds} + s\right) = \left(\hbar\frac{d}{ds} + s\right)^{\dagger} \tag{3c}$$

in equation (3b) gives

$$\left(\hbar\frac{d}{ds} + s\right)^{\dagger}\left(\hbar\frac{d}{ds} + s\right)\psi_0 = 0 \tag{3d}$$

which on multiplying from the left by the ($\hbar$-sign reversed) Hermitian conjugate

$\psi_0^\dagger(s)$ of the lowest order eigenfunction $\psi_0(s)$ takes the form

$$\psi_0^\dagger(s)\left(\hbar\frac{d}{ds} + s\right)^\dagger\left(\hbar\frac{d}{ds} + s\right)\psi_0(s) = 0 \quad \Rightarrow \quad \left\|\left(\hbar\frac{d}{ds} + s\right)\psi_0(s)\right\|^2 = 0. \quad (3e)$$

The basic equation for the lowest order eigenfunction $\psi_0(s)$ then follows from equation (3e) in the form

$$\left(\hbar\frac{d}{ds} + s\right)\psi_0(s) = 0 \tag{4a}$$

with a simple solution

$$\psi_0(s) = e^{-\frac{1}{2\hbar}s^2} \tag{4b}$$

noting that the integration constant evaluated at $s = 0$ is $\psi_0(0) = 1$.

Eigenfunctions $\psi_n(s)$ of general order are generated through repeated application of the conjugate operator $\left(-\hbar\frac{d}{ds} + s\right)$ on the lowest order eigenfunction $\psi_0(s)$ according to

$$\psi_n(s) = \left(-\hbar\frac{d}{ds} + s\right)^n \psi_0(s); \quad n = 0, 1, 2, 3, ..., \infty \tag{4c}$$

which on substituting $\psi_0(s)$ from equation (4b) and evaluating for $n = 0, 1$ gives the first two lower order eigenfunctions in the form

$$\psi_0(s) = e^{-\frac{1}{2\hbar}s^2}, \quad \psi_1(s) = 2s\psi_0(s). \tag{4d}$$

To evaluate higher order eigenfunctions $\psi_n(s)$, $n \geq 2$, we derive a simplifying formula for any functions $f(s)$, $\phi(s)$ in the form

$$\left(-\hbar\frac{d}{ds} + s\right)f\phi = f\left(-\hbar\frac{d}{ds} + s\right)\phi - \hbar\frac{df}{ds}\phi \tag{4e}$$

and apply the general relation

$$\psi_{n+1}(s) = \left(-\hbar\frac{d}{ds} + s\right)\psi_n(s); \quad n = 0, 1, 2, 3, ... \tag{4f}$$

which follows easily from equation (4c) by setting $n \to n + 1$.

For $\psi_2(s)$, equation (4f) gives

$$\psi_2(s) = \left(-\hbar \frac{d}{ds} + s\right)\psi_1(s) \qquad (5a)$$

which on substituting $\psi_1(s) = 2s\psi_0(s)$ from equation (4d) and applying the formula (4e) with $f = 2s$, $\phi = \psi_0(s)$, then using equation (4f) in the final step gives

$$\psi_2(s) = 2(s\psi_1(s) - \hbar\psi_0(s)). \qquad (5b)$$

Proceeding in the same manner for

$$\psi_3(s) = \left(-\hbar \frac{d}{ds} + s\right)\psi_2(s); \quad \psi_4(s) = \left(-\hbar \frac{d}{ds} + s\right)\psi_3(s) \qquad (5c)$$

easily gives the forms

$$\psi_3(s) = 2(s\psi_2(s) - 2\hbar\psi_1(s)); \quad \psi_4(s) = 2(s\psi_3(s) - 3\hbar\psi_2(s)). \qquad (5d)$$

We arrive at the important general result that higher order eigenfunctions are obtained in the form of a recurrence relation

$$\psi_{n+1}(s) = 2(s\psi_n(s) - n\hbar\psi_{n-1}(s)), \quad n = 0, 1, 2, 3, ..., \infty. \qquad (5e)$$

Setting $n = 0, 1, 2, 3, ...$ in equation (5e) and substituting lower order eigenfunctions as appropriate, recalling $\psi_0(s)$ from equation (4b) or (4d), we obtain the general eigenfunction $\psi_n(s)$ in the form

$$\psi_n(s) = H_n(s, \hbar)e^{-\frac{1}{2\hbar}s^2}, \quad n = 0, 1, 2, 3, ..., \qquad (6a)$$

where $H_n(s, \hbar)$ is a polynomial depending explicitly on the parameter s and Planck's quantization constant $\hbar$. We shall call $H_n(s, \hbar)$ the quantized oscillator polynomial. Using equation (4b) in equation (4c) and substituting the result on the l.h.s. of equation (6a) provides the general relation for generating the quantized oscillator polynomials in the form

$$H_n(s, \hbar) = e^{\frac{1}{2\hbar}s^2}\left(-\hbar \frac{d}{ds} + s\right)^n e^{-\frac{1}{2\hbar}s^2}, \quad n = 0, 1, 2, ..., \infty. \qquad (6b)$$

Using equation (4b) together with its ($\hbar$-sign reversed) Hermitian conjugate

$$\psi_0^\dagger(s) = e^{\frac{1}{2\hbar}s^2} \tag{6c}$$

in equation (6b) defines the quantized oscillator polynomials in terms of the lowest order eigenfunction according to

$$H_n(s, \hbar) = \psi_0^\dagger(s)\left(-\hbar\frac{d}{ds} + s\right)^n \psi_0(s). \tag{6d}$$

Explicit forms of $H_n(s, \hbar)$ are easily obtained using a recurrence relation derived in the next subsection.

2.1. Recurrence relations for $H_n(s, \hbar)$

Setting $n \to n+1$ in equation (6b) and inserting $e^{-\frac{1}{2\hbar}s^2} e^{\frac{1}{2\hbar}s^2} = 1$ as appropriate, then using equation (6b) gives the relation

$$H_{n+1}(s, \hbar) = e^{\frac{1}{2\hbar}s^2}\left(-\hbar\frac{d}{ds} + s\right)H_n(s, \hbar)e^{-\frac{1}{2\hbar}s^2} \tag{7a}$$

which is easily evaluated to obtain the first recurrence relation for the polynomials $H_{n+1}(s, \hbar)$ in the form

$$H_{n+1} = 2sH_n - \hbar\frac{dH_n}{ds}, \quad H_m = H_m(s, \hbar), \; m = n, \, n+1. \tag{7b}$$

