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PAINLEVE PROPERTY ANALYSIS OF TWO TYPES OF THREE SPECIES SELF INTERACTING FOOD CHAIN MODELS

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Abstract

The long time scale stability analysis of two types of three species food chain models are studied by using Painleve Property analysis of nonlinear dynamical systems. These modified models are obtained by adding self interacting terms in the earlier studied models. It is found that these modified three species models are of non Painleve types and so lack of long time scale stability. The control parameters that produce instability are identified. Results are compared with well known two species Lotka-Volterra's modified models with self interacting terms.

1. Introduction

During the first world war in 1926 an Italian Mathematician Vito

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Volterra closely observed the fish population of Adriatic sea and found that increase in the predator fish population as and when corresponding decrease in the prey fish population. From that he proposed two mathematical model equations of nonlinear ordinary first order differential equations

$$\frac{dX}{dt} = -(A - BY)X; \quad A > 0, \ B > 0 \tag{1.1}$$

and

$$\frac{dY}{dt} = -(C - DX)Y; \quad C > 0, \ D > 0. \tag{1.2}$$

But, Alfred Lotka, a US mathematician already derived the same set of above model equations independently in 1925 to describe a hypothetical chemical reaction in which the chemical concentrations oscillates. This set of Lotka-Volterra model equations is the first somewhat successful attempt to model an ecological system [1], [2], [3].

In the above equations (1.1) and (1.2), the parameter A is the growth rate of population number or its density of a prey X and it is increasing in the absence of interaction with predator species Y. It is assumed that per capita growth rate of prey X decrease linearly with increase of predator population Y.

The parameter B is the measure of the impact of predation and D is the death or migration rate of the predator population number or density of CY in the absence of interaction with prey species X. Then CX denotes the net ratio of growth or migration of the predator population Y in response to the variation of the prey population X.

There are many weak points like, ignored the competition among prey or predator populations. So prey population may increase indefinitely without any resource limits. Moreover, predator species have unlimited consumption rate at the same time prey consumption is proportional to their population density. Also does not include the population size or density effects, gender differences and external influences, presence of other types of predator that can consume X or Y or both, etc. So mathematically it is a closed system. In effect both prey and predator population follow the same cycles indefinitely until external factors interact and produce a shift in population density of any one of the population, then new cycles begins. In fact, in real situations due to environmental factors produces erratic patterns of population density and these patterns are neither stable nor cyclical but chaotic.

Many people suggested corrections to the Lotka-Volterra model of two species interacting food chain model [1], [2], [3], [4], [5], [6]. One of the modified Lotka-Volterra model is the following [1]. Let us consider X and Y are the prey and its predator population numbers or their measure of densities, parameter A to be the intrinsic growth rate of the prey X. Then the growth rate of the prey population X in the absence of the predator Y is given by

$$\frac{dX}{dt} = AX\left(1 - \frac{X}{K}\right); \quad A > 0,\ K > 0, \tag{1.3}$$

where K is the maximum prey population allowed by the limited resources available in the system, so the self interaction among the individuals of the prey population is also included.

In the presence of predator Y, the mortality of prey X due to predation or migration is to be subtracted. For that we consider Holling type functional response [2], [3].

$$f(X) = \frac{BX}{X + C}; \quad B > 0,\ C > 0, \tag{1.4}$$

where $f(X)$ is the predation rate per capita and B is the maximum of $f(X)$ can reach when predator never consume prey even when it is available. The parameter C is related to the predator's handling time or time required for predator to catch and consume prey X. So modified equation (1.1) becomes

$$\frac{dX}{dt} = AX\left(1 - \frac{X}{K}\right) - \frac{BXY}{X + C}. \tag{1.5}$$

So when handling time parameter C is zero, then mortality of prey X is directly proportional to the population of predator Y. Also if predator population is very near to zero, then mortality of prey X is very less, so their population grows fast. Obviously, resultant system is an oscillating population density growth of both X and Y.

2. Three Species Interacting Food Chain Model

Another attempt of modifying Lotka-Volterra model is addition of one more predator. The first attempt in this aspect was made by Gilpin [5], in which one predator and two competing preys were reported. After this many more studies reported [1], [4], [5], [6], [7].

