Fundamental Journal of Mathematics and Mathematical Sciences
Volume 12, Issue 2, 2019, Pages 27-41
This paper is available online at http://www.frdint.com/
Published online November 21, 2019

ON PEDAL POLYGONS IN LORENTZIAN PLANE

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Abstract

In this paper, we characterize the pedal pure non-null polygon of a pure non-null polygon in Lorentzian plane. In addition, we study the pure timelike triangles and the relationships with its orthic triangle. Finally, we show a family of convex pure timelike trapezoids whose pedal polygon vertices are collinear points.

1. Introduction

Archimedes discovered the center of gravity of a triangle, centroid, and he was the first to consider the so-called medial triangle, whose vertices are the midpoints of the sides of a given triangle, [3].

Keywords and phrases: Lorentzian plane, pure polygon, pedal polygon, orthic triangle.

2010 Mathematics Subject Classification: 53B20, 53C50.

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Received September 14, 2019; Accepted September 28, 2019

Orthic triangle is the answer to a famous optimization problem in plane geometry, the Fagnano's Problem, [3].

These triangles are only two examples of pedal triangles of a point P with respect to a given triangle in Euclidean geometry.

It is well known that polygons in Lorentzian plane have some properties of their own that do not agree with those of Euclidean geometry, [1], [2].

The aim of this paper is to characterize the pedal pure non-null polygon of a pure timelike polygon as pure spacelike polygon in Lorentzian plane.

In order to do it, we study the pedal triangles of pure triangles. We show relationships between an isosceles pure timelike triangle and its orthic triangle, involving angles (middle and not middle), Lorentzian sides and circles. In addition, we find a family of convex pure timelike trapezoids whose pedal polygon vertices are collinear points.

2. Preliminaries

Let x and y be two vectors in the vector space $\mathbb{R}^2$. As it is well known, [1], the Lorentzian inner product of x and y is defined by

$$\langle x, y \rangle = x_1 y_1 - x_2 y_2.$$

Thus, the square ds^2 of an element of arc-length is given by

$$ds^2 = dx_1^2 - dx_2^2.$$

The space $\mathbb{R}^2$ equipped with this metric is called the bidimensional Lorentzian space. We write L^2 instead of $(\mathbb{R}^2, ds)$.

We say that a vector $x \in L^2$ is timelike if $\langle x, x \rangle < 0$, spacelike if $\langle x, x \rangle > 0$ and null if $\langle x, x \rangle = 0$. The null vectors also said to be lightlike.

In L^2, let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. A timelike vector x is future-pointing (resp., past-pointing) if $\langle x, e_2 \rangle < 0$ (resp., $\langle x, e_2 \rangle > 0$).

Let x be a vector in L^2, then $\|x\| = \sqrt{|\langle x, x\rangle|}$ is called the Lorentzian norm of x. We say that x is orthogonal to y if $\langle x, y\rangle = 0$, $x \neq y \neq 0$.

When x and y are future-pointing timelike vectors, the angle α (oriented or unoriented) for x and y satisfies $\cosh \alpha = -\dfrac{\langle x, y\rangle}{\|x\|.\|y\|}$.

Let P be a point in Lorentzian plane and $r > 0$. The curve $(Q; \langle \overrightarrow{PQ}, \overrightarrow{PQ} \rangle = r^2)$ has two branches, $\mathcal{C}^+ y \, \mathcal{C}^-$, and each of them is called the Lorentzian circle of center P and radius r. If $P = (p_1, p_2)$, the corresponding equation is $(x_1 - p_1)^2 - (x_2 - p_2)^2 = r^2$.

In classical way, $T[A, B, C]$ denotes the triangle with vertices A, B, C which are non collinear points in Lorentzian plane. According to Birman and Nomizu, [1], by a pure timelike triangle, we mean a triangle $T[A, B, C]$ such that $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$ are timelike vectors. We will call the middle vertex to the vertex B such that the angle $\widehat{B}$ between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ looks more like the exterior angle to Euclid. That is, the vertex B is a middle vertex of $T[A, B, C]$ if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same timelike orientation.