Setting $n = 0$ in equation (6b) gives

$$H_0 = 1. \tag{7c}$$

Setting $n = 0, 1, 2, 3$ in equation (7b) then provides the first five quantized oscillator polynomials as

$$H_0 = 1; \; H_1 = 2s; \; H_2 = 4s^2 - 2\hbar; \; H_3 = 8s^3 - 12\hbar s; \; H_4 = 16s^4 - 48\hbar s^2 + 12\hbar^2 \tag{7d}$$

taking the general expansion

$$H_n(s, \hbar) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n!(-\hbar)^m}{(n-2m)!\,m!}(2s)^{(n-2m)} \tag{7e}$$

which clearly displays the explicit dependence of the polynomials on the quantization constant $\hbar$, thus suggesting the reference quantized oscillator polynomials. The symbol $\left[\dfrac{1}{2}\right]$ in the summation means that m runs over integer values up to the integer part of $\dfrac{n}{2}$, e.g., $n = 3$, $m = 0, 1$. The quantized oscillator polynomials take the same form as the Hermite polynomials [1, 8-10], but differ only on the $\hbar$ factors in the various terms. Setting $\hbar = 1$ in equation (7e) gives the corresponding Hermite polynomials $H_n(s)$ in the general expansion form [9]

$$H_n(s) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n!(-1)^m}{(n-2m)!\,m!}(2s)^{(n-2m)} \tag{7f}$$

which would arise if we defined the parameter s in the dimensionless form $s = \sqrt{\dfrac{m\omega}{\hbar}}x$ instead of the form in equation (1b).

   Substituting

$$\psi_{n+1}(s) = H_{n+1}(s,\hbar)e^{-\frac{1}{2\hbar}s^2}\;;\;\psi_n(s) = H_n(s,\hbar)e^{-\frac{1}{2\hbar}s^2}\;;\;\psi_{n-1}(s) = H_{n-1}(s,\hbar)e^{-\frac{1}{2\hbar}s^2}$$

into equation (5e) gives the second recurrence relation for the quantized oscillator polynomials in the form

$$H_{n+1}(s,\hbar) = 2sH_n(s,\hbar) - 2n\hbar H_{n-1}(s,\hbar). \tag{7f}$$

Comparing the first recurrence relation (7b) and the second recurrence relation (7f) easily provides the third recurrence relation for the quantized oscillator polynomials in the form

$$\frac{dH_n}{ds} = 2nH_{n-1}, \quad H_m = H_m(s,\hbar). \tag{7g}$$

Applying $\hbar\dfrac{d}{ds}$ on equation (7g) gives

$$\hbar\frac{d^2H_n}{ds^2} = 2n\hbar\frac{dH_{n-1}}{ds}. \tag{8a}$$

Using equation (7e) together with the result of setting $n \to n+1$ in equation (7g) gives

$$2n\hbar \frac{dH_{n-1}}{ds} = 2s \frac{dH_n}{ds} - 2nH_n \tag{8b}$$

which we substitute into equation (8a) to obtain the differential equation for the quantized oscillator polynomials in the form

$$\hbar \frac{d^2 H_n}{ds^2} - 2s \frac{dH_n}{ds} + 2nH_n = 0, \quad H_n = H_n(s, \hbar) \tag{8c}$$

which differs from the familiar Hermite differential equation [1, 8-10] only due to the factor $\hbar$. Setting $\hbar = 1$ reduces equation (8c) to the Hermite differential equation.

2.2. Positive energy spectrum

Substituting

$$H_n(s, \hbar) = \psi_n(s) e^{\frac{1}{2\hbar} s^2} \tag{9a}$$

from equation (6a) into equation (8c) and reorganizing gives the final result

$$\left(-\hbar^2 \frac{d^2}{ds^2} + s^2 \right) \psi_n(s) = \hbar(2n+1)\psi_n(s) \tag{9b}$$

which confirms that the eigenfunctions $\psi_n(s)$ satisfy the original Schroedinger equation in the form (1c), with $\frac{2E}{\omega}$ taking the corresponding discrete form $\hbar(2n+1)$ specifying quantization.

Comparing equations (1c) and (9b), noting $E \to E_n$ gives

$$\frac{2E_n}{\omega} = \hbar(2n+1) \tag{9c}$$

which provides the positive energy spectrum for the quantized linear harmonic oscillator in the usual form

$$E_n = \hbar\omega\left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, 3, ..., \infty. \tag{9d}$$

This is the positive energy spectrum arising from the solution of the Schroedinger

equation factorized in the normal order form (2a).

2.3. Algebraic operations with $\hat{a}$, $\hat{a}^\dagger$

Applying the operator $\hbar \dfrac{d}{ds} + s$ on the general eigenfunction $\psi_n(s) = H_n(s, \hbar)e^{-\frac{1}{2\hbar}s^2}$ and evaluating gives

$$\left(\hbar \frac{d}{ds} + s\right)\psi_n = \hbar \frac{dH_n}{ds}e^{-\frac{1}{2\hbar}s^2} \tag{10a}$$

which on using the third recurrence relation (7f) and then substituting

$$\psi_{n-1}(s) = H_{n-1}(s, \hbar)e^{-\frac{1}{2\hbar}s^2} \tag{10b}$$

takes the final form

$$\left(\hbar \frac{d}{ds} + s\right)\psi_n(s) = 2n\hbar\psi_{n-1}(s). \tag{10c}$$

This operation with $\hbar \dfrac{d}{ds} + s$ is essentially the reverse of the operation with its conjugate $-\hbar \dfrac{d}{ds} + s$ on $\psi_n(s)$ obtained earlier in equation (4f). Bringing the two together gives the complete pair of reverse algebraic operations on the general eigenfunction according to

$$\left(-\hbar \frac{d}{ds} + s\right)\psi_n(s) = \psi_{n+1}(s); \quad \left(\hbar \frac{d}{ds} + s\right)\psi_n(s) = 2\hbar n\psi_{n-1}(s). \tag{10d}$$

It follows that

$$\left(-\hbar \frac{d}{ds} + s\right)\psi_{n-1}(s) = \psi_n(s); \quad \left(\hbar \frac{d}{ds} + s\right)\psi_{n+1}(s) = 2\hbar(n+1)\psi_n(s). \tag{10e}$$

Successive operations in normal and anti-normal order then easily give eigenvalue equations

$$\left(-\hbar \frac{d}{ds} + s\right)\left(\hbar \frac{d}{ds} + s\right)\psi_n(s) = 2\hbar n\psi_n(s);$$

$$\left(\hbar \frac{d}{ds} + s\right)\left(-\hbar \frac{d}{ds} + s\right)\psi_n(s) = 2\hbar(n+1)\psi_n(s). \tag{10f}$$

Dividing these through by 2 and introducing the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ as defined in equation (2c) gives

$$\hat{a}^\dagger \hat{a}\psi_n(s) = \hbar n\psi_n(s); \quad \hat{a}\hat{a}^\dagger \psi_n(s) = \hbar(n+1)\psi_n(s). \tag{11a}$$

Effective operations with $\hat{a}$ and $\hat{a}^\dagger$ consistent with the set of equations (10d), (10f) and (11a) take the algebraic form $(\psi = \psi(s))$

$$\hat{a}^\dagger \psi_n = \psi_{n+1}; \quad \hat{a}\psi_n = \hbar n\psi_{n-1}. \tag{11b}$$

Notice that substituting equation (2c) for $\hat{a}$ and $\hat{a}^\dagger$ separately into equation (10d) would leave factors $\dfrac{1}{\sqrt{2}}$ and $\sqrt{2}$ as appropriate on the r.h.s. of each equation in (11b), which would effectively yield the result of successive operations in equation (11a). Hence, the effective algebraic operations with the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ take the form in equation (11b).

