ON WORMHOLES IN SPACETIMES OF EMBEDDING CLASS ONE

PETER K. F. KUHFITTIG

Department of Mathematics

Milwaukee School of Engineering

Milwaukee, Wisconsin 53202-3109

USA

e-mail: kuhfitti@msoe.edu

Abstract

An n-dimensional Riemannian space is said to be of embedding class m if $n + m$ is the lowest dimension of the flat space in which the given space can be embedded. A spherically symmetric spacetime of class two can be reduced to class one by a suitable transformation of coordinates. Applied to wormholes, given a well-defined shape function $b = b(r)$, the resulting wormhole has an event horizon and is therefore nontraversable. On a macroscopic scale, $b(r)$ can be replaced by $m(r)$, the effective mass of a spherical star of radius r with $m(0) = 0$, to yield a valid solution. Spacetimes of embedding class one have been used successfully for modeling compact stellar objects. On a microscopic scale, one can invoke noncommutative geometry to obtain a charged nontraversable wormhole, i.e., an Einstein-Rosen bridge, and hence a model for a charged particle.

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1. Introduction

Wormholes are handles or tunnels in spacetime connecting widely separated regions of our Universe or different universes altogether. Morris and Thorne [1] proposed the following static and spherically symmetric line element for a wormhole spacetime:

$$ds^2 = -e^{\nu(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

using units in which $c = G = 1$. Here $\nu = \nu(r)$ is called the *redshift function*, which must be everywhere finite to avoid the appearance of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram [1]. (The embedding diagram will play a critical role in Section 2.) The spherical surface $r = r_0$ is the radius of the throat of the wormhole. The shape function must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$ and $b'(r_0) \leq 1$, called the *flare-out condition*. This condition can only be met by violating the null energy condition (NEC)

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0 \tag{2}$$

for all null vectors k^α, where $T_{\alpha\beta}$ is the energy-momentum tensor. Matter that violates the NEC is called "exotic" in Ref. [1]. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation has the form

$$T_{\alpha\beta}k^\alpha k^\beta = \rho + p_r < 0. \tag{3}$$

Here $T^t{}_t = -\rho$ is the energy density, $T^r{}_r = p_r$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t$ is the lateral pressure. A final requirement is asymptotic flatness: $\lim_{r\to\infty}\nu(r) = 0$ and $\lim_{r\to\infty}b(r)/r = 0$.

Much of our discussion is based on the assumption that our spacetime is of embedding class one. So we need to recall that an n-dimensional Riemannian space is said to be of embedding class m if $n + m$ is the lowest dimension of the flat space in which the given space can be embedded [2, 3, 4, 5, 6, 7]. We also need to recall that the exterior Schwarzschild solution is a Riemannian space of embedding class two.

We continue our discussion with the following static and spherically symmetric line element from Ref. [2], but using the signature from line element (1):

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (4)$$

It is shown in Ref. [2] that this metric of class two can be reduced to a metric of class one; the spacetime is thereby embedded in a five-dimensional flat spacetime. This reduction can be accomplished by means of the following coordinate transformation: $z^1 = r\sin\theta\cos\phi$, $z^2 = r\sin\theta\sin\phi$, $z^3 = r\cos\theta$, $z^4 = \sqrt{K}e^{\nu/2}\cosh\dfrac{t}{\sqrt{K}}$, and $z^5 = \sqrt{K}e^{\nu/2}\sinh\dfrac{t}{\sqrt{K}}$. The result is

$$ds^2 = -e^{\nu}dt^2 + \left[1 + \frac{1}{4}Ke^{\nu}(\nu')^2\right]dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (5)$$

Metric (5) is equivalent to metric (4) if

$$e^{\lambda} = 1 + \frac{1}{4}Ke^{\nu}(\nu')^2, \qquad (6)$$

where $K > 0$ is a free parameter. Eq. (6) can also be obtained from the Karmarkar condition [8]

$$R_{1414} = \frac{R_{1212}R_{3434} + R_{1224}R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0,$$

which is equivalent to the above reduction. In fact, Eq. (6) is a solution to the differential equation (readily solved by separation of variables)

$$\frac{\nu'\lambda'}{1 - e^\lambda} = \nu'\lambda' - 2\nu'' - (\nu')^2,$$

so that K is actually an integration constant [3, 9].