In [2], for each $m \geq 1$, it is defined a polygonal path of order m as a set of points of the form $[P_0, P_1, \ldots, P_{m+1}] = \overline{P_0P_1} \cup \overline{P_1P_2} \cup \cdots \cup \overline{P_mP_{m+1}}$, with the points $P_0, P_1, \ldots, P_{m+1} \in L^2$ as vertices and the named segments as sides. If the polygonal path $[P_0, P_1, \ldots, P_{m+1}]$ is closed and if no three of its vertices lie on a line, then it is called a polygon and it is denoted $\mathbf{P}[P_0, P_1, \ldots, P_m]$.

A polygon $\mathbf{P} = \mathbf{P}[P_0, P_1, \ldots, P_m]$ is said to be pure spacelike, pure timelike or pure lightlike if every side of **P** is spacelike, timelike or lightlike, respectively.

Let us recall that the polygon with at most two different vertices is usually called a degenerated polygon. In what follows, we will consider non degenerated polygons.

3. Pedal Triangles in Lorentzian Plane

In similar way to Euclidean geometry, we will define pedal triangles in Lorentzian plane.

Definition. 1 Let $T = T[A, B, C]$ be a triangle in Lorentzian plane, the pedal triangle T_P of a point P is the triangle whose vertices $A', B',$ and C' are the orthogonal projections of P on the lines that contains the sides AB, BC, and AC, respectively.

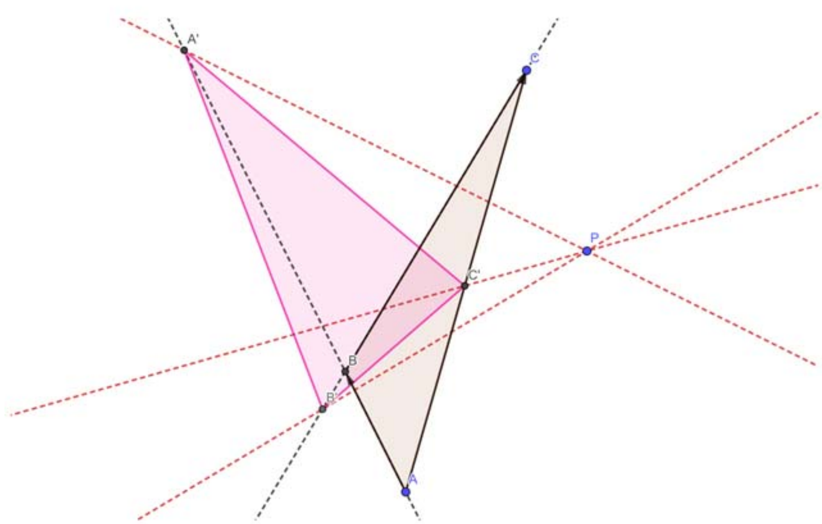

Figure 1. Pure timelike triangle $T[A, B, C]$ and its pedal triangle $T_P[A', B', C']$.

3.1. Pedal triangles of pure triangles

We will characterize the pedal pure triangle of a pure triangle in Lorentzian plane.

Theorem 2. *Let* $T = T[A, B, C]$ *be a triangle and* P *a point in Lorentzian plane*.

(a) *If T is pure timelike triangle then its pedal triangle* T_P *is*:

(i) *pure timelike if and only if* $\overrightarrow{BP}, \overrightarrow{AP},$ *and* $\overrightarrow{CP}$ *are spacelike vectors*.

(ii) *pure spacelike if and only if* $\overrightarrow{BP}, \overrightarrow{AP},$ *and* $\overrightarrow{CP}$ *are timelike vectors*.

(b) *If T is pure spacelike triangle then its pedal triangle* T_P *is*:

(i) *pure spacelike if and only if* $\overrightarrow{BP}, \overrightarrow{AP},$ *and* $\overrightarrow{CP}$ *are timelike vectors*.

(ii) *pure timelike if and only if* $\overrightarrow{BP}, \overrightarrow{AP},$ *and* $\overrightarrow{CP}$ *are spacelike vectors*.