As usual, equation (11b) reveals that the creation operator $\hat{a}^\dagger$ is a raising operator, while the annihilation operator $\hat{a}$ is a lowering operator for the positive energy eigenfunctions $\psi_n(s)$ obtained as solutions of the normal order form (2a). The form of the algebraic operations in equation (11b) is different from the Dirac form in standard quantum mechanics textbooks [8], but it is more useful in the physical interpretation of operations with the creation and annihilation operators. The transition to the familiar Dirac algebraic form is presented in [1].

First we note an important feature that in the basic operator algebraic relations obtained in equation (11b), the quantity $n\hbar$ is equal to the well known quantized orbital angular momentum. Secondly, according to equation (11b), the operation of a creation operator $\hat{a}^\dagger$ on $\psi_n(s)$ represents photon absorption process at the n-th quantum state, where the number of photons absorbed as the quantum state is raised to the $(n+1)$-th state described by $\psi_{n+1}(s)$ remains hidden, i.e., absorbed photons are internal and are not externally observable. On the other hand, the operation of an annihilation operator $\hat{a}$ on $\psi_n(s)$ represents photon emission process at the n-th quantum state, where the number of photons emitted (n) as the quantum state is lowered to the $(n-1)$-th state described by $\psi_{n-1}(s)$ is revealed, i.e., emitted photons are external and are directly observable.

3. Anti-normal Order Form: Conjugate Quantized Oscillator
Polynomials and Negative Energy Spectrum

The anti-normal order form in equation (2b) is an eigenvalue equation with eigenvalue $\dfrac{2E}{\omega} + \hbar$. It has an upper bound of zero eigenvalue obtained as

$$\frac{2E}{\omega} + \hbar \equiv \frac{2\overline{E_0}}{\omega} + \hbar = 0 \quad \Rightarrow \quad \overline{E_0} = -\frac{1}{2}\hbar\omega, \tag{12a}$$

where $\overline{E_0}$ denotes the highest value of E obtained at zero eigenvalue in the anti-normal order form.

The negative value $\overline{E_0} = -\dfrac{1}{2}\hbar\omega$ represents a negative energy at the highest level in the anti-normal order energy spectrum. The corresponding highest order negative energy eigenfunction (highest order anti-eigenfunction) $\overline{\psi}_0(s)$ at zero eigenvalue $\left(\dfrac{2E_0}{\omega} + \hbar = 0\right)$ is determined through equation (2b) under the condition (12a) according to

$$E = \overline{E_0} = -\frac{1}{2}\hbar\omega; \quad \psi(s) = \overline{\psi}_0(s) \Rightarrow \left(\hbar\frac{d}{ds} + s\right)\left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_0(s) = 0. \tag{12b}$$

Applying $\hbar$-sign reversed Hermitian conjugation of the operators $\left(\hbar\dfrac{d}{ds} + s\right)$ and $\left(-\hbar\dfrac{d}{ds} + s\right)$ according to

$$\left(\hbar\frac{d}{ds} + s\right) = \left(-\hbar\frac{d}{ds} + s\right)^{\dagger} \tag{12c}$$

in equation (12b) gives

$$\left(-\hbar\frac{d}{ds} + s\right)^{\dagger}\left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_0 = 0 \tag{12d}$$

which on multiplying from the left by the ($\hbar$-sign reversed) Hermitian conjugate $\overline{\psi}_0^{\dagger}(s)$ of the highest order anti-eigenfunction $\overline{\psi}_0(s)$ takes the form

$$\overline{\psi}_0^\dagger(s)\left(-\hbar\frac{d}{ds}+s\right)^\dagger\left(-\hbar\frac{d}{ds}+s\right)\overline{\psi}_0(s)=0 \quad\Rightarrow\quad \left|\left(-\hbar\frac{d}{ds}+s\right)\overline{\psi}_0(s)\right|^2=0. \quad (12e)$$

The basic equation for the highest order anti-eigenfunction $\overline{\psi}_0(s)$ then follows from equation (13e) in the form

$$\left(-\hbar\frac{d}{ds}+s\right)\overline{\psi}_0(s)=0 \tag{13a}$$

with a simple solution

$$\overline{\psi}_0(s)=e^{\frac{1}{2\hbar}s^2} \tag{13b}$$

noting that the integration constant evaluated at $s=0$ is $\overline{\psi}_0(0)=1$.

Anti-eigenfunctions $\overline{\psi}_n(s)$ of general order are generated through repeated application of the conjugate operator $\left(\hbar\frac{d}{ds}+s\right)$ on the highest order anti-eigenfunction $\overline{\psi}_0(s)$ according to

$$\overline{\psi}_0(s)=\left(\hbar\frac{d}{ds}+s\right)^n\overline{\psi}_0(s); \quad n=0,1,2,3,,...,\infty \tag{13c}$$

which on substituting $\overline{\psi}_0(s)$ from equation (13b) and evaluating for $n=0,1$ gives the first two highest order anti-eigenfunctions in the form

$$\overline{\psi}_0(s)=e^{\frac{1}{2\hbar}s^2}, \quad \overline{\psi}_0(s)=2s\overline{\psi}_0(s). \tag{13d}$$

To evaluate lower order anti-eigenfunctions $\overline{\psi}_n(s)$, $n\geq 2$, we derive a simplifying formula as in equation (4e) for any functions $f(s)$, $\phi(s)$ in the form

$$\left(\hbar\frac{d}{ds}+s\right)f\phi=f\left(\hbar\frac{d}{ds}+s\right)\phi+\hbar\frac{df}{ds}\phi \tag{13e}$$

and apply the general relation

$$\overline{\psi}_{n+1}(s)=\left(\hbar\frac{d}{ds}+s\right)\overline{\psi}_n(s), \quad n=0,1,2,3,... \tag{13f}$$

which follows easily from equation (13c) by setting $n\rightarrow n+1$.