In this study we use the following models

$$\frac{dX}{dt} = AX\left(1 - \frac{X}{K_0}\right) - \frac{CXY}{X + D_0}, \tag{2.1}$$

$$\frac{dY}{dt} = BY\left(1 - \frac{Y}{K_1}\right) - \frac{DYZ}{Y + D_1} \tag{2.2}$$

and

$$\frac{dZ}{dt} = -EZ + GYZ. \tag{2.3}$$

In the above model the self interaction of both X and Y species are also taken into considerations, that others ignored. All parameters A, B, C, D_0, D_1, K_0, K_1, E, G are nonzero and positive valued. So self competition among X and Y populations is also included.

Another model also we are studying in which self interactions of all three species X, Y and Z are considered, is the following

$$\frac{dX}{dt} = AX\left(1 - \frac{X}{K_0}\right) - \frac{CXY}{X + D_0}, \tag{2.4}$$

$$\frac{dY}{dt} = BY\left(1 - \frac{Y}{K_1}\right) - \frac{DYZ}{Y + D_1} \tag{2.5}$$

and

$$\frac{dZ}{dt} = EZ\left(1 - \frac{Z}{K_2}\right) - GYZ. \tag{2.6}$$

As in the previous model all parameters are nonzero and positive valued real numbers. We are adopting the standard method of Painleve Property Analysis [9] designed for ordinary differential equations.

3. Painleve Property Analysis (PPA)

In the context of deterministic dynamical systems, chaos is measured by an extreme sensitivity of the solutions to the choice of initial conditions. Equations whose solutions are free from any chaotic behavior are called '*completely integrable*' and are characterized by regular or predictable behavior for all initial conditions and for all time. Whereas, in '*nonintegrable systems*', the regions in the phase space of their dependent variables where the motion is irregular and chaotic. This type of study initiated by Sonya Kowaleveskaya (see [9]) in the context of rotating bodies, is the first attempt in dynamical systems associated with singularities.

There are two types of singularities, first one called *fixed singularity*, because their location is determined by the equation itself, second is called *movable singularity* where its location depends on the initial conditions. First type is absent in linear equations, but in nonlinear differential equations, both can be formed. Painleve for the first time reported the singularity analysis of three second order nonlinear ordinary differential equations [10]. Then Gambier (see [10]) added three more such differential equations with movable singularity, those six second order differential equations are called Painleve type equations. Such classifications are still unknown in third and higher ordered equations.

An algorithm developed by Ablowitz, Ramai and Segur called ARS algorithm for nonlinear ordinary first order differential equations (see [9]) admits movable branch points, either algebraic or logarithmic using in this study.

Let us consider a system of ordinary differential equations (ODEs) of the form

$$\frac{dW_i}{dt} = F_i(W_1, \ldots, W_n, t), \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Then we are searching for dominant behavior of the solutions in the neighborhood of a movable singularity is of the form

$$W_i = \alpha_i \tau^{\rho_i}, \qquad \tau = (t - t_0). \tag{3.2}$$

Then there are three steps of the ARS algorithm.

Step 1.

Substitute eq. (3.2) in eq. (3.1) and find all possible ρ_i, for which two or more terms in each equation balance each other, and the rest can be ignored as arising at higher powers of $(t - t_0)$. For each such choice of the ρ_i, the balance of these so-called leading terms also find the respective values of the α_i.

Step 2.

In Step 2, we keep only the leading terms of Step 1, and substitute

$$W_i = \alpha_i \tau^{\rho_i}. \tag{3.3}$$

All product terms of γ_i are to be omitted, then system reduced to

$$Q(r) \cdot \gamma = 0, \quad \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n), \tag{3.4}$$

where $Q(r)$ is an $n \times n$ matrix, with r entering only in its diagonal elements, atmost linearly. Then find roots of

$$\det Q(r) = 0. \tag{3.5}$$

These values of r are called resonances. Resonances determines the number of arbitrary constants exist in the given system of ODEs, equation (3.1).

In the resonances $r = -1$ is related to the one free constant, namely the location t_0 of the singularity.

The resonance $r = 0$ corresponds to the coefficient of one of the leading terms being arbitrary. Any resonance with real $r > 0$ but not integer indicates that at $t = t_0$ is a movable singularity. One has to find all $(n - 1)$ nonnegative and real valued integer resonances.

Less than $(n - 1)$ nonnegative and real valued integer resonances implies system of equations (3.1) has no Painleve property.