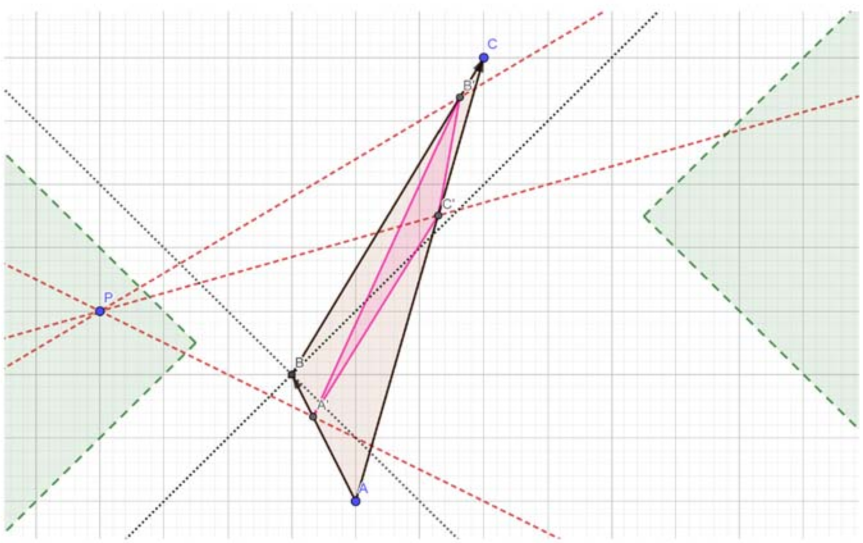

Figure 2. Set of points P for which T and T_P are pure timelike triangles.

Proof. (a) Without loss of generality, we consider the pure timelike triangle $T = T[A, B, C]$ with vertices $A = (a_1, -a_2)$, $B = (0, 0)$ and $C = (c_1, c_2)$, $a_1 > 0$, $a_2 > 0$, $c_1 > 0$, $c_2 > 0$.

We have $\overrightarrow{AB} = (-a_1, a_2)$, $\overrightarrow{BC} = (c_1, c_2)$, and $\overrightarrow{AC} = (c_1 - a_1, c_2 + a_2)$. Hence $\| \overrightarrow{AB}\|^2 = a_2^2 - a_1^2$, $\| \overrightarrow{AC}\|^2 = (c_2 + a_2)^2 - (c_1 - a_1)^2$, and $\| \overrightarrow{BC}\|^2 = c_2^2 - c_1^2$.

Let $L_A : y = -\frac{a_2}{a_1} x$ be the line which contains the side AB of T. In order to determine the orthogonal projection of P on the line L_A, A', we find the perpendicular line to L_A that passes through P, $L'_A : y = -\frac{a_1}{a_2} x + \frac{1}{a_2}\langle \overrightarrow{BA}, \overrightarrow{BP}\rangle$. Then,

$$A' = \left(-\frac{a_1}{\| \overrightarrow{AB}\|^2}\langle \overrightarrow{BA}, \overrightarrow{BP}\rangle, \frac{a_2}{\| \overrightarrow{AB}\|^2}\langle \overrightarrow{BA}, \overrightarrow{BP}\rangle \right) \tag{1}$$

is the intersection of L_A and L'_A.

Analogously, we find B' :

$$B' = \left(-\frac{c_1}{\| \overrightarrow{BC}\|^2}\langle \overrightarrow{BC}, \overrightarrow{BP}\rangle, -\frac{c_2}{\| \overrightarrow{BC}\|^2}\langle \overrightarrow{BC}, \overrightarrow{BP}\rangle \right), \tag{2}$$

and

$$C' = (c'_1, c'_2), \tag{3}$$

with

$$c'_1 = \frac{c_2 + a_2}{\| \overrightarrow{AC}\|^2}(a_1 c_2 + a_2 c_1) - \frac{c_1 - a_1}{\| \overrightarrow{AC}\|^2}\langle \overrightarrow{AC}, \overrightarrow{BP}\rangle,$$

and

$$c_2' = \frac{c_1 - a_1}{\| \overrightarrow{AC} \|^2}(a_1c_2 + a_2c_1) - \frac{c_2 + a_2}{\| \overrightarrow{AC} \|^2}\langle \overrightarrow{AC}, \overrightarrow{BP} \rangle.$$

