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PHASE CHANGE FROM BALLISTIC DIFFUSION TO SATURATED DIFFUSION

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Abstract

In order to describe diverse non-stationary diffusive phenomena, the non-stationary Markovian replication process has been introduced in recent. In it, the replication probability with which a walk decides the direction of a motion changing with time results in long-range correlations between steps. We used the replication probability of harmonic function decaying with time from 1 to $1/2$ and confirmed that the mean squared displacement increases like ballistic diffusion at early times and then is saturated, which is totally different from that of the normal diffusion. Also by measuring the autocorrelation functions between steps and the distribution of the persistent length size, we found that the correlations have been drastically changed by time as well as time interval and the distribution follows the power-law behavior indicating the existence of persistent length of large sizes.

Keywords and phrases: random walk, anomalous diffusion, stochastic process.

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1. Introduction

Random walks has been received much attention in various scientific fields, providing a theoretical framework for various phenomena from diffusions of Brownian particles to various movements of humans and animals and even bacteria [1-5]. Specially, diffusive phenomena deviated from the normal random walk has been dealt with the anomalous diffusions in which the mean squared displacement (MSD) is not linearly depend on time unlike the normal diffusion and follows power-law behavior [6, 7]

$$\langle x(t)^2 \rangle \sim t^{2H}, \tag{1}$$

where H is the Hurst exponent and for $H > 1/2$ persistence in a walk is exhibited and for $H < 1/2$ anti-persistence is exhibited.

In recent, the non-stationary diffusions affected by complex temporal stimuli coming from natural environments have been drawing attention in various fields. The non-stationary Markovian replication process (NMRP) was being introduced to describe these non-stationary diffusive phenomena, where the probability of hopping to the right or the left in a step is explicitly dependent on time and is denoted as the replication probability $\alpha(t)$, and it follows the generalized telegrapher equation (GTE) [8],

$$\frac{\partial \rho(x, t)}{\partial t} + \mathcal{R}(t) \frac{\partial^2 \rho(x, t)}{\partial t^2} = \mathcal{D}(t) \frac{\partial^2 \rho(x, t)}{\partial x^2}, \tag{2}$$

where

$$\mathcal{R}(t) = \frac{\tau}{2}\left[\frac{3\alpha(t) - 2}{1 - \alpha(t)}\right] \tag{3}$$

and

$$\mathcal{D}(t) = D_0\left[\frac{\alpha(t)}{1 - \alpha(t)}\right], \tag{4}$$

where $\rho(x, t)$ is the probability density function (PDF) of the displacement x at time

t and τ is a time interval for which a walker moves to one of the two neighboring sites and D_0 is the diffusion coefficient for the normal diffusion. The coefficients $\mathcal{R}(t)$ and $\mathcal{D}(t)$ are related to the replication probability $\alpha(t)$ by Eq. (3) and Eq. (4), which indicates that the larger $\alpha(t)$ is, the larger coefficients are. Note that when $\alpha(t)$ approaches to 1, $\mathcal{R}(t)$ diverges and therefore the second term in the left hand side can be more dominant compared to the first term in the left hand side of Eq. (2). However, the second term in the left hand side could be ignored in the most asymptotic limit and Eq. (2) is reduced to the diffusion equation with the time varying diffusion coefficient.

Especially, the case with oscillatory replication probability has exhibited the step-like behavior of the MSD [8], which is totally different from the anomalous diffusion with power-law behavior of the MSD. It may play a key role in describing the non-stationary behaviors of living organisms represented by responding the temporally alternating stimuli. Therefore we study the characteristics of phase change of diffusion by measuring autocorrelation function between steps, avalanche size distribution as well as the MSD in detail.