2. Seeking a Complete Wormhole Solution

The strategy adopted by Morris and Thorne in Ref. [1] was to satisfy the geometric requirements for a traversable wormhole by specifying $b(r)$ and $\nu(r)$ and then either manufacture or search the Universe for matter or fields that can produce the desired energy-momentum tensor. In this section, we will consider the case where $b = b(r)$ is a legitimate shape function, which may actually be known for physical reasons, such as a noncommutative-geometry background. (This possibility will be explored further in Section 4.) Another possibility is to start with a constant energy density, as in Ref. [10]. Returning to Eq. (6), let us rewrite the equation as

$$\frac{1}{\sqrt{K}} \sqrt{e^\lambda - 1} = e^{\frac{1}{2}\nu} \nu'.$$

Integrating, we obtain from $e^{\lambda(r)} = [1 - b(r)/r]^{-1}$,

$$\sqrt{K} e^{\frac{1}{2}\nu(r)} = \int \sqrt{e^{\lambda(r)} - 1}\, dr = \int \left(\frac{r}{b(r)} - 1 \right)^{-\frac{1}{2}} dr \tag{7}$$

and

$$e^{\nu(r)} = \frac{1}{K} \left(\int \left(\frac{r}{b(r)} - 1 \right)^{-\frac{1}{2}} dr \right)^2. \tag{8}$$

This integral exists as long as $b(r)$ is a continuous function since

$$\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}} = \left(\frac{r}{b(r)}\right)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)\left(\frac{r}{b(r)}\right)^{-\frac{3}{2}}(-1)$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{r}{b(r)}\right)^{-\frac{5}{2}}(-1)^2 + \dots .$$

This series converges for $r > r_0$, resulting in a removable discontinuity at $r = r_0$. Unfortunately, the integral in Eq. (7) is nothing more than the profile curve $z(r)$ in the standard embedding diagram in Ref. [1]: $z(r)$ is rotated about the z-axis and in the resulting tunnel-like figure, the circle $r = r_0$ lies in the plane $z = 0$. We conclude that there is an event horizon for any shape function; so we do not get a traversable wormhole.

An interesting special case is provided by $b(r) = 2M$, a zero-density wormhole:

$$e^{\frac{1}{2}\nu(r)} = \frac{1}{\sqrt{K}} \int_{2M}^{r} \sqrt{\frac{1}{\frac{r'}{2M} - 1}}\, dr' = \frac{4M}{\sqrt{K}} \sqrt{\frac{r}{2M} - 1}. \tag{9}$$

The wormhole spacetime is not asymptotically flat and will have to be cut off at some $r = a$ and joined to an external Schwarzschild spacetime. In other words,

$$e^{\nu(a)} = \frac{16M^2}{K}\left(\frac{a}{2M} - 1\right) = 1 - \frac{2M}{a}.$$

Since K is a free parameter, we can let

$$K = \frac{16M^2\left(\frac{a}{2M} - 1\right)}{1 - \frac{2M}{a}}.$$

The result is

$$e^{\nu(r)} = \frac{1 - \dfrac{2M}{a}}{\dfrac{a}{2M} - 1}\left(\frac{r}{2M} - 1\right), \tag{10}$$

confirming the existence of an event horizon at $r = 2M$; also, observe that $e^{\nu(a)} = 1 - \dfrac{2M}{a}$. (A similar junction condition would be needed for any shape function.)

