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ON SOLUTIONS OF INTEGRO QUASI-DIFFERENTIAL EQUATIONS IN L^p -SPACES

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Abstract

A general quasi-differential expression τ of order n with complex coefficients and its formal adjoint τ^+ are considered in the space $L_w^p(a, b)$. In the case of one singular end-point and under suitable conditions on the function $F(t, y)$, we show that all solutions of a general integro quasi-differential equation $[\tau - \lambda I]y(t) = wF(t, y)$, $(\lambda \in \mathbb{C})$ are in $L_w^p(a, b) \cap L^\infty(a, b)$ for all $\lambda \in \mathbb{C}$ provided that all solutions of the homogeneous differential equations $(\tau - \lambda I)u = 0$ and $(\tau^+ - \overline{\lambda} I)v = 0$ are in $L_w^p(a, b) \cap L^\infty(a, b)$.

Keywords and phrases: quasi-differential expressions, Shin-Zettl matrix, L^p -spaces, quasi-derivatives and their adjoints, regular and singular end-points, integro quasi-differential equations, boundedness of L^p -solutions.

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1. Introduction

Wong et al. [14-17] considered the problem that all solutions of a perturbed linear differential equation belong to $L^2(0, b)$ assuming the fact that all solutions of the unperturbed equation possess the same property. For an ordinary linear differential equations with real coefficients and under suitable conditions on the function F, they showed that all solutions of the equation

$$\tau[y] - \lambda wy = wF(t, y), \quad (\lambda \in \mathbb{C}) \text{ on } [0, b), \tag{1.1}$$

are in $L_w^2(0, b)$ provided that all those of the equations

$$(\tau - \lambda I)u = 0 \quad \text{and} \quad (\tau^+ - \overline{\lambda} I)v = 0, \quad (\lambda \in \mathbb{C}) \tag{1.2}$$

are in $L_w^2(0, b)$.

In [7-9], Ibrahim extends their results for a general quasi-differential expression τ of arbitrary order n with complex coefficients, and considered the property of boundedness of solutions of a general integro quasi-differential equations.

Our objective in this paper is to extend the results in [6-9] and [14-18] to a general integro quasi-differential equations with their solutions in the space $L_w^p(a, b)$, $p \geq 2$. Also, we show in the case of one singular endpoint and under suitable conditions on the integrand function F that all solutions of the general integro quasi-differential equation (1.1) are in $L_w^p(a, b) \cap L^\infty(a, b)$ provided that all solutions of the homogeneous integro quasi-differential equations in (1.2) are in $L_w^p(a, b) \cap L^\infty(a, b)$.

2. Quasi-differential Operators on L^p-spaces

We deal, throughout this paper, with a quasi-differential expression τ of an arbitrary order n defined by a Shin-Zettl matrix in the L^p-space. The left-hand end-point of the interval $I = [a, b)$ is assumed to be regular but the right-hand end-point may be either regular or singular.

First, we define the L^p -space.

Let $\mathbb{K}$ denote either $\mathbb{R}$, the field of real numbers, or $\mathbb{C}$, the field of complex numbers. For some positive integers n and m, let $\mathbb{M}_{n,m}$ denote the vector space of $n \times m$ matrices with $\mathbb{K}$-valued entries and GL_m the subset of $\mathbb{M}_m := \mathbb{M}_{m,m}$ consisting of all non-singular matrices. For $A \in \mathbb{M}_{n,m}$, let A^T denote the transpose and A^* the adjoint, i.e., the complex conjugate transpose of A.

If A is a subset of $\mathbb{M}_{n,n}$ and I is an interval, $B(I, A)$ denotes the set of Lebesgue measurable maps of I into A and $AC_{loc}(I, A)$ the set of locally absolutely continuous maps. Measurable maps are regarded as equal if they are equal almost everywhere on I. Further we define

$$L^p(I, A) := \{ y \in B(I, A) \mid |y|^p \text{ is Lebesgue-integrable} \},$$

$$\|y\|_{p,I} := \left(\int_I |y|^p \right)^{\frac{1}{p}} \text{ for all } y \in L^p(I, A) \text{ and } p \in [1, \infty),$$

$$L^\infty(I, A) := \{ y \in B(I, A) \mid y \text{ is essential bounded} \},$$

$$\|y\|_{\infty, I} := \operatorname{ess\,sup}_{x \in I} |y(x)| \text{ for all } y \in L^\infty(I, A),$$

$$L^p_{loc}(I, A) := \{ y \in B(I, A) \mid y|K \in L^p(K, A) \text{ for all compact subinterval } K \text{ of } I,\ p \in [1, \infty) \}.$$

If $r \in [1, \infty)$, then $r' \in [1, \infty)$ is always chosen such that $\frac{1}{r} + \frac{1}{r'} = 1$. We always assume that $p, q \in [1, \infty)$. If $L^p := L^p(I, \mathbb{K}^s)$ for some positive integer s, then $(L^p)^* = L^{p'}$ for $p \in [1, \infty)$ and L^1 is a subspace of $(L^\infty)^*$, where $(.)^*$ denotes the complex conjugate transpose. We refer to [5] for more details.

Let I be an interval with end-points $a, b\ (-\infty \le a < b \le \infty)$, let n, s be positive integers and $p, q \in [1, \infty)$. The quasi-differential expressions are defined in terms of a Shin-Zettl matrix $Z_{n,s}^{p,q}(I)$ on an interval *I*.

