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OPEN, CLOSED, REGULARLY OPEN, REGULARLY CLOSED SUBSPACES AND REGULARLY OPEN T_1 SPACES

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Abstract

In this paper, we investigate and resolve questions concerning subspaces of regularly open T_1 spaces.

1. Introduction

Within topological studies questions concerning subspaces have been and continue to be raised. Until recently, the question concerning subspaces for a property P has been "Does a space have property P if and only if every subspace has property P? Within this paper, properties for which the above statement is true are called subspace properties. The proofs of the converse statement for the subspace theorem cited above are all the same with the property itself only mentioned: "Since the space is a subspace of itself and every subspace has the property, the space has the property." As a result proper subspace properties were introduced and investigated giving the properties themselves a new, meaningful role in subspace questions [1].

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Definition 1.1. Let (X, T) be a space and let P be a property of topological spaces. If (X, T) has property P when every proper subspace of (X, T) has property P, then P is said to be a proper subspace inherited property [1].

Since singleton set spaces satisfy many topological properties, within the recent paper [1] only spaces with three or more elements were considered. Each of the subspace properties T_0, R_0, T_1, R_1, T_2, weakly Urysohn, Urysohn, regular, and T_3 proved to be proper subspace inherited properties giving new characterizations for each of those properties.

Theorem 1.1. *A space* (X, T) *has property* P *iff every proper subspace of* (X, T) *has property* P, *where* P *can be each of the properties cited above* [1].

The results above raised the question of whether or not topological properties could be further characterized using only certain types of sets within the space. With the important role of open and closed sets in the study of topology, a natural place to start such an investigation would be with open sets and closed sets subspaces, which led to new characterizations of T_0 spaces [2] followed by new characterizations of T_1 spaces [3], which are used in this paper.

Theorem 1.2. *Let* (X, T) *be a space. Then the following are equivalent*: (*a*) (X, T) *is* T_1, (*b*) *for each open set* O, (O, T_O) *is* T_1, (*c*) *for each closed set* C, (C, T_C) *is* T_1, (*d*) *for each nonempty proper closed set* C, (C, T_C) *is* T_1 *and for each x in* X, $Cl_T(\{x\})$ *is not* X, *and* (*e*) *for each nonempty proper open set* O, (O, T_O) *is* T_1, *for each x in* X, $Cl_T(\{x\})$ *is not* X, *and for each nonempty proper open set U in* X, *for each x in* U, $Cl_T(\{x\})$ *is a subset of U.*

Theorem 1.3. *Let* (X, T) *be a space. Then the following are equivalent*: (*a*) (X, T) *is* T_1, (b) *for each open set O in X*, *for each open set W in* O, (W, T_W) *is* T_1, (c) *for each nonempty proper open set O in X*, *for each nonempty proper open set W in* O, (W, T_W) is T_1, *for each nonempty proper open set U in X*, *for each x in* U, $Cl_T(\{x\})$ *is a subset of U*, *and for each open set Z in X with two or more elements*, *for each x in Z*, *the* T_Z*-closure of* $\{x\}$ *is not* Z, (d) *for each closed set C in X*, *for each closed set D in* C, (D, T_D) *is* T_1, *and for each closed set E in X with* *two or more elements, for each x in E, the* T_E*-closure of* $\{x\}$ *is not E, and* (*e*) *for each nonempty proper closed set C in X, for each nonempty proper closed set D in C,* (D, T_D) *is* T_1 *and for each closed set E in X with two or more elements, for each x in E,* the T_E*-closure of* $\{x\}$ *is not E.*

The successes using open set and closed set subspaces for the T_0 and T_1 separation axioms raised questions about other types of sets in a topological space that could be considered for subspace questions leading to the consideration of regularly open sets.

Regularly open sets were introduced in 1937 [6].

Definition 1.2. Let (X, T) be a space and let A be a subset of X. Then A is regularly open if and only if $A = Int(Cl(A))$.

In the 1937 paper [6], it was shown that the regularly open sets of a space (X, T) form a base for a topology Ts on X coarser than T and the space (X, Ts) was called the semiregularization space of (X, T). The space (X, T) is semiregular if and only if the regularly open sets of (X, T) is a base for T [6].

The introduction of regularly open sets led to the introduction of regularly closed sets.

Definition 1.3. Let (X, T) be a space and let C be a subset of X. Then C is regularly closed iff one of the following equivalent conditions is satisfied: (1) $X \setminus C$ is regularly open and (2) $C = Cl(Int(C))$ [7].

The investigation of subspace questions for regularly open sets [4] led to the following discoveries, which are used in this paper. For a space (X, T), the regularly open sets of (X, T) equal the regularly open sets of (X, Ts). The semiregularization process generates at most one new topology. Thus a space (X, T) is semiregular if and only if $T = Ts$. For an open set O in a space (X, T), $(Ts)_O = (T_O)s$, which led to the discovery that semiregular is an open set subspace property, but not a proper open set inherited property. Examples are known showing that semiregular is not a closed set subspace property and that for a closed set C in a space (X, T), $(Ts)_C$ need not be $(T_C)s$.

