Fundamental Journal of Mathematics and Mathematical Sciences
Vol. 6, Issue 1, 2016, Pages 25-31
This paper is available online at http://www.frdint.com/
Published online July 22, 2016

ON THE DIOPHANTINE EQUATION $\sqrt[n]{x} + \sqrt[n]{y} = \sqrt[n]{z}$

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Abstract

In this paper, the surd equations of the form $\sqrt[n]{x} + \sqrt[n]{y} = \sqrt[n]{z}$ have been discussed for different integral values of $n > 1$. This type of surd equation has also been discussed for rational values of n of the form $n = \frac{2}{q}$ for some integral values of $q > 2$.

Keywords and phrases: diophantine equation, surd equation and integral solution.

2010 Mathematics Subject Classification: 11D45.

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Received May 22, 2016; Revised June 3, 2016; Accepted June 7, 2016

1. Introduction

Pierre de Fermat in 1637 wrote in the margin of a copy of Arithmetica that no three positive integers *a, b* and *c* satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* greater than 2. He claimed that he had a proof that was too large to be fitted in the margin. This result is known as Fermat's Last Theorem or Fermat's Conjecture in the literature. Andrew Wiles in 1994 proved this result successfully. He formally published it in 1995. This theorem is among the most notable theorems in the history of mathematics and was in the Guinness Book of World Records for most difficult mathematical problems prior to its proof.

Billionaire banker Andrew Beal (1993) while investigating the generalization of Fermat's Last Theorem proposed a conjecture. This conjecture is known as Beal's Conjecture. Gregorio (2013) presented a proof for the the Beal's conjecture and a new proof for the Fermat's last theorem. Gola, L.W. (2014) presented a proof of Beal's conjecture. Ghanouchi (2014) presented an elementary proof of Fermat-Wiles theorem and generalization to Beal's conjecture. Thiagrajan (2014) provided computational results and a proof of Beal's conjecture. Gopalan, Sumathi and Vidhyalakshmi (2013) discussed the transcendental equation with five unknowns $3\sqrt[3]{x^2 + y^2} - 2\sqrt[4]{x^2 + y} = (r^2 + s^2)^2 z^6$. Pandichelvi, V. (2013) discussed an exclusive transcendental equation $\sqrt[3]{x^2 + y^2} + \sqrt[3]{z^2 + w^2} = (k^2 + 1)R^2$. Gopalan, Vidhyalakshmi and Usha Rani (2013) discussed the integral solutions of the surd equation $\sqrt[3]{x^2 - y^2} + \sqrt[3]{X^2 + Y^2} + 2\sqrt[3]{z^2 + w^2} = 6p^2$.

Here, the surd equations of the form $\sqrt[n]{x} + \sqrt[n]{y} = \sqrt[n]{z}$ have been discussed for different integral values of $n > 1$. This type of surd equation has also been discussed for rational values of *n* of the form $n = \dfrac{2}{q}$ for some integral values of $q > 2$.

2. Analysis

(a) For $n = 2,$ the given surd equation reduces to $\sqrt[2]{x} + \sqrt[2]{y} = \sqrt[2]{z}$

If we take $x = (a^2 - b^2)^4$, $y = 16a^4b^4$ and $z = (a^2 + b^2)^4$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(b) For $n = 3$, the given surd equation reduces to $\sqrt[3]{x} + \sqrt[3]{y} = \sqrt[3]{z}$

If we take $x = (a^2 - b^2)^6$, $y = 64a^6b^6$ and $z = (a^2 + b^2)^6$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(c) For $n = 4$, the given surd equation reduces to $\sqrt[4]{x} + \sqrt[4]{y} = \sqrt[4]{z}$

If we take $x = (a^2 - b^2)^8$, $y = 256a^8b^8$ and $z = (a^2 + b^2)^8$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(d) For $n = 5$, the surd equation is $\sqrt[5]{x} + \sqrt[5]{y} = \sqrt[5]{z}$

If we take $x = (a^2 - b^2)^{10}$, $y = 4^5 a^{10}b^{10}$ and $z = (a^2 + b^2)^{10}$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(e) For $n = 6$, the surd equation is $\sqrt[6]{x} + \sqrt[6]{y} = \sqrt[6]{z}$

If we take $x = (a^2 - b^2)^{12}$, $y = 4^6 a^{12}b^{12}$ and $z = (a^2 + b^2)^{12}$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(f) For $n = 7$, the surd equation is $\sqrt[7]{x} + \sqrt[7]{y} = \sqrt[7]{z}$

If we take $x = (a^2 - b^2)^{14}$, $y = 4^7 a^{14}b^{14}$ and $z = (a^2 + b^2)^{14}$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(g) For general *n*, the surd equation is $\sqrt[n]{x} + \sqrt[n]{y} = \sqrt[n]{z}$