Definition 2.1 [5, 12]**.** The set $Z_{n,s}^{p,q}(I)$ of Shin-Zettl matrices on *I* consists of matrices are defined to be the sets of all lower triangular matrices $F = \{f_{j,k}\}$ of the form

$$F = \begin{pmatrix} f_{0,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ f_{n,1} & \cdots & f_{n,n+1} \end{pmatrix}$$

whose entries are complex-valued functions on *I* which satisfy the following conditions:

$$f_{0,1} \in L_{loc}^{p}(I, \mathbb{M}_s) \text{ and } f_{n,n+1} \in L_{loc}^{q'}(I, \mathbb{M}_s), \quad f_{j,k} \in L_{loc}^{p}(I, \mathbb{M}_s)$$

$$\text{for all } 1 \leq j \leq n \text{ and } 1 \leq k \leq \min\{j+1, n\}, \quad f_{j,j+1}(x) \in GL_s$$

$$\text{for all } 0 \leq j \leq n \text{ and } x \in I. \tag{2.1}$$

For $F \in Z_{n,s}^{p,q}(I)$, we define $\widetilde{F}$ as the $(n \times n)$ matrix obtained from *F* by removing the first row and the last column, i.e.,

$$\widetilde{F} = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ f_{n-1,1} & f_{n-1,2} & \cdots & f_{n-1,n} \\ f_{n,1} & f_{n,2} & \cdots & f_{n,n} \end{pmatrix}.$$

Definition 2.2 [5]**.** For $\widetilde{F} \in Z_{n,n}^{p,q}(I)$, the quasi-derivatives associated with $\widetilde{F}$ are defined by

$$y_{\widetilde{F}}^{[0]} := y_{\widetilde{F}},$$

$$y_{\widetilde{F}}^{[j]} := (f_{j,j+1})^{-1}\left\{\left(y_{\widetilde{F}}^{[j-1]}\right)' - \sum_{k=1}^{j} f_{j,k}\, y_{\widetilde{F}}^{[k-1]}\right\}, \ (1 \leq j \leq n-1),$$

$$y_{\tilde{F}}^{[n]} := \left\{\left(y_{\tilde{F}}^{[n-1]}\right)' - \sum_{k=1}^{n} f_{j,\,k}\, y_{\tilde{F}}^{[k-1]}\right\}, \tag{2.2}$$

where the prime $'$ denotes differentiation.

The quasi-differential expression $\tau_{\tilde{F}}$ associated with $\tilde{F}$ is given by:

$$\tau_{\tilde{F}}[.] := i^n y_{\tilde{F}}^{[n]},\ (n \geq 2), \tag{2.3}$$

this being defined on the set:

$$V(\tau_{\tilde{F}} := \{y_{\tilde{F}} : y_{\tilde{F}}^{[j-1]} \in AC_{loc}(I, \mathbb{K}^n), 1 \leq j \leq n\},$$

where $AC_{loc}(I, \mathbb{K}^n)$ denotes the set of functions which are locally absolutely continuous on every compact subinterval of I.

For $y \in V(\tau_{\tilde{F}})$, we define $Q_{\tilde{F}}\, y := \begin{pmatrix} y_{\tilde{F}}^{[0]} \\ \vdots \\ y_{\tilde{F}}^{[n-1]} \end{pmatrix}.$

Clearly the maps $\tau_{\tilde{F}} : V(\tau_{\tilde{F}}) \to B(I, \mathbb{K}^n)$ and $Q_{\tilde{F}} : V(\tau_{\tilde{F}}) \to AC_{loc}(I, \mathbb{K}^n)$ are linear.

In analogy to the adjoint and the transpose of a matrix, there are two different "(formal) adjoint" of a quasi-differential expression τ, we refer to [2-5] and [7-10] for more details.

In the following, we always assume that $\tilde{F} \in Z_{n,n}^{p,q}$ and $\tau_{\tilde{F}} := \tau_{p,q}$. The formal adjoint $\tau_{p,q}^{+}$ of $\tau_{p,q}$ is defined by the matrix $\tilde{F}^{+}$ given by

$$\tilde{F}^{+} = -J_n^{-1} \tilde{F}^{*} J_n, \tag{2.4}$$

where $\tilde{F}^{*}$ is the conjugate transpose of $\tilde{F}$ and J_n is the non-singular $(n \times n)$ matrix

$$J_n = ((-1)^j \delta_{j, n+1-k})_{\substack{1 \le j \le n \\ 1 \le k \le n}} \tag{2.5}$$

δ being the Kronecker delta. If $\tilde{F}^+ = f_{j,k}^+$, then it follows that

$$f_{j,k}^+ = (-1)^{j+k+1} \overline{f}_{n-k+1, n-j+1}. \tag{2.6}$$

The quasi-derivatives associated with the matrix $\tilde{F}^+$ in $Z_{n,n}^{p,q}(I)$ are therefore

$$y_+^{[0]} := y,$$

$$y_+^{[j]} := (\overline{f}_{n-j, n-j+1})^{-1} \left\{ \left(y_+^{[j-1]} \right)' - \sum_{k=1}^{j} (-1)^{j+k+1} \overline{f}_{n-k+1, n-j+1} y_+^{[k-1]} \right\}, \tag{2.7}$$

$$y_+^{[n]} := \left\{ \left(y_+^{[n-1]} \right)' - \sum_{k=1}^{n} (-1)^{n+k+1} \overline{f}_{n-k+1, 1} y_+^{[k-1]} \right\},$$

$$\tau_{q', p'}^+[.] := i^n y_+^{[n]} (n \ge 2) \text{ for all } y \in V(\tau_{q', p'}^+), \tag{2.8}$$

$$V(\tau_{q', p'}^+) := \{ y : y_+^{[j-1]} \in AC_{loc}(I, \mathbb{K}^n), 1 \le j \le n \}. \tag{2.9}$$

Note that: $(\tilde{F}^+)^+ = \tilde{F}$ and so $(\tau_{q', p'}^+)^+ = \tau_{p,q}$. We refer to [2-5], [7-10] and [19, 20] for a full account of the above and subsequent results on a quasi-differential expressions.