2. Model

A one dimensional lattice with the uniform spacing between neighboring sites that we denote as l is considered. With the time interval τ, a walker stochastically moves to one of the two neighboring sites. The relation between the walker's position at time t, x_t, and the step walker takes, σ_t, is defined as

$$x(t) = x(t-\tau) + \sigma_t. \tag{5}$$

The dynamics of the stochastic process is controlled by the time-varying probability $\alpha(t)$ in detail,

$$\sigma_t = \begin{cases} \sigma_{t-\tau}, & \text{with probability } \alpha(t), \\ -\sigma_{t-\tau}, & \text{with probability } 1-\alpha(t). \end{cases} \tag{6}$$

The first step (σ_τ) is randomly chosen between the two possibilities $\pm l$ with equal probabilities. Successive steps at $t \geq 2\tau$ are determined by the rule (6).

At each time t, a step σ_t just replicates or anti-replicates the latest step $\sigma_{t-\tau}$, which step process is Markovian, that is, the current step only affects the next step. It is the NMRP model. However, allowing $\alpha(t)$ varying with time by which the degree of replication or anti-replication is controlled results in long-term correlations between steps [9].

3. Results

We considered the case with the replication probability, $\alpha(t) = \cos^2(\pi t/4N)$, where N is the number of total steps. As shown in the inset of Figure 1, $\alpha(t)$ decreases with time from 1 to $1/2$. Figure 1 shows the plot of the MSD in which the

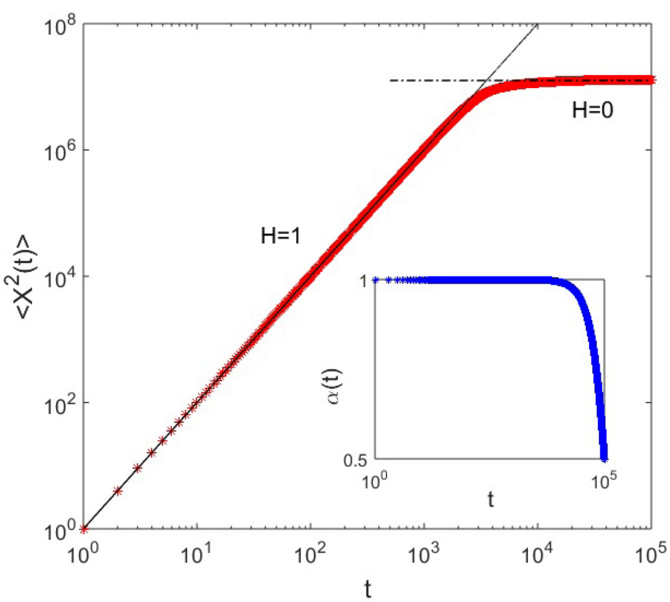

Figure 1. Plot of the MSD versus time. The solid line and the dashed line represent $H = 1$ and $H = 0$, respectively. The inset shows the replication probability $\alpha(t) = \cos^2(\pi t/4N)$.