Then, we have

$$\overrightarrow{A'B'} = (x_1', x_2'), \text{ with } x_1' = \left\langle -\frac{c_1}{\| \overrightarrow{BC} \|^2}\overrightarrow{BC} + \frac{a_1}{\| \overrightarrow{AB} \|^2}\overrightarrow{BA}, \overrightarrow{BP} \right\rangle,$$

$$x_2' = \left\langle -\frac{c_2}{\| \overrightarrow{BC} \|^2}\overrightarrow{BC} - \frac{a_2}{\| \overrightarrow{AB} \|^2}\overrightarrow{BA}, \overrightarrow{BP} \right\rangle;$$

$$\overrightarrow{B'C'} = (y_1', y_2'), \text{ with } y_1' = \frac{(c_2 + a_2)(a_1c_2 + a_2c_1)}{\| \overrightarrow{AC} \|^2} + \left\langle \frac{-(c_1 - a_1)}{\| \overrightarrow{AC} \|^2}\overrightarrow{AC} + \frac{c_1}{\| \overrightarrow{BC} \|^2}\overrightarrow{BC}, \overrightarrow{BP} \right\rangle,$$

$$y_2' = \frac{(c_1 - a_1)(a_1c_2 + a_2c_1)}{\| \overrightarrow{AC} \|^2} + \left\langle \frac{-(c_2 + a_2)}{\| \overrightarrow{AC} \|^2}\overrightarrow{AC} + \frac{c_2}{\| \overrightarrow{BC} \|^2}\overrightarrow{BC}, \overrightarrow{BP} \right\rangle;$$

$$\overrightarrow{A'C'} = (z_1', z_2'), \text{ with } z_1' = \frac{(c_2 + a_2)(a_1c_2 + a_2c_1)}{\| \overrightarrow{AC} \|^2} + \left\langle \frac{-(c_1 - a_1)}{\| \overrightarrow{AC} \|^2}\overrightarrow{AC} + \frac{a_1}{\| \overrightarrow{AB} \|^2}\overrightarrow{BA}, \overrightarrow{BP} \right\rangle,$$

$$z_2' = \frac{(c_1 - a_1)(a_1c_2 + a_2c_1)}{\| \overrightarrow{AC} \|^2} + \left\langle \frac{-(c_2 + a_2)}{\| \overrightarrow{AC} \|^2}\overrightarrow{AC} - \frac{a_2}{\| \overrightarrow{AB} \|^2}\overrightarrow{BA}, \overrightarrow{BP} \right\rangle.$$

Since

* $\langle \overrightarrow{A'B'}, \overrightarrow{A'B'} \rangle = \dfrac{(a_1c_2 + a_2c_1)^2}{\| \overrightarrow{BC} \|^2 \| \overrightarrow{BA} \|^2}(p_2^2 - p_1^2)$, then $\overrightarrow{A'B'}$ is timelike vector (resp., spacelike vector) if and only if $p_1^2 - p_2^2 > 0$, (resp., $p_1^2 - p_2^2 < 0$), is that

to say that $\overrightarrow{BP}$ is spacelike vector (resp., timelike vector).

$$* \quad \langle \overrightarrow{B'C'}, \overrightarrow{B'C'} \rangle = \frac{-[c_2(a_1 - c_1) - c_1(a_2 + c_2)]^2}{\| \overrightarrow{AC} \|^2 \| \overrightarrow{BC} \|^2} [(p_1 - c_1)^2 - (p_2 - c_2)^2],$$

then $\overrightarrow{B'C'}$ is timelike vector (resp., spacelike vector) if and only if $(p_1 - c_1)^2 - (p_2 - c_2)^2 > 0$ (resp., $(p_1 - c_1)^2 - (p_2 - c_2)^2 < 0$), is that to say that $\overrightarrow{CP}$ is spacelike vector (resp., timelike vector).

$$* \quad \langle \overrightarrow{A'C'}, \overrightarrow{A'C'} \rangle = \frac{-[a_1(c_2 + a_2) + a_2(c_1 - a_1)]^2}{\| \overrightarrow{AC} \|^2 \| \overrightarrow{AB} \|^2} [(p_1 - a_1)^2 - (p_2 + a_2)^2],$$

then $\overrightarrow{A'C'}$ is timelike vector (resp., spacelike vector) if and only if $(p_1 - a_1)^2 - (p_2 + a_2)^2 > 0$ (resp., $(p_1 - a_1)^2 - (p_2 + a_2)^2 < 0$), is that to say $\overrightarrow{AP}$ is spacelike vector (resp, timelike vector).