For $u \in V(\tau_{p,q}), v \in V(\tau_{q', p'}^+)$ and $\alpha, \beta \in I$, we have Green's formula

$$\int_{\alpha}^{\beta} \{ \overline{v} \tau_{p,q}[u] - u \overline{\tau_{q', p'}^+[v]} \} dx = [u, v](\beta) - [u, v](\alpha), \tag{2.10}$$

where

$$[u, v](x) = i^n \left(\sum_{r=0}^{n-1} (-1)^{r+n+1} u^{[r]} \overline{v_+^{[n-r-1]}} \right)(x)$$

$$= (-i)^n (Q_{\tilde{F}}^T u J_{n \times n} Q_{\tilde{F}} \overline{v}(x)$$

$$= (-i)^n (u, u^{[1]}, \ldots, u^{[n-1]}) J_{n\times n} \begin{pmatrix} \overline{v} \\ \vdots \\ \overline{v}_+^{[n-1]} \end{pmatrix} (x), \tag{2.11}$$

see [2-5] , [7-10] and [19]. Let $w : I \to \mathbb{R}$ be a non-negative weight function with $w \in L^1_{loc}(I)$ and $w > 0$ (for almost all $x \in I$). Then $H^r = L^r_w(I, \mathbb{K}^n)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that

$$\|y\|_{r,I} := \left(\int_I |y|^r w\right)^{\frac{1}{r}} \text{ for all } y \in L^r(I, \tilde{F}) \text{ and } r \in [1, \infty). \tag{2.12}$$

The equation

$$\tau_{p,q}[u] - \lambda w u = 0, \ (\lambda \in \mathbb{C}) \text{ on } I, \tag{2.13}$$

is said to be *regular* at the left end-point $a \in \mathbb{R}$, if for all $X \in [a, b)$,

$$a \in \mathbb{R}, \quad w, f_{j,k} \in L^1[a, X], \quad j, k = 1, 2, \ldots, n,$$

otherwise (2.13) is said to be *singular* at a. If (2.13) is regular at both end-points, then it is said to be regular; in this case we have

$$a, b \in \mathbb{R}, \quad w, f_{j,k} \in L^1[a, b], \quad j, k = 1, 2, \ldots, n. \tag{2.14}$$

We shall be concerned with the case when a is a regular end-point of the equation (2.13), the end-point b being allowed to be either regular or singular. Note that, in view of (2.6), an end-point of I is regular for (2.13), if and only if it is regular for the equation

$$\tau^+_{p,q}[v] - \overline{\lambda} w v = 0, \ (\lambda \in \mathbb{C}) \text{ on } I. \tag{2.15}$$

3. L^p_w-Solutions

In this section, we shall be concerned with L^p_w-solutions of the integro quasi-differential equations, and we denote for $\tau_{p,q}$ by τ and $\tau^{+}_{p,q}$ by τ^{+}.

Denote by $S(\tau)$ and $S(\tau^{+})$ $S(\tau^{+})$ the sets of all solutions of the equations

$$[\tau - \lambda_0 I]u = 0, \quad (\lambda_0 \in \mathbb{C}) \tag{3.1}$$

and

$$[\tau^{+} - \overline{\lambda}_0 I]v = 0, \quad (\lambda_0 \in \mathbb{C}), \tag{3.2}$$

respectively. Let $\varphi_j(t, \lambda)$, $j = 1, 2, \ldots, n$ be the solutions of the homogeneous equation $[\tau - \lambda I]u = 0, (\lambda \in \mathbb{C})$ satisfying

$$\varphi_j^{[k-1]}(t_0, \lambda) = \delta_{k, r+1} \text{ for all } t_0 \in [a, b), (j, k = 1, 2, \ldots, n)$$

for fixed t_0, $a < t_0 < b$. Then $\varphi_j(t, \lambda)$ is continuous in (t, λ) for $0 < t < b$, $|\lambda| < \infty$, and for fixed t it is entire in λ. Let $\varphi_k^{+}(t, \lambda)$, $k = 1, 2, \ldots, n$ denote the solutions of the adjoint homogeneous equation $[\tau^{+} - \overline{\lambda} I]v = 0, (\lambda \in \mathbb{C})$ satisfying:

$$(\varphi_k^{+})^{[r]}(t_0, \lambda) = (-1)^{k+r}\delta_{k, n-r} \text{ for all } t_0 \in [0, b),$$

$(k = 1, 2, \ldots, n;\ r = 0, 1, \ldots, n-1)$. Suppose $a < c < b$, by [3], [7-9] and [12-16], a solution of the equation

$$[\tau - \lambda I]u = wf, \quad (\lambda \in \mathbb{C}), \quad f \in L^1_w(a, b) \tag{3.3}$$

satisfying $u(c) = 0$ is given by

$$\varphi(t, \lambda) = \left(\frac{\lambda - \lambda_0}{i^n}\right)\sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t, \lambda)\int_a^t \overline{\varphi_k^{+}(s, \lambda)} f(s)w(s)ds,$$

where $\varphi_k^{+}(t, \lambda)$ stands for the complex conjugate of $\varphi_k(t, \lambda)$ and for each j, k, ξ^{jk} is constant which is independent of t, λ (but does depend in general on t_0).

The next lemma is a form of the variation of parameters formula for a general quasi-differential equation is given by the following Lemma.

Lemma 3.1. *Suppose* $f \in L_w^1(0, b)$ *locally integrable function and* $\varphi(t, \lambda)$ *is the solution of the equation* (3.3) *satisfying*:

$$\varphi^{[r]}(t_0, \lambda) = \alpha_{r+1} \text{ for } r = 0, 1, \ldots, n-1, t_0 \in [a, b).$$

Then

$$\varphi(t, \lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t, \lambda_0) + ((\lambda - \lambda_0)/i^n)$$

$$\times \sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t, \lambda_0)\int_a^t \overline{\varphi_k^+(s, \lambda_0)} f(s)w(s)ds \tag{3.4}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_n(\lambda) \in \mathbb{C}$, *where* $\varphi_j(t, \lambda_0)$ *and* $\varphi_k^+(t, \lambda_0)$, $j, k = 1, 2, \ldots, n$ *are solutions of the equations* (3.1) *and* (3.2), *respectively*, ξ^{jk} *is a constant which is independent of t*.