In a follow-up paper [5], regularly open T_0 and regularly open T_1 spaces were introduced and investigated. A space (X, T) is regularly open T_0 if and only if for *x* and *y* in X, x not *y*, there exists a regularly open set containing only one of *x* and *y*. A space (X, T) is regularly open T_1 if and only if for *x* and *y* in X, x not *y*, there exists a regular open set containing *x* and not *y*. Within this paper, regularly open T_1 spaces are further investigated using the results above. As in the initial investigations cited above, [1] and [2], all spaces will have three or more elements. Also, as was the case for the investigation [2], care will be taken for proper subspaces with more than one element to ensure the resulting subspace topology is not the indiscrete topology.

2. Regularly Open T_1 Spaces and their Subspaces

Within the paper [5], it was observed that regularly open T_1 is stronger than T_1 and it was proven that a space (X, T) is regularly open T_1 iff (X, Ts) is T_1. Thus, replacing the statement "(X, T) is T_1" in Theorems 1.2 and 1.3 by the statement "(X, T) is regularly open T_1" implies each of the other statements in Theorems 1.2 and 1.3. Combining a well-known characterization of T_1 spaces with the results above give the next result.

Corollary 2.1. *Let* (X, T) *be a space. Then* (X, T) *is regularly open* T_1 *if and only if* $Cl_{Ts}(\{x\}) = \{x\}$ *for each x in X.*

Theorem 2.1. *Let* (X, T) *be a space. Then* (a) (X, T) *is regularly open* T_1 *if and only if* (b) (X, Ts) is regularly open T_1.

Proof. (a) implies (b): Since (X, Ts) is T_1 and $(Ts)s = Ts$, $(X, (Ts)s)$ is T_1 and (X, Ts) is regularly open T_1.

(b) implies (a): Since (X, Ts) is regularly open T_1, (X, Ts) is T_1 and (X, T) is regularly open T_1.

Thus regularly open T_1 is a semiregularization property, i.e., a property simultaneously shared by both a space and its semiregularization space.

The next result follows immediately from the results above and the fact that T_1 is a subspace and a proper subspace inherited property as cited above.

Corollary 2.2. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is regularly open* T_1, (b) *for each subset Y of* X, $(Y, (Ts)_Y)$ *is* T_1, *and* (c) *for each nonempty proper subset Y of* X, $(Y, (Ts)_Y)$ *is* T_1.

Combining Theorems 1.2 and 1.3 with the results above give the next two results.

Corollary 2.3. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is regularly open* T_1, (b) *for each Ts-open set* O, $(O, (Ts)_O)$ *is* T_1, (c) *for each nonempty proper Ts-open set* O, $(O, (Ts)_O)$ is T_1, *for each x in* X, $Cl_{Ts}(\{x\})$ *is not X, and for each nonempty proper Ts-open set U in X, for each x in* U, $Cl_{Ts}(\{x\})$ *is a subset of* U, (d) *for each Ts-open set O in X, for each Ts-open set W in* O, $(W, (Ts)_W)$ *is* T_1, *and* (e) *for each nonempty proper Ts-open set O in X, for each nonempty proper Ts-open set W in* O, $(W, (Ts)_W)$ *is* T_1, *for each nonempty proper Ts-open set U in X, for each x in* U, $Cl_{Ts}(\{x\})$ *is a subset of U, and for each Ts-open set Z in X with two or more elements, for each x on Z, the* $(Ts)_Z$ *-closure of* $\{x\}$ *is not Z.*

Corollary 2.4. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is regularly open* T_1, (b) *for each Ts-closed set* C, $(C, (Ts)_C)$ is T_1, (c) *for each nonempty proper Ts-closed set* C, $(C, (Ts)_C)$ is T_1, *and for each x in* X, $Cl_{Ts}(\{x\})$ *is not* X, (d) *for each Ts-closed set C in X, for each Ts-closed set D in* C, $(D, (Ts)_D)$ is T_1, *and for each Ts-closed set E in X with two or more elements, for each x in E, the* $(Ts)_E$ *-closure of* $\{x\}$ *is not E, and* (e) *for each nonempty proper Ts-closed set C in X, for each nonempty proper Ts-closed set D in* C, $(D, (Ts)_D)$ is T_1, *and for each Ts-closed set E in X with two or more elements, for each x in E, the* $(Ts)_E$ *-closure of* $\{x\}$ *is not E.*

Theorem 2.2. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is regularly open* T_1, (b) *for each T-regularly closed set* C, $(C, (Ts)_C)$ *is* T_1, *and* (c) *for each nonempty proper T-regularly closed set* C, $(C, (Ts)_C)$ is T_1, *and for x in* X, $Cl_{Ts}(\{x\})$ *is not X.*

Proof. (a) implies (b): Since a *T*-regularly closed set is *Ts*-closed, (b) follows immediately.