(b) The condition (b) is followed by symmetry from the condition (a). □

3.2. Pedal triangles of isosceles pure timelike triangles

We will show same properties of pedal triangles of isosceles pure timelike triangles.

Theorem 3. *Let* $T = T[A, B, C]$ *be an isosceles pure timelike triangle and* $\mathcal{C}^+$ *the Lorentzian circle in which T is inscribed.*

(i) *If P lies in* $\mathcal{C}^+$ *or* $\mathcal{C}^-$, *then the vertices* A', B', *and* C' *of the pedal triangle* T_P *are collinear points.*

(ii) *If* T_P *is pure timelike triangle, then* T_P *is isosceles triangle with middle vertex* C'.

When A', B', and C' are collinear points, they determine the so-called Simson line.

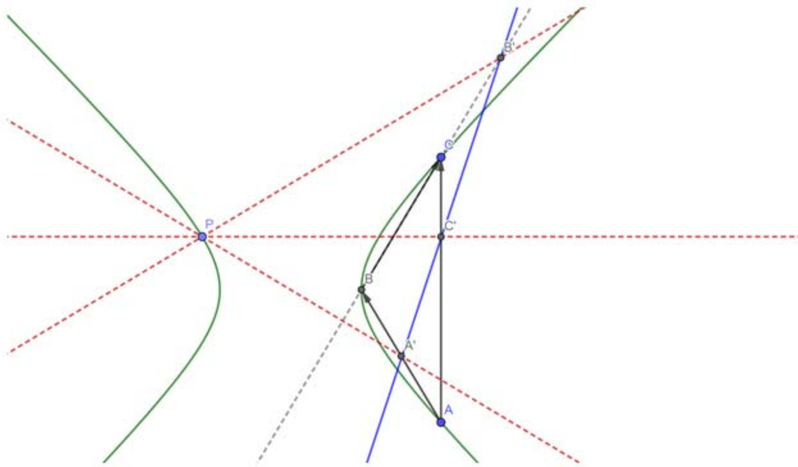

Figure 3. Simson line.

Proof. Without loss of generality, we suppose that $A = (a_1, -a_2)$, $B = (0, 0)$, and $C = (a_1, a_2)$, with $a_1 > 0$, $a_2 > 0$. Then by (1), (2) and (3), we have

$$A' = \left(-\frac{a_1}{\| \overrightarrow{AB} \|^2} \left\langle \overrightarrow{BA}, \overrightarrow{BP} \right\rangle, \frac{a_2}{\| \overrightarrow{AB} \|^2} \left\langle \overrightarrow{BA}, \overrightarrow{BP} \right\rangle \right), \tag{4}$$

$$B' = \left(-\frac{a_1}{\| \overrightarrow{BA} \|^2} \left\langle \overrightarrow{BC}, \overrightarrow{BP} \right\rangle, -\frac{a_2}{\| \overrightarrow{BA} \|^2} \left\langle \overrightarrow{BC}, \overrightarrow{BP} \right\rangle \right), \tag{5}$$

$$C' = (a_1, p_2). \tag{6}$$

(i) Since $T = T[A, B, C]$ is an isosceles pure timelike triangle inscribed in $\mathcal{C}^+$, $A, B,$ and C are points in the Lorentzian circle $\mathcal{C}^+$ with center $D = \left(\frac{-\| \overrightarrow{BA} \|^2}{2a_1}, 0 \right)$ and radius $\rho = \frac{\| \overrightarrow{BA} \|^2}{2a_1}$. In particular:

$$(a_1 + \rho)^2 - a_2^2 = \rho^2 \Rightarrow a_2^2 - a_1^2 = 2a_1\rho. \tag{7}$$