Lemma 3.2 [13] (Gronwall's inequality)**.** *Let* $u(t)$ *and* $v(t)$ *be two real-valued functions defined*, *non-negative and* $u, v \in L^1(t_0, t)$ *for* $t > t_0$, *and if*

$$u(t) \le c + \int_{t_0}^{t} u(s)v(s)ds, \; c > 0,$$

for some positive constant c, *then*

$$u(t) \le c \exp\left(\int_0^t v(s)ds\right). \tag{3.5}$$

Lemma 3.3. *Suppose that for some* $\lambda_0 \in \mathbb{C}$ *all solutions of the equations* (3.1) *and* (3.2) *are in* $L_w^2(a, b)$. *Then all solutions of the equations in* (1.2) *are in* $L_w^2(a, b)$ *for every complex number* $\lambda \in \mathbb{C}$.

Proof. The proof is similar to that in [8, Lemma 3.5].

Lemma 3.4. *If all solutions of the equation* $[\tau - \lambda_0 w]u = 0$ *are bounded on* $[a, b)$ *and* $\varphi_k^+(t, \lambda_0) \in L_w^1(a, b)$ *for some* $\lambda_0 \in \mathbb{C}$, $k = 1, \ldots, n$. *Then all solutions* *of the equation* $[\tau - \lambda w]u = 0$ *are also bounded on* $[a, b)$ *for every complex number* $\lambda \in \mathbb{C}$.

Lemma 3.5. *Suppose that for some complex number* $\lambda_0 \in \mathbb{C}$ *all solutions of the equation* (3.1) *are in* $L_w^p(a, b)$ *and all solutions of* (3.2) *are in* $L_w^q(a, b)$. *Suppose* $f \in L_w^p(a, b)$, *then all solutions of the equation* (3.3) *are in* $L_w^p(a, b)$ *for all* $\lambda \in \mathbb{C}$.

Proof. Let $\{\varphi_1(t, \lambda_0), \ldots, \varphi_n(t, \lambda_0)\}, \{\varphi_1^+(s, \lambda_0), \ldots, \varphi_n^+(s, \lambda_0)\}$ be two sets of linearly independent solutions of the equations (3.1) and (3.2), respectively. Then for any solutions $\varphi(t, \lambda)$ of the equation $[\tau - \lambda I]\varphi = wf$, $(\lambda \in \mathbb{C})$ which may be written as follows

$$[\tau - \lambda_0 w]\varphi = (\lambda - \lambda_0)w\varphi + wf$$

and it follows from (3.4) that

$$\varphi(t, \lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t, \lambda_0) + \frac{1}{i^n} \sum_{j,k=1}^{n} \xi^{jk} \varphi_j(t, \lambda_0)$$

$$\times \int_a^t \overline{\varphi_k^+(t, \lambda_0)}[(\lambda - \lambda_0)\varphi(s, \lambda) + f(s)]w(s)ds, \tag{3.6}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_n(\lambda) \in \mathbb{C}$. Hence

$$|\varphi(t, \lambda) \le \sum_{j=1}^{n} (|\alpha_j(\lambda)||\varphi_j(t, \lambda_0)|) + \sum_{j,k=1}^{n} |\xi^{jk}||\varphi_j(t, \lambda_0)|$$

$$\times \int_a^t \overline{\varphi_k^+(t, \lambda_0)}[|\lambda - \lambda_0||\varphi(s, \lambda)| + |f(s)|]w(s)ds. \tag{3.7}$$

Since $f \in L_w^p(a, b)$ and $\varphi_k^+(., \lambda_0) \in L_w^q(a, b)$ for some $\lambda_0 \in \mathbb{C}$, then $\varphi_k^+(., \lambda_0)f \in L_w^1(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and $k = 1, \ldots, n$. Setting

$$C_j(\lambda) = \sum_{j,k=1}^{n} |\xi^{jk}| \int_a^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right| |f(s)| w(s)ds, \quad j = 1, 2, \ldots, n, \tag{3.8}$$

then

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} (|\alpha_j(\lambda)| + C_j(\lambda))|\varphi_j(t, \lambda_0)| + |\lambda - \lambda_0|$$

$$\times \sum_{j,k=1}^{n} |\xi^{jk}||\varphi_j(t, \lambda_0)| \int_a^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right| |\varphi(s, \lambda)| w(s) ds. \qquad (3.9)$$

On application of the Cauchy-Schwartz inequality to the integral in (3.9), we get

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} (|\alpha_j(\lambda)| + C_j(\lambda))|\varphi_j(t, \lambda_0)|$$

$$+ |\lambda - \lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}||\varphi_j(t, \lambda_0)|$$

$$\times \left(\int_a^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right|^q w(s) ds \right)^{\frac{1}{q}} \left(\int_a^t |\varphi(s, \lambda)|^p w(s) ds \right)^{\frac{1}{p}}. \qquad (3.10)$$

From the inequality $(u + v)^p \leq 2^{(p-1)}(u^p + v^p)$, it follows that

$$|\varphi(t, \lambda)|^p \leq 2^{2(p-1)} \sum_{j=1}^{n} (|\alpha_j(\lambda)| + C_j(\lambda))^p |\varphi_j(t, \lambda_0)|^p$$

$$+ 2^{2(p-1)} |\lambda - \lambda_0|^p \sum_{j,k=1}^{n} |\xi^{jk}|^p |\varphi_j(t, \lambda_0)|^p$$

$$\times \left(\int_a^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right|^q w(s) ds \right)^{\frac{p}{q}} \left(\int_a^t |\varphi(s, \lambda)|^p w(s) ds \right). \qquad (3.11)$$