For several values $n \geq 1$ in equation (13f), we apply the formula (13e) and proceed in the same manner elaborated in the previous section to obtain general result for lower order anti-eigenfunctions in the form of a recurrence relation

$$\overline{\psi}_{n+1}(s) = 2(s\overline{\psi}_n(s) + n\hbar\overline{\psi}_{n-1}(s)), \quad n = 0, 1, 2, 3, ..., \infty. \tag{14}$$

Setting $n = 0, 1, 2, 3, ...$ in equation (14) and substituting higher order anti-eigenfunctions as appropriate, recalling $\overline{\psi}_0(s)$ from equation (13b) or (13d), we obtain the general anti-eigenfunction $\overline{\psi}_n(s)$ in the form

$$\overline{\psi}_n(s) = \overline{H}_n(s, \hbar)e^{\frac{1}{2\hbar}s^2}, \quad n = 0, 1, 2, 3, ... , \tag{15a}$$

where $\overline{H}_n(s, \hbar)$ is the conjugate quantized oscillator polynomial.

Using equation (13b) in equation (13c) and substituting the result on the l.h.s. of equation (15a) provides the general relation for generating the conjugate quantized oscillator polynomials in the form

$$\overline{H}_n(s, \hbar) = e^{-\frac{1}{2\hbar}s^2}\left(\hbar\frac{d}{ds} + s\right)^n e^{\frac{1}{2\hbar}s^2}, \quad n = 0, 1, 2, ..., \infty. \tag{15b}$$

Using equation (13b) together with its ($\hbar$-sign reversed) Hermitian conjugate

$$\overline{\psi}_0^{\dagger}(s) = e^{-\frac{1}{2\hbar}s^2} \tag{15c}$$

in equation (15b) defines the conjugate quantized oscillator polynomials in terms of the highest order anti-eigenfunction according to

$$\overline{H}_n(s, \hbar) = \overline{\psi}_0^{\dagger}(s)\left(\hbar\frac{d}{ds} + s\right)^n \overline{\psi}_0(s), \tag{15d}$$

Explicit forms of $\overline{H}_n(s, \hbar)$ are easily obtained using a recurrence relation derived in the next subsection.

3.1. Recurrence relations for $\overline{H}_n(s, \hbar)$

Setting $n \to n+1$ in equation (15b) and inserting $e^{\frac{1}{2\hbar}s^2} e^{-\frac{1}{2\hbar}s^2} = 1$ as appropriate, then applying equation (15b) gives the relation

$$\overline{H}_{n+1}(s,\,\hbar) = e^{-\frac{1}{2\hbar}s^2}\left(\hbar\frac{d}{ds} + s\right)\overline{H}_n(s,\,\hbar)e^{\frac{1}{2\hbar}s^2} \tag{16a}$$

which is easily evaluated to obtain the first recurrence relation for the polynomials $\overline{H}_n(s,\,\hbar)$ in the form

$$\overline{H}_{n+1} = 2s\overline{H}_n + \hbar\frac{d\overline{H}_n}{ds}, \quad \overline{H}_m = \overline{H}_m(s,\,\hbar),\, m = n,\, n+1. \tag{16b}$$

Setting $n = 0$ in equation (15b) gives

$$\overline{H}_0 = 1. \tag{16c}$$

Setting $n = 0,\, 1,\, 2,\, 3,\, ...$ in equation (16b) then provides the first five conjugate quantized oscillator polynomials as

$$\overline{H}_0 = 1; \quad \overline{H}_1 = 2s; \quad \overline{H}_2 = 4s^2 + 2\hbar;$$

$$\overline{H}_3 = 8s^3 + 12\hbar s; \quad \overline{H}_4 = 16s^4 + 48\hbar s^2 + 12\hbar^2 \tag{16d}$$

taking the general expansion

$$H_n(s,\,\hbar) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n!\,(\hbar)^m}{(n-2m)!\,m!}(2s)^{(n-2m)} \tag{16e}$$

which are evidently related to the corresponding quantized oscillator polynomials in equations (7d)-(7e) through $\hbar$-sign reversed conjugation $(\hbar \to -\hbar)$, leading us to refer to the polynomials $\overline{H}_n(s,\,\hbar)$ as the conjugate quantized oscillator polynomials. They seem to be a new set of polynomials (special functions), having arisen here for the first time in the solution of the anti-normal order form (2b) of the factorized Schroedinger equation which has never been achieved before in both physics and mathematics. Setting $\hbar = 1$ in equation (17d) gives the corresponding $\hbar$-independent case $\overline{H}_n(s)$, which we may call conjugate Hermite polynomials, arising from the solutions of the anti-normal order equation if we defined the parameter s in the dimensionless form $s = \frac{m\omega}{\hbar}x$ instead of the form in equation (1b).

Substituting

$$\overline{\Psi}_{n+1}(s) = \overline{H}_{n+1}(s,\,\hbar)e^{\frac{1}{2\hbar}s^2} \;;\; \overline{\Psi}_n(s) = \overline{H}_n(s,\,\hbar)e^{\frac{1}{2\hbar}s^2} \;;\; \overline{\Psi}_{n-1}(s) = \overline{H}_{n-1}(s,\,\hbar)e^{\frac{1}{2\hbar}s^2}$$

into equation (14) gives the second recurrence relation for the conjugate quantized oscillator polynomials in the form

$$\overline{H}_{n+1}(s,\,\hbar) = 2s\overline{H}_n(s,\,\hbar) + 2n\hbar\overline{H}_{n-1}(s,\,\hbar). \tag{16f}$$

Comparing the first recurrence relation (16b) and the second recurrence relation (16f) easily provides the third recurrence relation for the conjugate quantized oscillator polynomials in the form

$$\frac{d\overline{H}_n}{ds} = 2n\overline{H}_{n-1}, \quad \overline{H}_m = \overline{H}_m(s,\,\hbar). \tag{16g}$$

Applying $\hbar\dfrac{d}{ds}$ on equation (16g) gives

$$\hbar\frac{d^2\overline{H}_n}{ds^2} = 2n\hbar\frac{d\overline{H}_{n-1}}{ds}. \tag{17a}$$

Using equation (16f) together with the result of setting $n \to n+1$ in equation (16g) gives

$$2n\hbar\frac{d\overline{H}_{n-1}}{ds} = 2n\overline{H}_n - 2s\frac{d\overline{H}_n}{ds} \tag{17b}$$

which we substitute into equation (17a) to obtain the differential equation for the conjugate quantized oscillator polynomials in the form

$$\hbar\frac{d^2\overline{H}_n}{ds^2} + 2s\frac{d\overline{H}_n}{ds} - 2n\overline{H}_n = 0, \quad \overline{H}_n = \overline{H}_n(s,\,\hbar) \tag{17c}$$

which is clearly related to the differential equation for the quantized oscillator polynomials obtained earlier in equation (8c) by $\hbar$-sign reversed conjugation. It is a new second order ordinary differential equation, which emerges through the solution of the Schroedinger for a linear harmonic oscillator factorized in anti-normal order form (2b). We are not aware of its equivalent in the current physics and mathematics literature, in contrast to its $\hbar$-sign reversed conjugate differential equation (8c), which is equivalent to the Hermite differential equation for $\hbar = 1$.