Let $P = (p_1, p_2)$ be a point in $\mathcal{C}^+$ or in its symmetric branch. Then,

$$(p_1 + \rho)^2 - p_2^2 = \rho^2 \Rightarrow p_2^2 = p_1^2 + 2p_1\rho. \tag{8}$$

(a) If $p_2 \neq 0$, the line L which contains A' and B' is

$$L : y = -\frac{p_1}{p_2}\left(x + \frac{a_1p_1 + a_2p_2}{2\rho}\right) + \frac{a_2}{a_1}\left(\frac{a_1p_1 + a_2p_2}{2\rho}\right),$$

and it is verified that $C' \in L$.

(b) If $p_2 = 0$, then $p_1 = -2\rho$ and

$$L : x = -\frac{a_1p_1}{2\rho}.$$

Thus $C' \in L$.

(ii) By computing $\overrightarrow{A'C'}$ and $\overrightarrow{C'B'}$, we verified that $\overrightarrow{A'C'} + \overrightarrow{C'B'} = \overrightarrow{A'B'}$ and C' is the middle vertex. □

If H is the orthocenter of T, then T_H is called the orthic triangle.

Theorem 4. *The orthic triangle* $T_H = [B', C', A']$ *of an isosceles pure timelike triangle* $T = T[A, B, C]$ *is a pure timelike triangle and satisfies*:

(i) $\|\overrightarrow{B'A'}\| = 2\|\overrightarrow{AB}\| \cosh \widehat{A} \cosh \widehat{B}$ *and* $\|\overrightarrow{B'C'}\| = \|\overrightarrow{C'A'}\| = \|\overrightarrow{AB}\| \cosh \widehat{A}$.

(ii) *If* ρ *and* ρ' *are the radius of the circles that circumscribe to* T *and* T_H, *respectively*, *then* $\rho' = \frac{1}{2}\rho$.

(iii) $\cosh \widehat{B'} = \cosh \widehat{A'} = \dfrac{1}{2\| \overrightarrow{AB} \| \cosh \widehat{A}}$ *and* $\cosh \widehat{C'} = \dfrac{\| \overrightarrow{AB} \|^4}{2\rho^4} - 1.$

Proof. Without loss of generality, we suppose that $A = (a_1, -a_2)$, $B = (0, 0)$, and $C = (a_1, a_2)$, $a_1 > 0$, $a_2 > 0$, are vertices of an isosceles pure timelike triangle $T[A, B, C]$. By computing $H = \left(\dfrac{a_1^2 + a_2^2}{a_1}, 0 \right)$ and by (4), (5) and (6),

$$A' = \left(-\frac{1}{2\rho}(a_1^2 + a_2^2), \frac{a_2}{2a_1\rho}(a_1^2 + a_2^2) \right), \tag{9}$$

$$B' = \left(-\frac{1}{2\rho}(a_1^2 + a_2^2), -\frac{a_2}{2a_1\rho}(a_1^2 + a_2^2) \right), \tag{10}$$

$$C' = (a_1, 0). \tag{11}$$

According to Theorem 2, since $\overrightarrow{HA}$, $\overrightarrow{HB}$, and $\overrightarrow{HC}$ are spacelike vectors, then T_H is a pure timelike triangle.

(i) Since $2\rho a_1 = \| \overrightarrow{AB} \|^2$, then

$$\| \overrightarrow{B'A'} \| = \frac{a_2}{a_1\rho}(a_1^2 + a_2^2) \Rightarrow \| \overrightarrow{B'A'} \| = 2\| \overrightarrow{AB} \| \cosh \widehat{A} \cosh \widehat{B}.$$

Also,

$$\| \overrightarrow{B'C'} \| = \| \overrightarrow{C'A'} \| = a_2 \Rightarrow \| \overrightarrow{B'C'} \| = \| \overrightarrow{AB} \| \cosh \widehat{A}.$$