By hypothesis there exist positive constant K_0 and K_1 such that

$$\|\varphi_j(t, \lambda_0)\|_{L_w^p(a,b)} \leq K_0 \quad \text{and} \quad \left\|\overline{\varphi_k^+(s, \lambda_0)}\right\|_{L_w^q(a,b)} \leq K_1, \qquad (3.12)$$

$j, k = 1, 2, \ldots, n.$ Hence

$$|\varphi(t, \lambda)|^p \leq 2^{2(p-1)} \sum_{j=1}^{n} (|\alpha_j(\lambda)| + C_j(\lambda))^p |\varphi_j(t, \lambda_0)|^p$$

$$+ 2^{2(p-1)} K_1^p |\lambda - \lambda_0|^p \sum_{j,k=1}^{n} |\xi^{jk}|^p |\varphi_j(t, \lambda_0)|^p$$

$$\times\left(\int_a^t |\varphi(s, \lambda)|^p w(s)ds\right). \tag{3.13}$$

Integrating the inequality in (3.13) between a and t, we obtain

$$\int_0^t |\varphi(s, \lambda)|^p w(s)ds \le K_2 + \left(2^{2(p-1)}|\lambda - \lambda_0|^p \sum_{j,k=1}^n |\xi^{jk}|^p\right)$$

$$\times \int_a^t |\varphi_j(s, \lambda_0)|^p \left(\int_a^s |\varphi(x, \lambda)|^p w(x)dx\right) w(s)ds, \tag{3.14}$$

where

$$K_2 = 2^{2(p-1)} K_0^p \sum_{j=1}^n (|\alpha_j(\lambda)| + C_j(\lambda))^p. \tag{3.15}$$

Now, on using Gronwall's inequality, it follows that

$$\int_0^t |\varphi(s, \lambda)|^p w(s)ds \le K_2$$

$$\exp\left(2^{2(p-1)} K_1^p |\lambda - \lambda_0|^p \sum_{j,k=1}^n |\xi^{jk}|^p \int_a^t |\varphi_j(s, \lambda_0)|^p w(s)ds\right). \tag{3.16}$$

Since, $\varphi_j(t, \lambda_0) \in L_w^p(a, b)$ for some $\lambda_0 \in \mathbb{C}$ and for $j = 1, \ldots, n$, then $\varphi(t, \lambda) \in L_w^p(0, b)$ for all $\lambda \in \mathbb{C}$.

Remark. Lemma 3.5 also holds if the function f is bounded on $[a, b)$.

Lemma 3.6. *Let* $f \in L_w^p(0, b)$. *Suppose for some* $\lambda_0 \in \mathbb{C}$:

(i) *All solutions of* $(\tau^+ - \overline{\lambda} I)\varphi^+ = 0$ *are in* $L_w^q(a, b)$.

(ii) $\varphi_j(t, \lambda_0)$, $j = 1, \ldots, n$ *are bounded on* $[a, b)$.

Then all solutions $\varphi(t, \lambda)$ *of the equation* (3.3) *are in* $L_w^p(a, b)$ *for all* $\lambda \in \mathbb{C}$.

4. L_w^p-boundedness

In this section, we shall consider the question of determining conditions under which all solutions of the equation (1.1) are in $L_w^p(a, b) \cap L^\infty(a, b)$.

Suppose there exist non-negative continuous functions $k(t)$ and $h(t)$ on $[a, b)$, $a < b \leq \infty$ such that the function $F(t, y)$ in (1.1) satisfies:

$$|F(t, y)| \leq k(t) + h(t)|y(t)|^\sigma \quad \text{for} \quad t \geq 0, -\infty < y(t) < \infty, \tag{4.1}$$

for some $\sigma \in [0, 1]$; see [1, 8] and [18-19].

In the sequel, we shall require the following nonlinear integral inequality which generalizes those integral inequalities used in [1], [7-9], and [13-18].

Lemma 4.1 (cf. [8, 17]). *Let $u(t)$ and $v(t)$ be two non-negative functions, locally integrable on the interval $I = [a, b)$. Then the inequality*

$$u(t) \leq c + \int_0^t v(s)u^\sigma(s)dx, \quad c > 0,$$

for $0 \leq \sigma < 1$, implies that

$$u(t) \leq \left(c^{(1-\sigma)} + (1-\sigma)\int_0^t v(s)ds\right)^{\frac{1}{(1-\sigma)}}. \tag{4.2}$$

In particular, if $v(s) \in L^1(a, b)$, then (4.2) *implies that $u(t)$ is bounded.*

Theorem 4.2. *Suppose that F satisfies* (4.1) *with $\sigma = 1$, and that*

(i) $S(\tau) \cup S(\tau^+) \subset L^\infty(0, b)$ *for some* $\lambda_0 \in \mathbb{C}$,

(ii) $k(t)$ *and* $h(t) \in L_w^1(0, b)$ *for all* $t \in [a, b)$.