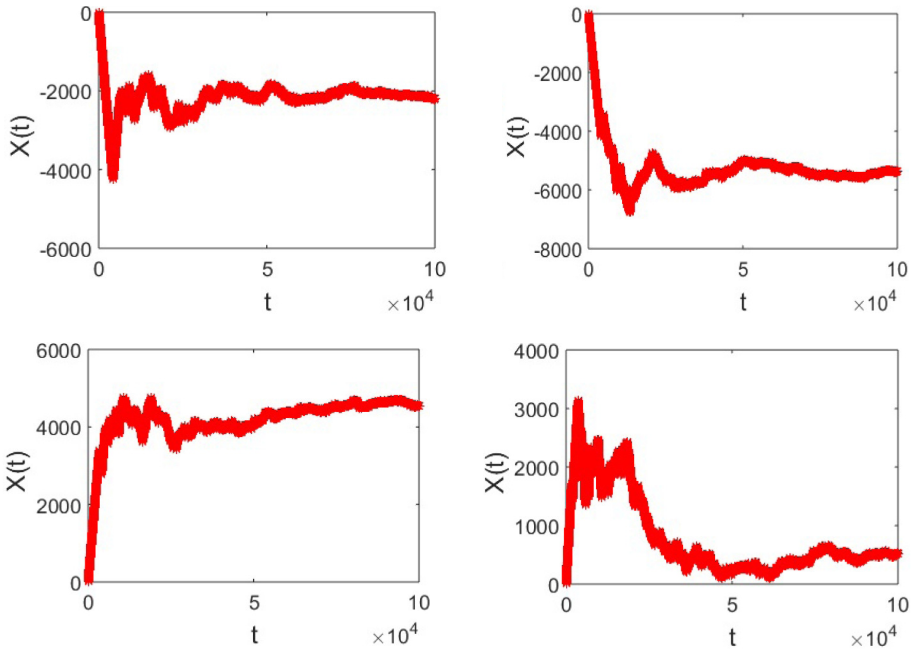

Figure 2. Plots of the displacement $x(t)$ versus time. At early time they exhibit ballistic motions and then alternating motion with decreasing amplitude with time resulting in constant fluctuations for later time.

MSD increases with $H = 1$ at early times and then saturates with $H = 0$. Early behavior of ballistic diffusion results from the value of the replication probability close to 1, while decreasing replication probability toward $1/2$ results in the deviation from the ballistic nature and changes in direction of motion. In the Figure 2, the plot of displacement $x(t)$ exhibits such a little alternating nature and eventually the fluctuation of the displacement does not depends on time on average.

We have also measured the correlation function $C(t, \Delta)$ to study how the replication probability affects to the correlations between steps. The correlation function is defined as $C(t, \Delta) = \langle \sigma_t \sigma_{t+\Delta} \rangle - \langle \sigma_t \rangle \langle \sigma_{t+\Delta} \rangle$, which does not depend on time for stationary stochastic processes but it only depend on the interval Δ between two times, while it varies with time for non-stationary stochastic processes. Considering non-stationary behaviors of the NMRP, we measured the correlation function at fixed time with various values of $t = 1, 10^2, 10^4, 5\times10^4,$ and 9×10^4.

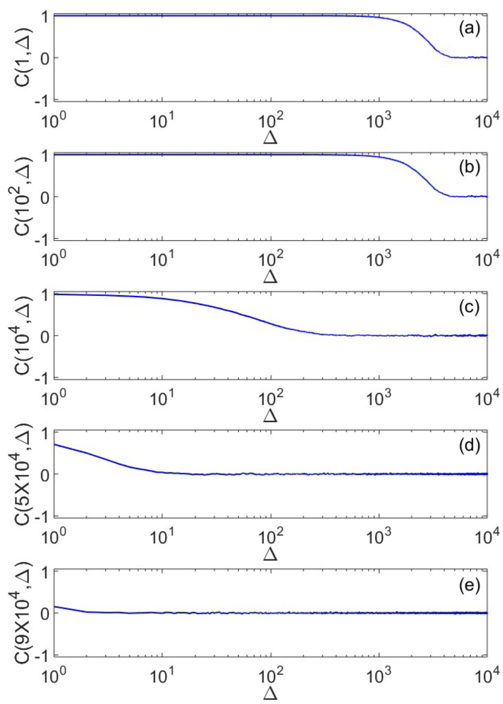

Figure 3. Plots of the autocorrelation function between steps $C(t, \Delta)$ versus the time interval Δ at fixed times, $t = 1, 10^2, 10^4, 5\times10^4$ and 9×10^4.

For $t < t_c$ where t_c represents the critical time after which the MSD is saturated, the correlation is maintained as about 1 for $\Delta < t_c$ because of the value of $\alpha(t)$ closing 1 (Figure 3(a) and (b)) and for $t > t_c$, the correlation decreases with Δ and eventually becomes zero as shown in most cases (Figure 3(c), (d), and (e)). Figure 4 shows the time dependence of the correlation function for the fixed time interval Δ. For $\Delta = 1$, the correlation follows directly $\alpha(t)$ by the definition (Figure 4(a)). The larger time interval Δ get, the shorter the time at that the correlation deviates from the value 1 is as shown in the Figure 4(b), (c) and (d). And the correlation becomes eventually zero for large time. Figure 4(e) shows that the correlation is close to 0 even at early time and for $\Delta = 5000,$ there is no correlations between steps regardless time as shown in the Figure 4(f). Thus the time-varying replication probability invokes the correlations between steps drastically depending on both time and time interval unlike the normal diffusion which have no correlation between steps.