(ii) By computing, $\rho' = \dfrac{\rho}{2}.$

(iii) By computing, $\widehat{B'} = \cosh \widehat{A'} = \dfrac{1}{2\| \overrightarrow{AB} \|} \cosh \widehat{A}.$

Since the angle at the middle vertex C' is equal to one half of the central angle for the chord $B'A'$, [1], then $\cosh \widehat{C'} = \left(\frac{2}{\rho}\right)^2 \left[-\left(\frac{\rho}{2}\right)^2 + \frac{\| \overrightarrow{AB} \|^4}{8\rho^2}\right] = \frac{\| \overrightarrow{AB} \|^4}{2\rho^4} - 1.$ □

If D is the circumcenter of T, then T_D is called the medial triangle.

Theorem 5. *The medial triangle* $T_D = T[A', C', B']$ *of an isosceles pure timelike triangle* $T = T[A, B, C]$ *is a pure timelike triangle and satisfies*:

(i) $\| \overrightarrow{A'B'} \| = \frac{\| \overrightarrow{AC} \|}{2}$ *and* $\| \overrightarrow{A'C'} \| = \| \overrightarrow{C'B'} \| = \frac{\| \overrightarrow{AB} \|}{2}$.

(ii) $\widehat{A'} = \widehat{A} = \widehat{C} = \widehat{B'}$ *and* $\widehat{C'} = \widehat{B}$.

(iii) *If* ρ *and* ρ'' *are the radius of the circles that circumscribe to T and* T_D, *respectively*, *then* $\rho'' = \frac{1}{2}\rho$.

Proof. Without loss of generality, we suppose that $A = (a_1, -a_2)$, $B = (0, 0)$, and $C = (a_1, a_2)$, $a_1 > 0$, $a_2 > 0$, are vertices of an isosceles pure timelike triangle $T = T[A, B, C]$.

The circumcenter of T is $D = \left(-\frac{\|\overrightarrow{BA}\|^2}{2a_1}, 0\right)$ and A', C', and B' are the midpoints of the sides AB, AC, and BC, respectively.

By applying Theorem 2, $T_D = T[A', C', B']$ is a pure timelike triangle.

The conditions (i), (ii) and (iii) are followed by computing. □

4. Pedal Polygons of pure non-null Polygons

Definition 6. Let $\boldsymbol{P} = \boldsymbol{P}[P_0, \ldots, P_m]$ be a polygon in the Lorentzian plane. The pedal polygon $\boldsymbol{P}_Q$ of a point Q is the polygon whose vertices $P_0', \ldots, P_m'$ are the orthogonal projections of Q on the lines that contains the sides $P_0P_1, \ldots, P_{m-1}P_m$ and P_0P_m, respectively.

Theorem 7. *Let $\boldsymbol{P} = \boldsymbol{P}[P_0, \ldots, P_m]$ be a polygon and Q a point in Lorentzian plane.*

(a) *If $\boldsymbol{P}$ is pure timelike polygon and $\overrightarrow{P_0Q}, \ldots, \overrightarrow{P_mQ}$ are spacelike vectors (timelike vectors, respectively), then the pedal polygon $\boldsymbol{P}_Q$ of Q is pure timelike (pure spacelike, respectively).*

(b) *If $\boldsymbol{P}$ is pure spacelike polygon and $\overrightarrow{P_0Q}, \ldots, \overrightarrow{P_mQ}$ are timelike vectors (spacelike vectors, respectively), then the pedal polygon $\boldsymbol{P}_Q$ of Q is pure spacelike (pure timelike, respectively).*

Proof. This result is obtained by applying Theorem 2 to each of triangles $T[P_i, P_{i+1}, P_{i+2}]$, $i = 0, 1, \cdots, m-2$. □

Theorem 8. *Let $\boldsymbol{P} = \boldsymbol{P}[P_0, \ldots, P_m]$ be a pure timelike polygon inscribed in a Lorentzian circle $\mathcal{C}$. If Q is a point of $\mathcal{C}$ and $\|\overrightarrow{P_0P_1}\| = \|\overrightarrow{P_1P_2}\|$, $\|\overrightarrow{P_2P_3}\| = \|\overrightarrow{P_0P_2}\|$, $\ldots$, $\|\overrightarrow{P_{m-1}P_m}\| = \|\overrightarrow{P_{m-2}P_m}\|$, then the orthogonal projections $P_0', \ldots, P_m'$ are collinear points.*