Then all solutions $\varphi(t, \lambda)$ of the equation (1.1) *are bounded on $[a, b)$ for all $\lambda \in \mathbb{C}$.*

Proof. Note that (4.1) and Lemma 3.6 implies that all solutions are defined on

$[a, b)$; see [2, Chapter 3], [7-9] and [13] and let $\{\varphi_1(t, \lambda_0), \varphi_2(t, \lambda_0), \ldots, \varphi_n(t, \lambda_0)\}$, $\{\varphi_1^+(s, \lambda_0), \{\varphi_2^+(s, \lambda_0), \ldots, \varphi_n^+(s, \lambda_0)\}$ be two sets of linearly independent solutions of the equations (3.1) and (3.2), respectively, and let $\varphi(t, \lambda)$ be any solution of (1.1) on $[a, b)$, then by Lemma 3.1, we have

$$\varphi(t, \lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t, \lambda_0) + \frac{1}{i^n}(\lambda - \lambda_0)\sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t, \lambda_0)$$

$$\times \int_a^t \overline{\varphi_k^+(s, \lambda_0)} F(s, y)w(s)ds.$$

Hence

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} |\alpha_j(\lambda)||\varphi_j(t, \lambda_0)| + |\lambda - \lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}||\varphi_j(t, \lambda_0)$$

$$\times \int_a^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right| \left(k(s) + h(s)|\varphi_j(s, \lambda)|\right) w(s)ds. \tag{4.3}$$

Since $k(s) \in L_w^1(a, b)$ and $\varphi_k^+(s, \lambda_0) \in L_w^\infty(a, b)$, $k = 1, 2, \ldots, n$ for some $\lambda_0 \in \mathbb{C}$, we have $\varphi_k^+(s, \lambda_0)k(s) \in L_w^1(a, b)$ for some $\lambda_0 \in \mathbb{C}$. Setting

$$C_j = |\lambda - \lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}| \int_a^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right| k(s)w(s)ds, \quad j = 1, 2, \ldots, n. \tag{4.4}$$

Then

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)|)|\varphi_j(t, \lambda_0)|$$

$$+ |\lambda - \lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}||\varphi_j(t, \lambda_0)$$

$$\times \int_0^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| h(s)|\varphi(s, y)| w(s)ds. \tag{4.5}$$

By hypothesis, there exist positive constants K_0 and K_1 such that

$$|\varphi_j(t, \lambda_0)| \leq K_0 \quad \text{and} \quad \left|\overline{\varphi_k^+(t, \lambda_0)}\right| \leq K_1 \text{ for all } t \in [0, b),$$

$j, k = 1, ..., n.$ Hence

$$|\varphi(t, \lambda_0)| \leq K_0 \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)|) + K_0 K_1 |\lambda - \lambda_0|$$

$$\times \sum_{j,k=1}^{n} |\xi^{jk}| \int_0^t h(s) |\varphi(s, \lambda)| w(s) ds. \tag{4.6}$$

Applying Gronwall's inequality to (4.6) and using (ii), we deduce that $|\varphi(t, \lambda)|$ is finite and hence the result.

Theorem 4.3. *Suppose that F satisfies* (4.1) *with* $\sigma = 1$, *and that*

(i) $S(\tau) \cup S(\tau^+) \subset L_w^\infty(a, b)$ *for some* $\lambda_0 \in \mathbb{C}$,

(ii) $k(t)$ *and* $h(t) \in L_w^q(a, b)$ *for all* $t \in [a, b)$.

Then all solutions $\varphi(t, \lambda)$ of the equation (1.1) are in $L_w^p(a, b)$ for all $\lambda \in \mathbb{C}$.

Proof. The proof follows on applying the Cauchy-Schwartz inequality for the integral in (4.5) as:

$$\int_a^t \left|\overline{\varphi_k^+(t, \lambda_0)}\right| |h(s)| |\varphi(s, \lambda)| w(s) ds$$

$$\leq \left(\int_a^t \left|\overline{\varphi_k^+(s, \lambda_0)}\right|^q |h(s)|^q w(s) ds \right)^{\frac{1}{q}} \left(\int_a^t |\varphi(s, \lambda)|^p w(s) ds \right)^{\frac{1}{p}}, \tag{4.7}$$

and hence the result. We refer to [1] and [16] for more details.

Corollary 4.4. *Suppose that* $|F(t, y)| = h(t)|y(t)|$, $S(\tau) \subset L_w^p(a, b)$, $S(\tau^+) \subset L_w^q(a, b)$ *for some* $\lambda_0 \in \mathbb{C}$ *and* $h(t) \in L_w^p(a, b)$ *for some* $p \geq 2$, $t \in [a, b)$. *Then all solutions* $\varphi(t, \lambda)$ *of the equation* (1.1) *are in* $L_w^p(a, b)$ *for all* $\lambda \in \mathbb{C}$.

Corollary 4.5. *Suppose that for some* $\lambda_0 \in \mathbb{C}$, $S(\tau) \subset L_w^p(0, b)$, $S(\tau^+) \subset L_w^q(a, b)$ *and* $k(t) \in L_w^p(a, b)$. *Then all solutions of the equations* $[\tau - \lambda w]\varphi = wk$ *are in* $L_w^p(a, b)$ *for every complex number* $\lambda \in \mathbb{C}$.

Next, for considering (4.1) with $0 \leq \sigma < 1$, we have the following.

Theorem 4.6. *Suppose that* $F(t, y)$ *satisfies* (4.1) *with* $0 \leq \sigma < 1$, $S(\tau) \cup S(\tau^{+}) \subset L_w^{\alpha}(a, b)$, $\alpha \geq 2$ *for some* $\lambda_0 \in \mathbb{C}$ *and that*

(i) $k(t) \in L_w^{\alpha}(a, b)$ *for all* $t \in [a, b)$,

(ii) $h(t) \in L_w^{\alpha/(\alpha-1-\sigma)}(a, b)$ *for all* $t \in [a, b)$.

Then all solutions $\varphi(t, \lambda)$ *of the equation* (1.1) *are in* $L_w^{\alpha}(a, b)$, $\alpha \geq 2$ *for all* $\lambda \in \mathbb{C}$.