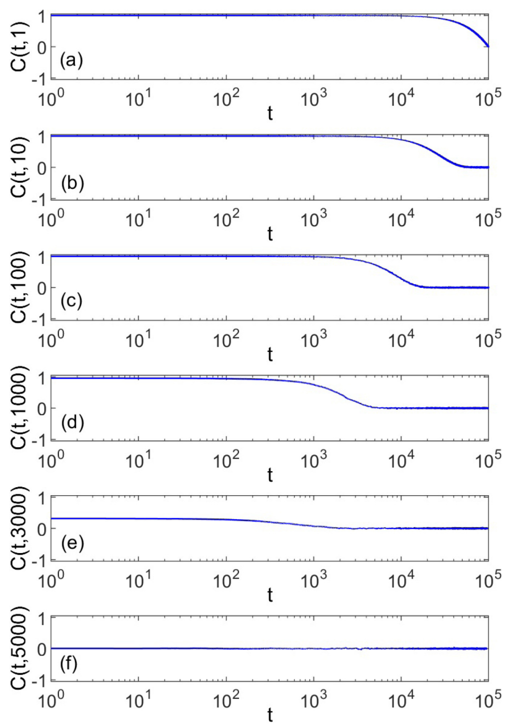

Figure 4. Plots of the autocorrelation function between steps $C(t, \Delta)$ versus the time t at fixed time intervals, $\Delta = 1, 10, 10^2, 10^3, 3 \times 10^3,$ and 5×10^3.

The difference between the NMRP model and the normal diffusion is also revealed by an another quantity, the distribution $P(s)$ of persistent length s that is the number of the successive steps into the same direction. Figure 5 shows the plot of $P(s)$ for this model. The solid line is the guide line whose slope is 2.5, which represents that the distribution follows the power-law behavior, $P(s) \sim s^{-\beta}$ with

$\beta \approx 2.5.$ It indicates that persistent lengths of large size exist in the NMRP unlike the normal diffusions where there is no large persistent length and also it makes a distinction from the model with $\beta \approx 2,$ where the replication probability follows the power-law form [9, 10].

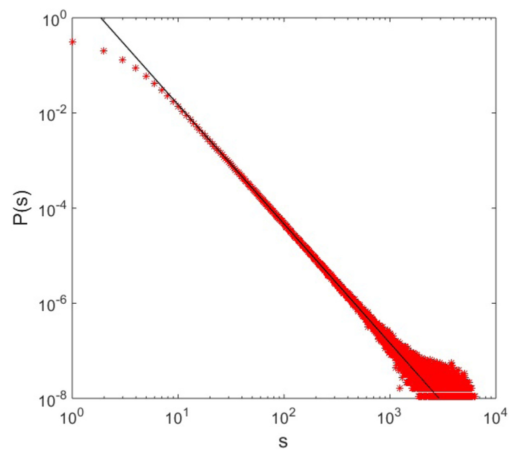

Figure 5. Plot of the distribution of the persistent length $P(s)$ versus the persistent length s. The straight guide line represents that $P(s)$ follows power-law behavior, $P(s) \sim s^{-\beta}$ with $\beta \approx 2.5.$

4. Summary

In summary, we have considered a non-stationary and non-Markovian walk model in which the steps are given by a Markov process replicating the current step with a time-varying probability. When the harmonic replication probability having value between 1 and $1/2$ have been chosen, the stochastic walk model has exhibited totally different behaviors from the normal random walk. The MSD shows the phase change from ballistic motion to saturation and the correlations between steps is non-stationary and has represented drastic natures depending on time as well as time interval. Also we have found that the distribution of persistent length follows the power-law behavior.

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