Setting $\hbar = 1$ reduces equation (17c) to the differential equation for the $\hbar$-

independent conjugate Hermite polynomials $\overline{H}_n(s)$ (new in physics and mathematics) taking the form

$$\hbar = 1: \quad \frac{d^2\overline{H}_n}{ds^2} + 2s\frac{d\overline{H}_n}{ds} - 2n\overline{H}_n = 0, \quad \overline{H}_n = \overline{H}_n(s); \quad s = \sqrt{\frac{m\omega}{\hbar}}x. \quad (17d)$$

As observed earlier, $\overline{H}_n(s)$, which are obtained here by setting $\hbar = 1$ in equation (15b) giving

$$\hbar = 1: \quad \overline{H}_n(s) = e^{-\frac{1}{2}s^2}\left(\frac{d}{ds} + s\right)^n e^{\frac{1}{2}s^2}, \quad n = 0, 1, 2, ..., \infty \quad (17e)$$

arise in the solution of the anti-normal order equation (2b) with parameter s defined in the dimensionless form given above in equation (17d). This is the usual definition of the parameter s which would lead to solutions of the normal order form (2a) in terms of the familiar Hermite polynomials. The solutions in terms of the Hermite polynomials, including Hermite differential equation, are easily obtained by setting $\hbar = 1$ in all the expressions in the previous section, but now defining $s = \sqrt{\frac{m\omega}{\hbar}}x$.

3.2. Negative energy spectrum

Substituting

$$\overline{H}_n(s, \hbar) = \overline{\psi}_n(s)e^{\frac{1}{2\hbar}s^2} \quad (18a)$$

from equation (15a) into equation (17c) and reorganizing gives the final result

$$\left(-\hbar^2\frac{d^2}{ds^2} + s^2\right)\overline{\psi}_n(s) = -\hbar(2n + 1)\overline{\psi}_n(s) \quad (18b)$$

which confirms that the anti-eigenfunctions $\overline{\psi}_n(s)$ satisfy the original Schroedinger equation in the form (1c), with $\frac{2E}{\omega}$ taking the corresponding discrete form $-\hbar(2n + 1)$ specifying quantization in the negative energy sector.

Comparing equations (1c) and (18b), noting $E \rightarrow \overline{E}_n$ gives

$$\frac{2\overline{E}_n}{\omega} = -\hbar(2n + 1) \quad (18c)$$

which provides the negative energy spectrum for the quantized linear harmonic oscillator in the unfamiliar form

$$\overline{E}_n = -\hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3, ..., \infty. \tag{18d}$$

This is the negative energy spectrum which arises from the solution of the Schroedinger equation factorized in the anti-normal order form (2b).

3.3. Algebraic operations with $\hat{a}$, $\hat{a}^\dagger$

Applying the operator $-\hbar\dfrac{d}{ds} + s$ on the general anti-eigenfunction $\overline{\psi}_n(s) = \overline{H}_n(s, \hbar)e^{\frac{1}{2\hbar}s^2}$ and evaluating gives

$$\left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_n = -\hbar\frac{d\overline{H}_n}{ds}e^{\frac{1}{2\hbar}s^2} \tag{19a}$$

which on using the third recurrence relation (16f) and then substituting

$$\overline{\psi}_{n-1}(s) = \overline{H}_{n-1}(s, \hbar)e^{\frac{1}{2\hbar}s^2} \tag{19b}$$

takes the final form

$$\left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_n(s) = -2n\hbar\overline{\psi}_{n-1}(s). \tag{19c}$$

This operation with $-\hbar\dfrac{d}{ds} + s$ is essentially the reverse of the operation with its conjugate $\hbar\dfrac{d}{ds} + s$ on $\overline{\psi}_n(s)$ obtained earlier in equation (13f). Bringing the two together gives the complete pair of reverse algebraic operations on the general anti-eigenfunction according to

$$\left(\hbar\frac{d}{ds} + s\right)\overline{\psi}_n(s) = \overline{\psi}_{n+1}(s); \quad \left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_n(s) = -2\hbar n\overline{\psi}_{n-1}(s). \tag{19d}$$

It follows that

$$\left(\hbar\frac{d}{ds} + s\right)\overline{\psi}_{n-1}(s) = \overline{\psi}_n(s); \quad \left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_{n+1}(s) = -2\hbar(n+1)\overline{\psi}_n(s). \tag{19e}$$

Successive operations in anti-normal and normal order then easily give eigenvalue equations

$$\left(\hbar\frac{d}{ds} + s\right)\left(-\hbar\frac{d}{ds} + s\right)\overline{\psi}_n(s) = -2\hbar n\overline{\psi}(s);$$

$$\left(-\hbar\frac{d}{ds} + s\right)\left(\hbar\frac{d}{ds} + s\right)\overline{\psi}_n(s) = -2\hbar(n+1)\overline{\psi}_n(s). \tag{19f}$$

Dividing these through by 2 and introducing the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ as defined in equation (2c) gives

$$\hat{a}\hat{a}^\dagger\,\overline{\psi}_n(s) = -\hbar n\overline{\psi}_n(s); \quad \hat{a}^\dagger\,\hat{a}\overline{\psi}_n(s) = -\hbar(n+1)\psi_n(s). \tag{20a}$$

Effective operations with $\hat{a}$ and $\hat{a}^\dagger$ consistent with the set of equations (19d), (19f) and (20a) take the algebraic form $(\overline{\psi} = \overline{\psi}(s))$

$$\hat{a}\overline{\psi}_n = \overline{\psi}_{n+1}; \quad \hat{a}^\dagger\,\overline{\psi}_n = -\hbar n\overline{\psi}_{n-1}. \tag{20b}$$

An important physical interpretation now emerges. According to equation (20b), the operation of a creation operator $\hat{a}^\dagger$ on an anti-eigenfunction $\overline{\psi}_n(s)$ describing negative energy states now represents negative energy photon emission process at the n-th negative energy quantum state. This emission of negative energy photons from the n-th negative energy state raises it to the higher $(n-1)$-th negative energy state. The emission of negative energy photons from a negative energy state is equivalent to absorption of positive energy photons into a positive energy state, both processes thus leading to excitation to a higher level within the corresponding energy state space, positive or negative. In this respect, the creation operator $\hat{a}^\dagger$ maintains its role as a state raising operator for both positive and negative energy states, causing the state raising effect through positive energy photon absorption according to equation (11b) in the positive energy state space and through negative energy photon emission according to equation (20b) in the negative energy state space.

On the other hand, equation (20b) reveals that the operation of an annihilation operator $\hat{a}$ on an anti-eigenfunction $\overline{\psi}_n(s)$ describing negative energy states now represents negative energy photon absorption process at the n-th negative energy quantum state. This absorption of negative energy photons into the n-th negative energy state lowers it to the lower $(n+1)$-th negative energy state. The absorption of

negative energy photons into a negative energy state is equivalent to emission of positive energy photons from a positive energy state, both processes thus leading to de-excitation to a lower level within the corresponding energy state space, positive or negative. In this respect, the annihilation operator $\hat{a}$ maintains its role as a state lowering operator for both positive and negative energy states, causing the state lowering effect through positive energy photon emission according to equation (11b) in the positive energy state space and through negative energy photon absorption according to equation (20b) in the negative energy state space.