Returning to Eq. (6), we have seen that whenever we assume a wormhole structure with a well-defined shape function, we cannot avoid an event horizon. On the other hand, if we reverse our point of view by assuming that $\nu(r)$ is finite (i.e., no event horizon), then Eq. (6) yields

$$b(r) = r\left(1 - \frac{1}{1 + \dfrac{1}{4}Ke^{\nu(r)}[\nu'(r)]^2}\right); \tag{11}$$

but in order to satisfy the condition $b(r_0) = r_0$, the fraction inside the parentheses must vanish. Since $e^{\nu(r_0)}$ is finite, we must have $\nu'(r_0+) = \pm\infty$. This is entirely possible, as can be seen from the choice $\nu(r) = \pm 2\sqrt{r - r_0}$. However, the resulting wormhole behaves much like a Schwarzschild black hole: if we denote $1 - \dfrac{2M}{r}$ in the Schwarzschild line element by $e^{\nu_1(r)}$, then $\nu_1(r) = \ln\left(1 - \dfrac{2M}{r}\right)$ and

$$\lim_{r\to 2M+}\nu_1'(r) = \lim_{r\to 2M+}\frac{\dfrac{2M}{r^2}}{1 - \dfrac{2M}{r}} = +\infty.$$

We conclude that a four-dimensional Riemannian space of embedding

class one does not allow a traversable wormhole. For traversability, some additional assumptions would be needed. For example, Ref. [11] assumes conformal symmetry, while Ref. [12] uses a modified shape function.

3. A Stellar Model

Since the embedding theory has failed to produce a macroscopic traversable wormhole, let us consider instead the stellar model [13]

$$ds^2 = -e^{\nu(r)}dt^2 + \frac{dr^2}{1 - m(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2);\tag{12}$$

here $m(r)$ is the effective mass inside a spherical star of radius r with $m(0) = 0$. For this model, we can return to Eq. (8) to deduce that

$$e^{\nu(r)} = \frac{1}{K}\left(\int_0^r \sqrt{\frac{1}{\frac{r'}{m(r')} - 1}}dr'\right)^2.\tag{13}$$

If the star has radius $r = R$, then the free parameter K once again allows a junction to an external Schwarzschild spacetime:

$$e^{\nu(R)} = \frac{1}{K}\left(\int_0^R \sqrt{\frac{1}{\frac{r}{m(r)} - 1}}\,dr\right)^2 = 1 - \frac{2M}{R},\tag{14}$$

where $M = \frac{1}{2}m(R)$. So

$$K = \frac{\left(\int_0^R \sqrt{\frac{1}{\frac{r}{m(r)} - 1}}\,dr\right)^2}{1 - \frac{m(R)}{R}}.\tag{15}$$

For the potential ν, we have

$$\nu = \ln\left(1 - \frac{m(R)}{R}\right) \approx -\frac{m(R)}{R}. \tag{16}$$

So for large r, $\nu = -\dfrac{m}{r}$, the Newtonian limit. Spacetimes of embedding class one have proved to be very effective for modeling compact stellar objects such as neutron stars and pulsars [2].

4. Microscopic Wormholes

A convenient way to study microscopic wormholes is by means of noncommutative geometry [14]. An important outcome of string theory is the realization that coordinates may become noncommutative operators on a D-brane [15, 16]. Noncommutativity replaces point-like objects by smeared objects [17, 18, 19] with the aim of eliminating the divergences that normally appear in general relativity. As a consequence, spacetime can be encoded in the commutator $[\mathbf{X}^{\mu}\ \mathbf{X}^{\nu}] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck's constant discretizes phase space [18]. An effective way to model the smearing is to assume that the energy density of the static and spherically symmetric and particle-like gravitational source is

$$\rho(r) = \frac{\mu\sqrt{\beta}}{\pi^2 (r^2 + \beta)^2}, \tag{17}$$

which can be interpreted to mean that the gravitational source causes the mass μ of a particle to be diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty; so $\sqrt{\beta}$ has units of length.