If we take $x = (a^2 - b^2)^{2n}$, $y = 4^n a^{2n} b^{2n}$ and $z = (a^2 + b^2)^{2n}$ then above surd equation is satisfies. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

The following transformations may also be considered for the solution

$$x = \alpha^{2n},\ y = \beta^{2n} \text{ and } z = \gamma^{(2u+1)n}.$$

These transformations reduces the given general surd equation to

$$\alpha^2 + \beta^2 = \gamma^{2u+1}.$$

This equation is satisfied by

$$\alpha = A(A^2 + B^2)^u,\ \beta = B(A^2 + B^2)^u \text{ and } \gamma = (A^2 + B^2).$$

Thus $x = A^{2n}(A^2 + B^2)^{2un}$, $y = B^{2n}(A^2 + B^2)^{2un}$ and $z = (A^2 + B^2)^{(2u+1)n}$ is the required solution. Putting values of *A*, *B*, *u* and *n* different solutions can be obtained.

Similarly, with the help of the transformation

$$x = \alpha^{2n},\ y = \beta^{2n} \text{ and } z = \gamma^{(2u+2)n},\ \gamma = a^2 + b^2.$$

It can be shown that

$$x = \frac{1}{2^{2n}} F^{2n}(u+1),\ y = \frac{1}{2^{2n} i^{2n}} G^{2n}(u+1) \text{ and } z = (a^2 + b^2)^{2(u+1)n}$$

where $F(u+1) = (a+ib)^{2(u+1)} + (a-ib)^{2(u+1)}$ and $G(u+1) = (a+ib)^{2(u+1)} -$

$(a-ib)^{2(u+1)}$. Putting different suitable values of *a*, *b*, *u* and *n* different required solutions can be obtained,

Example. Taking $u = 1$, we get $F(u+1) = 2(a^4 - 6a^2b^2 + b^4)$ and $G(u+1) =$ $4ab(a^2 - b^2)$. Therefore $x = (a^4 - 6a^2b^2 + b^4)^{2n}$, $y = (2ab\ \ (a^2 - b^2))^{2n}$ and $z = (a^2 + b^2)^{4n}$. Putting suitable values of *a* and *b*, we may get the required solutions.

(h) For $n = \frac{2}{3}$, the surd equation is $\sqrt{x} + \sqrt[\frac{2}{3}]{y} = \sqrt[\frac{2}{3}]{z}$

If we take $x = (a^2 - b^2)^3$, $y = 8a^3b^3$ and $z = (a^2 + b^2)^3$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(i) For $n = \frac{2}{5}$, the surd equation is $\sqrt{x} + \sqrt[\frac{2}{5}]{y} = \sqrt[\frac{2}{5}]{z}$

If we take $x = (a^2 - b^2)^5$, $y = 32a^5b^5$ and $z = (a^2 + b^2)^5$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(j) For $n = \frac{2}{7}$, the surd equation is $\sqrt{x} + \sqrt[\frac{2}{7}]{y} = \sqrt[\frac{2}{7}]{z}$

If we take $x = (a^2 - b^2)^7$, $y128a^7b^7$ and $z = (a^2 + b^2)^7$ then above surd equation is satisfied. Thus the above values of *x, y* and *z* provide the solution of the surd equation under consideration.

(k) For $n = \frac{2}{q}$, $q > 2$ the surd equation is $\sqrt{x} + \sqrt[\frac{2}{q}]{y} = \sqrt[\frac{2}{q}]{z}$

Considering the transformations

$$x = \alpha^q,\ y = \beta^q \text{ and } z = \gamma^{sq},\ \gamma = a^2 + b^2.$$

It can be shown that $x = \frac{1}{2q} F^q(s)$, $y = \frac{1}{2^q i^q} G^q(s)$ and $z = (a^2 + b^2)^{sq}$ is the required solution. By taking different suitable values of a, b, s and q the required solution can be obtained.

Example. Taking $s = 2$, we get $F(s) = 2(a^4 - 6a^2b^2 + b^4)$ and $G(s) =$

$4ab(a^2 - b^2)$. Therefore $x = (a^4 - 6a^2b^2 + b^4)^q$, $y = (2ab(a^2 - b^2)^q$ and $z = (a^2 + b^2)^{2q}$. Putting suitable values of a and b, we may get the required solutions.

3. Concluding Remarks

Here the given surd equation has been discussed for $n = 2, 3, 4, 5, 6, 7$ and general value of n. Solutions are also obtained for $n = \frac{2}{3}, \frac{2}{5}, \frac{2}{7}$ and $\frac{2}{q}$, $q > 2$. Infinite solutions of these surd equations are possible. Solutions for other values of n can also be obtained.

4. Acknowledgement

The authors are extremely thankful to the unknown referee for his fruitful comments and suitable suggestions for improvement of the paper.

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