If R is the largest positive resonance value of r, then substitute the Laurent series expansions

$$W_i = \alpha_i \tau^{\rho_i} + \sum_{m=1}^{R} \alpha_i^{(m)} \tau^{\rho_i + m}. \tag{3.6}$$

Then identify the terms order by order in powers of τ. For $m = r_1$, the smallest positive resonance, compatibility conditions must be satisfied.

4. Painleve Property Analysis of first Model

The first three species self interacting model equations (2.1), (2.2) and (2.3) can be simplified into the following forms

$$(X + D_0)\frac{dX}{dt} = D_0 AX + \left(A - \frac{AD_0}{K_0}\right)X^2 - \frac{A}{K_0}X^3 - CXY, \tag{4.1}$$

$$(Y + D_1)\frac{dY}{dt} = D_1 BY + \left(B - \frac{BD_1}{K_1}\right)Y^2 - \frac{B}{K_1}Y^3 - DZY \tag{4.2}$$

and

$$\frac{dZ}{dt} = -EZ^2 + GYZ. \tag{4.3}$$

Substitute

$$X = \alpha\tau^p, \quad Y = \beta\tau^q, \quad Z = \lambda t^s, \tag{4.4}$$

where $\tau = (t - t_0)$. Then select values of p, q and s so that two or more terms of each of the equations balancing.

For $p = q = -1$ and s and λ as arbitrary, two balanced leading order terms of equation (4.1) are

$$X\frac{dX}{dt} = \frac{-A}{K_0}X^3, \tag{4.5}$$

for which we get

$$\alpha^2 p\tau^{-3} = \frac{A}{K_0}\alpha^3\tau^{-3}. \tag{4.6}$$

For above value of $p = -1$, we get

$$\alpha = \frac{K_0}{A}; \quad K_0 > 0,\ A > 0. \tag{4.7}$$

For equation (4.2), the balancing leading order terms are

$$Y\frac{dY}{dt} = \frac{B}{K_1}Y^3 \tag{4.8}$$

for which we get

$$\beta = \frac{K_1}{B}; \quad K_1 > 0,\ B > 0. \tag{4.9}$$

For equation (4.3), the balancing leading order terms are

$$\frac{dZ}{dt} = GYZ \tag{4.10}$$

for which we get

$$s = \frac{GK_1}{B}; \quad G > 0,\ K_1 > 0,\ B > 0, \tag{4.11}$$

where λ is arbitrary.

For finding resonances put

$$X = \alpha\tau^p(1 + \gamma\tau^r), \tag{4.12}$$

$$Y = \beta\tau^q(1 + \delta\tau^r), \tag{4.13}$$

$$Z = \lambda\tau^s(1 + \eta\tau^r), \tag{4.14}$$

in the balanced leading order terms of equations of (4.5), (4.8) and (4.10). Then after simplification we get, for eq. (4.5)

$$\frac{K_0^2}{A^2}\cdot(r+1)r = 0. \tag{4.15}$$

For eq. (4.8)

$$\frac{K_1^2}{B^2}\cdot(r+1)\rho = 0 \tag{4.16}$$

and for eq. (4.10), we get

$$r\eta - \rho = 0, \text{ for } s = +1. \tag{4.17}$$

This implies

$$\begin{pmatrix} (r+1)\dfrac{K_0^2}{A^2} & 0 & 0 \\ 0 & (r+1)\dfrac{K_1^2}{B^2} & 0 \\ 0 & -1 & r \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.18}$$

and

$$\begin{vmatrix} (r+1)\dfrac{K_0^2}{A^2} & 0 & 0 \\ 0 & (r+1)\dfrac{K_1^2}{B^2} & 0 \\ 0 & -1 & r \end{vmatrix} = 0. \tag{4.19}$$

So the resonances are

$$r = 0, \quad r = -1, \quad r = -1. \tag{4.20}$$

Hence only two distinct real valued resonances $r = 0$ and $r = -1$ when λ is arbitrary.

We suppose to have three distinct resonances including $r = -1$ but found only two. Hence PPA test is to be terminated at this step and concluded that system of self interacting three species model equations (2.1), (2.2) and (2.3) are not Painleve type.