Next, let us list the Einstein field equations (replacing ν by 2ϕ):

$$\rho(r) = \frac{b'}{8\pi r^2}, \tag{18}$$

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{19}$$

$$p_t(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[\Phi'' - \frac{b'r - b}{2r(r - b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)}\right]. \tag{20}$$

Then from Eqs. (17) and (18), we obtain the shape function

$$b(r) = \int_{r_0}^{r} 8\pi(r')^2 \rho(r') dr'$$

$$= \frac{4m\sqrt{\beta}}{\pi}\left(\frac{1}{\sqrt{\beta}}\tan^{-1}\frac{r}{\sqrt{\beta}} - \frac{r}{r^2 + \beta} - \frac{1}{\sqrt{\beta}}\tan^{-1}\frac{r_0}{\sqrt{\beta}} + \frac{r_0}{r_0^2 + \beta}\right) + r_0; \tag{21}$$

observe that $b(r_0) = r_0$, as required.

In this section, we would like to consider microscopic wormholes with electric charge Q. Following Kim and Lee [22], we take the line element to be

$$ds^2 = -e^{\nu(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{22}$$

where $b(r)$ is given by Eq. (21). In line element (22), the effective shape function $b_{\text{eff}}(r)$ is

$$b_{\text{eff}}(r) = b(r) - \frac{Q^2}{r}, \tag{23}$$

where $b_{\text{eff}}(r_1) = r_1$ and r_1 is the solution of the equation $b(r) = r + Q^2/r$. From Eq. (8), we now get

$$e^{\nu(r)} = \frac{1}{K}\left(\int_{r_1}^{r}\left(\frac{r'}{b_{\text{eff}}(r')} - 1\right)^{-1/2} dr'\right)^2.$$ (24)

Here we return to the discussion in Section 2. Because of the event horizon at $r = r_1$, the wormhole is not traversable, thereby constituting an Einstein-Rosen bridge.

Ref. [23] discusses microscopic charged wormholes in the context of quadratic Palatini gravity. According to this theory, the solution can be interpreted as an electric flux going through one mouth of the wormhole and coming out of the other mouth. The result is a negative charge on one side and a positive charge on the other. Referring to Ref. [24], for all practical purposes, there is no difference between the kind of charge described as a wormhole (i.e., by means of a nontrivial topology) and a standard point-like charge, suggesting that spacetime could have a foam-like structure.

For a detailed discussion of the microstructure in conjunction with entanglement and the $ER = EPR$ conjecture, see Ref. [23].

5. Summary

An n-dimensional Riemannian space is said to be of embedding class m if $n + m$ is the lowest dimension of the flat space in which the given space can be embedded. A spherically symmetric spacetime of embedding class two can be reduced to class one by a suitable transformation of coordinates. From the resulting metric (5), we have

$$e^{\lambda} = 1 + \frac{1}{4} Ke^{\nu}(\nu')^2, \quad K > 0,$$

where K is a free parameter. If we start with a well-defined shape function $b = b(r)$, then the resulting wormhole has an event horizon and

is therefore not traversable. Replacing $b(r)$ by $m(r)$, the effective mass inside a spherical star of radius r and with $m(0) = 0$, we obtain a valid expression for $e^{\lambda(r)}$ [Eq. (13)], thereby avoiding an event horizon. If R is the radius of the star, then the free parameter K allows a junction to an external Schwarzschild spacetime at $r = R$. The potential $\nu(r)$ reduces to the Newtonian limit $\nu = -m/r$ for large r. It is therefore not surprising that spacetimes of embedding class one have been used successfully for modeling compact stellar objects. As we have seen, however, such spacetimes cannot be used to model traversable wormholes without introducing some additional conditions.

Making use of a noncommutative-geometry background, we can consider microscopic wormholes with electric charge. The presence of an event horizon results in an Einstein-Rosen bridge and is therefore a viable model for a charged particle. The reason is that, according to Ref. [24], for all practical purposes, there is no difference between a point-like charge and a wormhole structure arising from a nontrivial toplogy. Ref. [23] discusses the microstructure in conjunction with entanglement and the $ER = EPR$ conjecture.

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