Proof. For $0 \leq \alpha < 1$, the proof is the same up to (1.1). In this case (4.5) becomes

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)|)|\varphi_j(t, \lambda_0)| + |\lambda - \lambda_0|$$

$$\times \sum_{j, k=1}^{n} |\xi^{jk}||\varphi_j(t, \lambda_0)| \int_a^t \left|\overline{\varphi_k^{+}(s, \lambda_0)}\right| h(s)|\varphi(s, \lambda)|^{\sigma} w(s) ds. \tag{4.8}$$

Applying the Cauchy-Schwartz inequality to the integral in (4.8) we get

$$\int_0^t \left|\overline{\varphi_k^{+}(s, \lambda_0)}\right| |h(s)||\varphi(s, \lambda)|^{\sigma} w(s) ds,$$

$$\leq \left(\int_a^t \left|\overline{\varphi_k^{+}(s, \lambda_0)}\right|^{\mu} |h(s)|^{\mu} w(s) ds\right)^{\frac{1}{\mu}} \left(\int_a^t |\varphi(s, \lambda)|^{\alpha} w(s) ds\right)^{\frac{\sigma}{\alpha}}, \tag{4.9}$$

where $\mu = \alpha/(\alpha - \sigma)$, $\alpha \geq 2$. Since $\varphi_k^{+}(t, \lambda_0) \in L_w^{\alpha}(a, b)$ for some $\lambda_0 \in \mathbb{C}$, $k = 1, 2, \ldots, n$ and $h(s) \in L_w^{\alpha/(\alpha-1-\sigma)}(a, b)$ by hypothesis, then we have $\varphi_k^{+}(t, \lambda_0|h(t)|) \in L_w^{\mu}(a, b)$, for some $\lambda_0 \in \mathbb{C}$, $k = 1, 2, \ldots, n$. Using this fact and (4.9), we obtain

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)|)\varphi_j(t, \lambda_0)| + K_0|\lambda - \lambda_0|$$

$$\times \sum_{j,k=1}^{n} |\xi^{jk}| |\varphi_j(t, \lambda_0)| \left(\int_a^t |\varphi(s, \lambda)|^{\alpha} w(s) ds \right)^{\frac{\sigma}{\alpha}}, \quad (4.10)$$

where $K_0 = \|\varphi_k^+(t, \lambda_0) h(t)\|_{\mu}$, $\|\cdot\|_{\mu}$ denotes the norm in $L_w^{\mu}(a, b)$. The inequality

$$(u + v)^{\alpha} \leq 2^{(\alpha - 1)} (u^{\alpha} + v^{\alpha})$$

implies that

$$|\varphi(t, \lambda)|^{\alpha} \leq 2^{2(\alpha-1)} \sum_{j=1}^{n} (C_j^{\alpha} + |\alpha_j(\lambda)|^{\alpha}) |\varphi_j(t, \lambda_0)|^{\alpha} + 2^{2(\alpha-1)} K_0^{\alpha}$$

$$\times |\lambda - \lambda_0|^{\alpha} \sum_{j,k=1}^{n} |\xi^{jk}|^{\alpha} |\varphi_j(t, \lambda_0)|^{\alpha} \left(\int_a^t |\varphi(s, \lambda)|^{\alpha} w(s) ds \right)^{\sigma}. \quad (4.11)$$

Setting $K_1 = \int_a^t |\varphi_j(t, \lambda_0)|^{\alpha} w(s) ds$ for some $\lambda_0 \in \mathbb{C}$, $j = 1, ..., n$ and integrating (4.11), we obtain

$$\int_0^t |\varphi(t, \lambda)|^{\alpha} w(s) ds \leq K_2 + 2^{2(\alpha-1)} K_0^{\alpha} |\lambda - \lambda_0|^{\alpha}$$

$$\times \sum_{j,k=1}^{n} |\xi^{jk}|^{\alpha} \int_a^t |\varphi_j(s, \lambda_0)|^{\alpha} \left[\left(\int_a^s |\varphi(x, \lambda)|^{\alpha} w(x) dx \right)^{\sigma} \right] w(s) ds, \quad (4.12)$$

where $K_2 = 2^{2(\alpha-1)} \sum_{j=1}^{n} (C_j^{\alpha} + |\alpha_j(\lambda)|^{\alpha}) K_1$.

An application of Lemma (4.1) for $0 \leq \sigma < 1$ and of Gronwall's inequality to (4.12) for $\sigma = 1$, yields the result.

Theorem 4.7. *Suppose that* F *satisfies* (4.1) *with* $0 \leq \sigma < 1$, $S(\tau) \cup S(\tau^+) \subset L_w^{\alpha}(0, b) \cap L^{\infty}(a, b)$, $\alpha \geq 2$ *for some* $\lambda_0 \in \mathbb{C}$ *and that*

(i) $k(t) \in L_w^{\alpha}(a, b)$ *for all* $t \in [a, b)$,

(ii) $h(t) \in L_w^{p}(a, b)$ *for some* p, $1 \leq p \leq 2/(1 - \sigma)$.

Then all solutions $\varphi(t, \lambda)$ of the equation (1.1) are in $L_w^{\alpha}(a, b) \cap L^{\infty}(a, b)$ for all

$\lambda \in \mathbb{C}$.

Proof. Since $S(\tau) \cup S(\tau^+) \subset L_w^\alpha(a, b) \cap L^\infty(a, b)$ for some $\lambda_0 \in \mathbb{C}$, then $\varphi_j(s, \lambda_0) \in L_w^p(a, b)$ and $\varphi_k^+(t, \lambda_0) \in L_w^q(a, b)$, $j, k = 1, ..., n$ for every $p, q \geq 2$ and for some $\lambda_0 \in \mathbb{C}$.