Proof. This result is obtained by applying Theorem 3 to each of triangles $T[P_i, P_{i+1}, P_{i+2}]$, $i = 0, 1, \cdots, m-2$. □

Theorem 9. *Let* $\overline{AB}$ *be a timelike segment. Then there exists a family of convex pure timelike trapezoids* $\{\boldsymbol{I}_n = \boldsymbol{P}[A, A_n, B_n, B]\}_{n\geq 1}$, *whose pedal polygons vertices are collinear points and such that* $\boldsymbol{I}_n$ *is contained inside* $\boldsymbol{I}_{n+1}$ *for all n.*

Proof. Let L be the perpendicular line to $\overleftrightarrow{AB}$ that passes through the midpoint M of $\overline{AB}$. If $P_0 \in L - \{M\}$, then the pure timelike triangle $T_0 = [A, P_0.B]$ is an isosceles triangle and there exists a Lorentzian circle $\mathcal{C}_1$ for which T_0 is inscribed in $\mathcal{C}_1$.

Let S_A, S_{P_0}, and S_B be the tangent lines to $\mathcal{C}_1$ at A, P_0, and B, respectively. If $\{A_1\} = S_A \cap S_{P_0}$, $\{P_1\} = S_A \cap S_B$ and $\{B_1\} = S_{P_0} \cap S_B$, then $\boldsymbol{I}_1 = \boldsymbol{P}[A, A_1, B_1, B]$ is a convex pure timelike trapezoid and the orthogonal projections of P_1 on the lines containing the sides of $\boldsymbol{I}_1$ are points in L.

Analogously, we determine a succession of points $\{P_n\}_{n\geq 2}$ and construct a succession of convex pure timelike trapezoids $\{\boldsymbol{I}_n = \boldsymbol{P}[A, A_n, B_n, B]\}_{n\geq 2}$ such that the orthogonal projections of P_n on the lines containing the sides of $\boldsymbol{I}_n$ are points to be in L. Let us note that $\boldsymbol{I}_n$ is contained inside $\boldsymbol{I}_{n+1}$ for $n \geq 1$. □

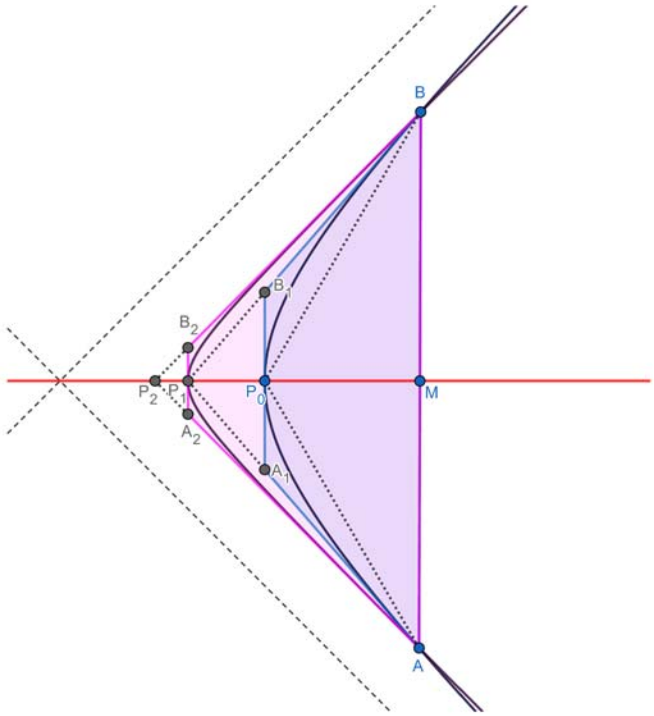

Figure 4. Trapezoids I_1 and I_2.

Acknowledgement

This work is supported by the Universidad Nacional del Sur (Grant no. PGI 24/ZL12).

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