4. Quantum Conjugation

To determine the physical connection between the positive and negative energy state spaces, we start by recognizing that the only fundamental quantum mechanical parameter defining the basic operators $\hbar \dfrac{d}{ds} + s$ and $-\hbar \dfrac{d}{ds} + s$ or $\hat{a}, \hat{a}^\dagger$ arising from the factorization in equations (2a)-(2b) is the Planck's quantization constant $\hbar$. We therefore develop an appropriate conjugation rule based on the sign reversal of Planck's quantization constant $\hbar$. We call the conjugation rule effected by $\hbar$-sign reversal the quantum conjugation, in contrast to the usual mathematical complex conjugation effected by sign reversal of the imaginary number $i = \sqrt{-1}$.

We apply the quantum conjugation according to the rule

$$\hbar \rightarrow -\hbar. \tag{21}$$

Hence, we define quantum conjugation as the reversal of the sign of Planck's quantization constant $\hbar$ according to $\hbar \rightarrow -\hbar$ everywhere in operators, eigenfunctions and related quantities to obtain the corresponding quantum conjugates.

In this theory of quantum conjugation, we define quantum Hermitian conjugation of a matrix by taking the quantum conjugation of every entry and then taking the transpose. We shall generally treat quantum conjugates as quantum hermitian conjugates, denoted by superscript †. The usual mathematical rules of Hermitian conjugation then apply, except now we replace mathematical complex conjugation $(i \rightarrow -i)$ with quantum conjugation $(\hbar \rightarrow -\hbar)$.

Quantum conjugation seems natural and distinctly different from the usual mathematical complex conjugation, since it involves the change of sign of a

fundamental physical parameter $\hbar$ and applies to both real and complex quantities. Any operator, eigenfunction or quantity which is independent of $\hbar$ or depends on $\hbar$ only through $\hbar^{2j}$, $j = 0, \pm 1, \pm 2, \pm 3, \ldots$ is not changed by quantum conjugation and is said to be quantum Hermitian. Since we carry out quantum conjugation separately, a quantum Hermitian quantity may still be a mathematically complex quantity with a complex conjugate.

We establish below that quantum conjugation applies fully in the quantum theory of a linear harmonic oscillator, transforming the positive energy states into their partner negative energy states, thus providing a procedure for transforming photon quantum states into their partner anti-photon quantum states.

4.1. Quantum conjugation of positive energy eigenfunctions

For the lowest order positive energy eigenfunction

$$\psi_0(s) = e^{-\frac{2}{2\hbar}s^2},$$

we apply the rule in equation (21) to obtain the quantum conjugate as

$$\psi_0^\dagger(s) = e^{\frac{2}{2\hbar}s^2}. \tag{22a}$$

Recognizing this result to be the highest order anti-eigenfunction $\overline{\psi}_0(s)$ of the negative energy spectrum, we express equation (22a) as

$$\psi_0^\dagger(s) = \overline{\psi}_0(s). \tag{22b}$$

The general positive energy eigenfunction $\psi_n(s)$ is generated from $\psi_0(s)$ and expressed in terms of the quantized oscillator polynomials $H_n(s, \hbar)$ according to

$$\psi_n(s) = \left(-\hbar\frac{d}{ds} + s\right)^n \psi_0(s) = H_n(s, \hbar)\psi_0(s)$$

which on taking the quantum conjugation and using equation (22b) becomes

$$\psi_0^\dagger(s) = \left(\hbar\frac{d}{ds} + s\right)^n \overline{\psi}_0(s) = H_n^\dagger(s, \hbar)\overline{\psi}_0(s) \tag{22c}$$

from which follows the quantum conjugation of the quantized oscillator polynomials in the form

$$H_n^\dagger(s, \hbar)\overline{\Psi}_0(s) = \left(\hbar \frac{d}{ds} + s\right)^n \overline{\Psi}_0(s). \tag{22d}$$

Multiplying equation (22d) by the quantum conjugate $\overline{\Psi}_0^\dagger(s)$ of $\overline{\Psi}_0(s)$ from the left and using

$$\overline{\Psi}_0^\dagger(s)\overline{\Psi}_0(s) = 1 \tag{22e}$$

gives

$$H_n^\dagger(s, \hbar) = \overline{\Psi}_0^\dagger(s)\left(\hbar \frac{d}{ds} + s\right)^n \overline{\Psi}_0(s) \tag{22f}$$

which on recalling that the r.h.s. provides the conjugate quantized oscillator polynomials $\overline{H}_n(s, \hbar)$ as defined in equation (15d) in the negative energy state space takes the form

$$H_n^\dagger(s, \hbar) = \overline{H}_n(s, \hbar). \tag{22g}$$

Substituting this result into the last step of equation (22c) and introducing the general negative energy anti-eigenfunction

$$\overline{\Psi}_n(s) = \overline{H}_n(s, \hbar)\overline{\Psi}_0(s)$$

provides the quantum conjugation result

$$\psi_{0s}^\dagger(s) = \overline{\Psi}_n(s). \tag{22h}$$

The quantum conjugate of the positive energy spectrum

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

is obtained as

$$E_n^\dagger = \left(\hbar\omega\left(n + \frac{1}{2}\right)\right)^\dagger = -\hbar\omega\left(n + \frac{1}{2}\right) \tag{23a}$$

which on introducing the negative energy spectrum

$$\overline{E}_n = -\hbar\omega\left(n + \frac{1}{2}\right)$$

becomes

$$E_n^\dagger = \overline{E}_n. \tag{23b}$$

The quantum conjugate of the quantized oscillator polynomial differential equation

$$\hbar \frac{d^2 H_n}{ds^2} - 2s \frac{dH_n}{ds} + 2n H_n = 0; \quad H_n = H_n(s, \hbar)$$

is obtained as

$$-\hbar \frac{d^2 H_{n2}}{ds^2} - 2s \frac{dH_n^\dagger}{ds} + 2n H_n^\dagger = 0; \quad H_n^\dagger = H_n^\dagger(s, \hbar) \tag{23c}$$

which on substituting equation (22g) takes the final form

$$\hbar \frac{d^2 \overline{H}_n}{ds^2} + 2s \frac{d\overline{H}_n}{ds} - 2n \overline{H}_n = 0; \quad \overline{H}_n = \overline{H}_n(s, \hbar). \tag{23d}$$

We recognize this as the differential equation for the conjugate quantized oscillator polynomials obtained earlier.