(b) implies (c): Clearly, if C is a nonempty proper *T*-regularly closed set, $(C, (Ts)_C)$ is T_1. Since *X* is a *Ts*-regularly closed set, $(X, (Ts)_X) = (X, Ts)$ is T_1 and $Cl_{Ts}(\{x\}) = \{x\}$, which is not *X*, for each *x* in *X*.

(c) implies (a): Suppose (X, T) is not regularly open T_1. Let *x* be in *X* such that $Cl_{Ts}(\{x\})$ is not $\{x\}$. Let *a* be in $Cl_{Ts}(\{x\})$, a not *x*. Since $Cl_{Ts}(\{x\})$ is not *X*, let *y* be in $X \backslash Cl_{Ts}(\{x\})$. Let *U* be *Ts*-open such that *y* is in *U* and *x* is not in *U*. Then *a* is not in *U* and both *a* and *x* are in $X \backslash U$. Let *W* be *T*-regularly open containing *y* inside *U*. Then a and x are in $Z = X \backslash W$, which is *T*-regularly closed, and $(Z, (Ts)_Z)$ is T_1. Let *O* be $(Ts)_Z$-open containing *a* and not *x*. Let *Y* be *Ts*-open such that *O* is the intersection of *Z* and *O*. Then *a* is in *O*, which is *Ts*-open and *x* is not in *O*, which contradicts *a* is in $Cl_{Ts}(\{x\})$. Thus (X, Ts) is regularly open T_1.

Theorem 2.3. *Let (X, T) be a space. Then the following are equivalent*: (a) (X, T) *is regularly open* T_1, (b) *for each T-open set* O, (O, T_O) *is regularly open* T_1, (c) *for each T-open set O in X, for each T-open set W in* O, (W, T_W) *is regularly open* T_1, (d) *for each nonempty proper T-open set O in* X, (O, T_O) *is regularly open* T_1 *and for each x in* X, $Cl_T(\{x\}) = \{x\}$, *and* (e) *for each nonempty proper T-open set O in X, for each nonempty proper T-open set W in* O, (W, T_W) *is regularly open* T_1, *for each nonempty proper T open set U in X, for each x in* U, $Cl_T(\{x\})$ *is a subset of U, and for each T-open set Z in X with two or more elements, for each x in Z, the* T_Z*-closure of* $\{x\}$ *is not Z.*

Proof. (a) implies (b): Let *O* be *T*-open. Then $(O, (Ts)_O)$ is T_1. Since $(Ts)_O = (T_O)s$, $(O, (T_O)s$ is T_1, which implies (O, T_O) is regularly open T_1.

(b) implies (c): Since *X* is *T*-open, (X, T) is regularly open T_1, which implies (c).

(c) implies (d): Since *X* is a *T*-open set in itself, (X, T) is regularly open T_1. Thus (X, T) is T_1 and $Cl_T(\{x\}) = \{x\}$ for each *x* in *X* and statement (d) is satisfied.

(d) implies (e): Let *O* be a nonempty proper *T*-open set in *X*. Each nonempty proper T-open set W in O is a nonempty proper T-open set in X and (W, T_W) is regularly open T_1. Since for each x in X, $Cl_T(\{x\}) = \{x\}$, the remainder of statement (e) is satisfied.

(e) implies (a): Since statement (e) implies statement (c) in Theorem 1.3, (X, T) is T_1. If X is finite, T is the discrete topology on X and (X, T) is regularly open T_1. Thus consider the case that X is infinite. Let x be an element in X. Let y be an element in X distinct from x. Let a and b be distinct elements of X different from x and y. Since singleton sets are T closed, $O = X \backslash \{a\}$ is a proper T-open set in X and $W = X \backslash \{a, b\}$ is a proper T-open set in O. Then x and y are in W and (W, T_W) is regularly open T_1. Let Z be a T_W-regularly open set containing x and not y. Since $(T_W)s = (Ts)_W$, Z is the intersection of W and a Ts-open open set V. Then V is Ts-open containing x and not y. Hence $Cl_{Ts}(\{x\}) = \{x\}$ and (X, T) is regularly open T_1.

The following example shows that T-open in Theorem 2.3 cannot be replaced by T-closed.

Example 2.1. Let X be an infinite set and let T be the finite complement topology on X. Then (X, T) is T_1. Let C be a nonempty proper set in C. Then C is finite, T_C is the discrete topology on C, and $(C, (T_C)s)$ is regularly open T_1. Since Ts is the indiscrete topology on X, $(X, Ts)_C$ is not regularly open T_1.

The last result in this paper uses regularly open T_1 spaces to further topologically characterize nonempty finite sets.

Theorem 2.4. *Let X be a nonempty set. Then X is finite if and only if for a topology T on* X, (X, T) is T_1 if and only if (X, T) is regularly open T_1.

The proof is straightforward using Example 2.1 and is omitted.

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