5. Painleve Property Analysis of Second Model

The second model of three species self interacting system equations (2.4), (2.5) and (2.6) can be simplified to the following form

$$(X + D_0)\frac{dX}{dt} = D_0 AX + \left(A - \frac{AD_0}{K_0}\right)X^2 - \frac{A}{K_0}X^3 - CXY, \tag{5.1}$$

$$(Y + D_1)\frac{dY}{dt} = D_1 BY + \left(B - \frac{BD_1}{K_1}\right)Y^2 - \frac{B}{K_1}Y^3 - DZY \tag{5.2}$$

and

$$K_2\frac{dZ}{dt} = K_2 EZ - EZ^2 - K_2 GYZ. \tag{5.3}$$

As in the previous case substitute

$$X = \alpha\tau^p, \quad Y = \beta\tau^q, \quad Z = \lambda t^s, \quad \tau = (t - t_0). \tag{5.4}$$

For the values $p = q = -1$, λ and s as arbitrary, we found for equation (5.1), the balancing leading order terms are

$$X\frac{dX}{dt} = \frac{-A}{K_0}X^3 \tag{5.5}$$

yields

$$\alpha = \frac{K_0}{A}; \quad K_0 > 0,\ A > 0. \tag{5.6}$$

For the equation (5.2), the balancing leading order terms are

$$Y\frac{dY}{dt} = \frac{-A}{K_1}Y^3 \tag{5.7}$$

yields

$$\beta = \frac{K_1}{B}; \quad K_1 > 0,\ B > 0. \tag{5.8}$$

For the equation (5.4), the balancing leading order terms are

$$K_2\frac{dZ}{dt} = GK_2 YZ \tag{5.9}$$

yields

$$s = \frac{-GK_1}{B} \text{ for arbitrary value of } \lambda. \tag{5.10}$$

For finding resonances substitute

$$X = \alpha\tau^p(1 + \gamma\tau^r), \tag{5.11}$$

$$Y = \beta\tau^q(1 + \delta\tau^r), \tag{5.12}$$

$$Z = \lambda\tau^s(1 + \eta\tau^r), \tag{5.13}$$

in the above balancing leading order terms of each equations.

Then we get

$$\frac{K_0^2}{A^2} \cdot (r+1)r = 0; \quad K_0 > 0,\ A > 0, \tag{5.14}$$

$$\frac{K_1^2}{B^2} \cdot (r+1)\delta = 0; \quad K_1 > 0,\ B > 0 \tag{5.15}$$

and

$$\delta + r\eta = 0 \tag{5.16}$$

or its matrix form is

$$\begin{pmatrix} (r+1)\dfrac{K_0^2}{A^2} & 0 & 0 \\ 0 & (r+1)\dfrac{K_1^2}{B^2} & 0 \\ 0 & +1 & r \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.18}$$

Then

$$\begin{vmatrix} (r+1)\dfrac{K_0^2}{A^2} & 0 & 0 \\ 0 & (r+1)\dfrac{K_1^2}{B^2} & 0 \\ 0 & +1 & r \end{vmatrix} = 0 \tag{5.18}$$

gives the resonance values as

$$r = 0, \quad r = -1 \text{ and } r = -1.$$

So we have only two distinct resonance values $r = 0$ and $r = -1$ instead of three.

So no need of proceeding further steps of PPA and concluding that system of self interacting three species food chain model equations (2.4), (2.5) and (2.6) are of not Painleve type.

6. Discussions

In the above studies, we found the absences of Painleve Property in the both modified models of three species food chain are due to the values of resonances. In equations (4.7), (4.9), (5.6) and (5.8), we found the values of α and β are depended on the parameters K_0, A_0, B, K_1. So these parameters are the control parameter of both the systems to verify the chaos studies.

If we put $Z = 0$, then both models are reduced to well known two species Lotka-Volterra food chain with self interacting terms. In that case the matrices $Q(r)$ of both models reduced to 2×2 matrices and respective $\det Q(r) = 0$ gives two values of resonances. For a two variables problem those two values of resonances are sufficient to satisfy the Painleve Property and so two species self interacting Lotka-Volterra models are of Painleve types deterministic dynamical systems. Whereas, in the three species models no such stability is possible for any values of control parameters.

These instabilities of three species models are due to the particular types of product terms selected in the modeling of differential equations. So if we are able to find some other types of product terms for Holling types functional response equations (1.4) and (1.5), we may able to get rid of the absences of Painleve Property or long time scale instability in the dynamics of the three species models.

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