4.2. Quantum conjugation of the basic operators

For the basic operators $\hbar \dfrac{d}{ds} + s$ and $-\hbar \dfrac{d}{ds} + s$, we apply the quantum conjugation rule in equation (21) to obtain

$$\left(\hbar \frac{d}{ds} + s \right)^\dagger = \left(-\hbar \frac{d}{ds} + s \right); \quad \left(-\hbar \frac{d}{ds} + s \right)^\dagger \left(\hbar \frac{d}{ds} + s \right). \tag{24a}$$

These basic operators are quantum conjugates. It also follows that the annihilation and creation operators defined by

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\hbar \frac{d}{ds} + s \right); \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(-\hbar \frac{d}{ds} + s \right) \tag{2c}$$

are quantum conjugates according to

$$(\hat{a})^\dagger = \hat{a}^\dagger; \quad (\hat{a}^\dagger)^\dagger = \hat{a}. \tag{24b}$$

We use equation (2c) to obtain

$$s = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger); \quad \frac{d}{ds} = \frac{1}{\hbar\sqrt{2}} (\hat{a} - \hat{a}^\dagger) \tag{24c}$$

which on applying equations (21) and (24b) are established as quantum Hermitian according to

$$s^\dagger = s; \quad \left(\frac{1}{ds}\right)^\dagger = \frac{d}{ds}. \tag{24d}$$

Let us now introduce the position coordinate and linear momentum operators

$$\hat{x} = x; \quad \hat{p} = -i\hbar\frac{d}{dx} \tag{25a}$$

to express the annihilation and creation operators in (2c) in the form

$$\hat{a} = \frac{1}{\sqrt{2m\omega}}\left(i\frac{\hat{p}}{m\omega} + \hat{x}\right); \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega}}\left(-i\frac{\hat{p}}{m\omega} + \hat{x}\right) \tag{25b}$$

giving

$$\hat{x} = \frac{1}{\sqrt{2m\omega}}(\hat{a} + \hat{a}^\dagger); \quad \hat{p} = -i\sqrt{\frac{m\omega}{2}}(\hat{a} + \hat{a}^\dagger). \tag{25c}$$

Taking quantum conjugation of equation (25c) and applying equation (24b) gives

$$\hat{a}^\dagger = \hat{x}; \quad \hat{p}^\dagger = -\hat{p} \tag{25d}$$

which shows that the displacement operator $\hat{x}$ is quantum Hermitian, while the linear momentum operator $\hat{p}$ is quantum anti-Hermitian. Not that the quantum conjugation of $\hat{p}$ as defined in equation (25c) involves only the operation $\hbar \rightarrow -\hbar$, leaving the imaginary number intact. We observe that a combined quantum and mathematical complex conjugation $(\hbar \rightarrow -\hbar, i \rightarrow -i)$ would give Hermitian form $\hat{p} \rightarrow \hat{p}$, but we are not pursing the combined conjugation operation here, since its physical meaning in the context of the transformation of positive energy states into negative energy partner states may not be clear.

For the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{25e}$$

quantum conjugation gives

$$H^\dagger = \frac{(-\hat{p})^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{25f}$$

which on substituting equation (25e) shows that the Hamiltonian is quantum Hermitian according to

$$H^\dagger = H. \tag{25g}$$

4.3. Quantum conjugation of the basic operator algebraic operations

Taking the quantum conjugation of the positive energy algebraic operations

$$\hat{a}\psi_n(s) = \hbar n \psi_{n-1}(s); \quad \hat{a}^\dagger \psi_n(s) = \psi_{n+1}(s)$$

gives

$$(\hat{a}\psi_n(s))^\dagger = (\hbar n \psi_{n-1}(s))^\dagger \Rightarrow \hat{a}^\dagger \overline{\psi}_n(s) = -\hbar n \overline{\psi}_{n-1}(s), \tag{26a}$$

$$(\hat{a}\psi_n(s))^\dagger (\psi_{n+1}(s))^\dagger \Rightarrow \hat{a}\overline{\psi}_n(s) = \overline{\psi}_{n+1}(s). \tag{26b}$$

The Hamiltonian of the quantized linear Harmonic oscillator is expressed in terms of the annihilation and creation operators in the form

$$H = \frac{1}{2}\omega(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger); \quad H^\dagger = H \tag{26c}$$

which is easily confirmed to be quantum Hermitian $(H^\dagger = H)$. The quantum Hermitian property means that the Hamiltonian is the same in both positive and negative energy state spaces.

In the positive energy state space, the Hamiltonian acts on the eigenfunctions $\psi_n(s)$ according to equation (11b) to produce the positive energy spectrum in the form

$$H\psi_n(s) = \frac{1}{2}\omega(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)\psi_n(s) = E_n\psi_n(s); \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{26d}$$

while in the negative energy state space, the Hamiltonian acts on the anti-eigenfunctions $\overline{\psi}_n(s)$ according to equation (20b) to produce the negative energy spectrum in the form

$$H\overline{\psi}_n(s) = \frac{1}{2}\omega(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)\overline{\psi}_n(s) = \overline{E}_n\overline{\psi}_n(s); \quad \overline{E}_n = -\hbar\omega\left(n + \frac{1}{2}\right). \tag{26e}$$

The commutator $[\hat{a}, \hat{a}^\dagger]$ acts within the positive and negative energy state spaces

according to

$$[\hat{a}, \hat{a}^\dagger]\psi_n(s) = \hbar(n + 1 - n)\psi_n(s); \quad [\hat{a}, \hat{a}^\dagger]\overline{\psi}_n(s) = -\hbar\{n - (n + 1)\}\overline{\psi}_n(s) \quad (26f)$$

from which it follows that the commutation bracket remains the same in the quantum conjugation transformation relating the positive and negative energy state spaces, i.e., the commutation bracket is quantum Hermitian according to

$$([\hat{a}, \hat{a}^\dagger])^\dagger = [\hat{a}, \hat{a}^\dagger] = \hbar. \tag{26g}$$

5. General Interpretation

All the results of the quantum conjugation show that the positive energy eigenfunctions, quantized oscillator polynomials, energy spectrum, basic operators and operator algebraic operations transform into their negative energy partners. This leads to a fundamental feature of quantum dynamics that the general quantum state space of an oscillator is composed of two conjugate state spaces, namely, the positive energy state space and the negative energy state space. The two quantum state spaces are related by quantum conjugation transformation effected through sign reversal of Planck's quantization constant $\hbar$ (i.e., $\hbar \rightarrow -\hbar$). Positive and negative energy states are therefore interpreted as quantum conjugation partners.