First, suppose that $h(t) \in L_w^p(a, b)$ for some $p, 1 \leq p \leq 2$. Setting

$$K_0 = \|\varphi_j(t, \lambda_0)\|_\infty \quad \text{and} \quad K_1 = \left\|\overline{\varphi_k^+(s, \lambda_0)}\right\|_\infty, \quad j, k = 1, 2, ..., n, \tag{4.13}$$

we have from (4.8),

$$|\varphi(t, \lambda)| \leq K_0 \sum_{j=1}^n (C_j + |\alpha_j(\lambda)|) + K_0 K_1 |\lambda - \lambda_0|$$

$$\times \sum_{j,k=1}^n |\xi^{jk}| \int_a^t h(s) |\varphi(s, \lambda)|^\sigma w(s) ds.$$

Since $h(t) \in L_w^p(a, b)$ for some p, $1 \leq p \leq 2$, then Lemma 4.1 together with Gronwall's inequality implies that $\varphi(t, \lambda) \in L^\infty(a, b)$ for all $\lambda \in \mathbb{C}$, i.e., there exists a positive constant K_2 such that

$$|\varphi(t, \lambda)| \leq K_2 \text{ for all } \lambda \in \mathbb{C}, t \in [a, b). \tag{4.14}$$

From (4.8) and (4.14), we obtain

$$|\varphi(t, \lambda)| \leq K_0 \sum_{j=1}^n (C_j + |\alpha_j(\lambda)| + K_3) |\varphi_j(t, \lambda_0)|$$

for an appropriate constant K_3. Since $\varphi_j(t, \lambda_0) \in L_w^2(a, b)$ for some $\lambda_0 \in \mathbb{C}$, this proves $\varphi(t, \lambda) \in L_w^p(a, b)$ for all $\lambda \in \mathbb{C}, 1 \leq p \leq 2$.

Next, suppose that $h(t) \in L_w^p(a, b)$ for some $p, 2 < p \leq 2/(1 - \sigma)$. Define $q \geq 2$ by

$$\frac{1}{q} = \frac{\alpha - \sigma}{\alpha} - \frac{1}{p}$$

(which is possible because of the restriction on q).

Thus $\varphi_j(t, \lambda_0)\varphi_k^+(s, \lambda_0) \in L_w^q(a, b)$ and $\varphi_k^+(s, \lambda_0)h(t) \in L_w^\mu(a, b)$, $\mu = \frac{\alpha}{\alpha - \sigma}$, $\alpha \geq 2$; $j, k = 1, \ldots, n$. Repeating the same argument from (4.8) to (4.12) in the proof of Theorem 4.6, we obtain that $\varphi(t, \lambda) \in L_w^\alpha(a, b)$. Returning to (4.9), we find that the integral on the left-hand side is bounded, which implies by (4.8) that

$$|\varphi(t, \lambda)| \leq \sum_{j=1}^{n} (C_j + |\alpha_j(\lambda)| + K_3)|\varphi_j(t, \lambda_0)|$$

for an appropriate constant K_3. Since $\varphi_j(t, \lambda_0) \in L^\infty(a, b)$, this completes the proof. We refer to [1], [7-9] and [17, 19] for more details.

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References

[1] R. P. Agarwal, A note on Grace and Lalli, s paper, J. Math. Anal. Appl. 86 (1982), 471-475.

[2] D. E. Edmunds and W. D. Evans, Spectral Theory and Differential Operators, Oxford University Press, 1987.

[3] W. N. Everitt, The number of integrable-square solutions of products of differential expressions, Proc. Royal Soc. of Edinburgh 76A (1977), 215-226.

[4] W. N. Everitt and D. Race, Some remarks on linear ordinary quasi-differential expressions, Proc. London Math. Soc. 54(3) (1987), 300-320.

[5] H. Frentzen, Quasi-differential operators in L^p spaces, Bull. London Math. Soc. 31(3) (1999).

[6] S. R. Grace and B. S. Lalli, Asymptotic behavior of certain second order integro-differential equations, J. Math. Anal. Appl. 76 (1980), 84-90.

[7] S. E. Ibrahim, Boundedness for solutions of general ordinary quasi-differential equations, J. Egypt. Math. Soc. 2 (1994), 33-44.

[8] S. E. Ibrahim, On L_w^2-quasi derivatives for solutions of perturbed general quasi-differential equations, Czech. Math. J. 49(124) (1999), 877-890.

[9] S. E. Ibrahim, On the boundedness of solutions for products of quasi-integro differential equations, IJMMS 10(16) (2003), 639-659.

[10] S. E. Ibrahim, On the boundary conditions for products of general quasi-differential operators in direct sum spaces, IJAM 22(1) (2009), 55-87.

[11] S. E. Ibrahim, On the product of self-adjoint Sturm-Liouville differential operators in direct sum spaces, J. Inform. Math. Sci. 1 (2012), 93-109.

[12] S. E. Ibrahim, The regularly solvable operators in L^p spaces, Fundamental J. Math. Math. Sci. 2(1) (2015), 1-28.

[13] B. S. Lalli, On boundedness of solutions of certain second-order differential equations, J. Math. Anal. Appl. 25 (1969), 182-188.

[14] J. S. W. Wong, Square integrable solutions of perturbed linear differential equations, Proc. Royal Soc. of Edinburgh 73A (1974/75), 251-254.

[15] E. H. Yang, On asymptotic behavior of certain second order integro-differential equations, Proc. Amer. Math. Soc. 90 (1984), 271-276.

[16] E. H. Yang, Asymptotic behavior of certain second order integro-differential equations, J. Math. Anal. Appl. 106 (1985), 132-139.

[17] A. Zettl, Square integrable solutions of $Ly = f(t, y)$, Proc. Amer. Math. Soc. 26 (1970), 635-639.

[18] A. Zettl, Perturbation of the limit circle case, Quart. J. Math. Oxford 26(3) (1975), 355- 360.

[19] A. Zettl, The limit point and limit circle cases for polynomials in differential expressions, Proc. Royal Soc. of Edinburgh 72A (1975), 219-224.

[20] A. Zettl, Formally self-adjoint quasi-differential operators, Rocky Mountain J. Math. 5 (1975), 453-474.