Each quantum state space is specified by its physical elements, essentially consisting of state eigenfunctions and quantum operators which provide information on the dynamics through algebraic operations within the state space. In general, if the physical elements of the positive energy state space are denoted by q, then the corresponding physical elements of the negative energy state space denoted by $\overline{q}$ are obtained through quantum conjugation according to

$$\hbar \rightarrow -\hbar: \quad \overline{q} = q^\dagger. \tag{27a}$$

A quantum state physical element which does not change under the quantum conjugation operation according to

$$q^\dagger = q \Rightarrow \overline{q} = q \tag{27b}$$

is quantum Hermitian. Such quantum Hermitian elements take the same form in both positive and negative energy state spaces and they may be called universal physical elements. In addition, physical elements such as annihilation and creation operators

which under quantum conjugation simply interchange roles are also universal elements, since they maintain their forms within the two conjugate state spaces.

The negative energy state anti-eigenfunctions $\overline{\psi}_n(s)$, annihilation operator $\overline{a}$ and creation operator $\overline{a}^\dagger$ are obtained from the corresponding positive energy eigenfunctions $\psi_n(s)$, annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ through quantum conjugation according to

$$\overline{\psi}_n(s) = \psi_n^\dagger(s); \quad \overline{a} = \hat{a}^\dagger; \quad \hat{a}^\dagger = \hat{a}. \tag{28a}$$

The corresponding energy eigenvalues and quantized oscillator polynomials are obtained as

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right); \quad \overline{E}_n = E_n^\dagger = -\hbar\omega\left(n + \frac{1}{2}\right); \quad \overline{H}_n(s, \hbar) = H_n^\dagger(s, \hbar). \tag{28b}$$

The annihilation and creation operators act on respective eigenfunctions within the positive and negative energy state spaces according to

$$\hat{a}\psi_n(s) = \hbar n \psi_{n-1}(s); \quad \hat{a}^\dagger \psi_n(s) = \psi_{n+1}(s), \tag{28c}$$

$$\overline{a}\,\overline{\psi}_n(s) = -\hbar n \overline{\psi}_{n-1}(s); \quad \overline{a}^\dagger \overline{\psi}_n(s) = \overline{\psi}_{n+1}(s) \tag{28d}$$

which are related by quantum conjugation.

According to equations (28c)-(28d), the action of the annihilation operator $\hat{a}$ within the positive energy state space causes emission of positive energy photons from the state $\psi_n(s)$, lowering it to the state $\psi_{n-1}(s)$, while the action of the annihilation operator $\overline{a}$ within the negative energy state space causes emission of negative energy photons from the state $\overline{\psi}_n(s)$, raising it to the state $\overline{\psi}_{n-1}(s)$. On the other hand, the action of the creation operator $\hat{a}^\dagger$ within the positive energy state space causes absorption of positive energy photons into the state $\psi_n(s)$, raising it to the state $\psi_{n+1}(s)$, while the action of the creation operator $\overline{a}^\dagger$ within the negative energy state space causes absorption of negative energy photons into the state $\overline{\psi}_n(s)$, lowering it to the state $\overline{\psi}_{n+1}(s)$.

Emission of positive energy photons from a positive energy state, leading to positive energy state lowering is thus seen to be the reverse process relative to the

emission of negative energy photons from a negative energy state, leading to negative energy state raising. Likewise, absorption of positive energy photons into a positive energy state, leading to positive energy state raising is seen to be the reverse process relative to the absorption of negative energy photons from a negative energy state, leading to negative energy state lowering. These reversed roles of state lowering $\rightarrow$ raising or raising $\rightarrow$ lowering is due to the interchange of creation and annihilation operators under quantum conjugation transformation from positive to negative energy state space, which is evident in equation (28a).

It follows from the above that the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ are universal operators, acting in both positive and negative energy state spaces. Indeed, they satisfy universal commutation property

$$[\bar{a}, \bar{a}^\dagger] = ([\hat{a}, \hat{a}^\dagger])^\dagger = [\hat{a}, \hat{a}^\dagger] = \hbar. \tag{28e}$$

Substituting $\bar{a} = \hat{a}^\dagger$, $\bar{a}^\dagger = \hat{a}$ from equation (28a) into equation (28d) gives their action within the negative energy state space according to

$$\hat{a}^\dagger \, \overline{\psi}_n(s) = -\hbar n \overline{\psi}_{n-1}(s); \quad \hat{a}\overline{\psi}_n(s) = \overline{\psi}_{n+1}(s). \tag{28f}$$

It is clear from equations (28c) and (28f) that positive energy photon emission from a positive energy state and negative energy photon absorption into a negative energy state due to the action of the annihilation operator $\hat{a}$ leads to state lowering in both positive and negative energy state spaces, while positive energy photon absorption into a positive energy state and negative energy photon emission from a negative energy state due to the action of the creation operator $\hat{a}^\dagger$ leads to state raising in both positive and negative energy state spaces. We arrive at the fundamental quantum mechanical feature of the dynamics of a linear harmonic oscillator that emission of positive energy photons from a positive energy quantum state is equivalent to absorption of negative energy photons into a negative energy quantum state, leading to state lowering, while absorption of positive energy photons into a positive energy quantum state is equivalent to emission of negative energy photons from a negative energy quantum state, leading to state raising.

The quantum Hermitian property of the displacement $\hat{x}$ and Hamiltonian H according to

$$\bar{x} = \hat{x}^\dagger = \hat{x}; \quad \overline{H} = H^\dagger = H \tag{28g}$$

and the quantum anti-Hermitian property of the momentum $\hat{p}$ according to

$$\overline{p} = \hat{p}^{\dagger} = -\hat{p} \qquad (28h)$$

lead to the important interpretation that photons occupying negative energy states $(\overline{\psi}_n(s))$ have the same displacement $\hat{x}$ and Hamiltonian H, but opposite momentum $-\hat{p}$ compared to photons occupying the corresponding positive energy partner states $(\psi_n(s))$. The Hamiltonian H has positive energy eigenvalue spectrum $E_n = \hbar\omega\left(n + \dfrac{1}{2}\right)$ in the positive energy state space and negative energy eigenvalue spectrum $\overline{E}_n = -\hbar\omega\left(n + \dfrac{1}{2}\right)$ in the negative energy state space.

6. Conclusion

We have established that the full energy spectrum of a non-relativistic quantized linear harmonic oscillator is composed of positive and negative energy states related by quantum conjugation effected through sign-reversal of Planck's quantization constant $\hbar$. Photons occupying negative energy states have the same displacement $\hat{x}$ and Hamiltonian H, but opposite momentum $-\hat{p}$ compared to photons occupying the corresponding positive energy partner states. Emission of positive energy photons from a positive energy quantum state is equivalent to absorption of negative energy photons into a negative energy quantum state, leading to state lowering, while absorption of positive energy photons into a positive energy quantum state is equivalent to emission of negative energy photons from a negative energy quantum state, leading to state raising. These fundamental quantum features are similar to the well established Dirac's particle-hole or particle-antiparticle theory in relativistic quantum mechanics. We have introduced a new pair of quantum conjugate $\hbar$-dependent polynomials which specify the eigenfunctions and anti-eigenfunctions in the positive and negative energy state spaces.

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