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ON HEAT-FLOW DIRECTION INERTIA EFFECT

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Abstract

There is proposed a theoretical description of inertia effect of the heat-flow direction, which for a finite period of time τ maintains the same after the mean temperature-gradient in the heating system has been instantly reversed. Analyzing solutions of the heat equation with appropriate initial and boundary conditions the inertia time is shown to be proportional of the squared heat-conductor length L, that is its size in the temperature-gradient direction, and inversely proportional of the material thermal diffusivity $\alpha : \tau \sim L^2 / \alpha$. Numerical estimate made for the heat-conductor in form of a steel bar is in satisfactory agreement with the experimental results.

Introduction

Metals, which are good heat-conductors, are often subjected to thermal treatments involving stages of both heating and cooling. The temperature-field

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distribution in a sample produced during the thermal treatment largely determines the required functional properties of the material. In some cases, such as hardening of steels, the sign of the mean temperature-gradient in the thermal system changes over time.

In the hardening experiments on metal bars, one can frequently see a phenomenon, which we can call the heat-flow direction inertia in a sign-reversible temperature-gradient [1]. The essence of the effect is as follows. Let one end of the bar of length L made from a heat-conductive material (e.g., metal) is in thermal contact with a medium (e.g., air) at the temperature T_0 less than the temperature T_L' of another medium (e.g., boiling water), in which other end of the bar is placed, $T_0 < T_L'$. If now second end of the bar is quickly cooled by putting it down in a medium (e.g., cold water) with a temperature T_L'' lower than $T_0, T_0 > T_L''$, then within a finite period τ heat will be released at the first end, despite the fact that the sign of the mean temperature-gradient in the heating system has been instantly reversed.

In this paper, we propose a quantitative description of the heat-flow direction inertia effect. Obviously, it is associated with non-stationary thermal process occurring in a heat-conductor. For this reason, our consideration will be based on analysis of solutions of the non-stationary heat equation for appropriate combinations of initial and boundary conditions.

2. Model

Let us formulate the basic assumptions simplifying the model of unsteady heat-flow, which will help clearly imagine the mechanism of inertia of the heat-flow direction.

The first assumption is the 1D heat-flow. In other words, we suppose that the cross-section of the heat-conductor is infinite or its lateral surfaces are thermally insulated. Of course, in real conditions lateral surfaces of any heat-conductor are traversed by certain heat-fluxes, the magnitude and direction of which depend on the heat-conductor geometry, as well as the quality of the thermal insulation. But, in a bar-shaped sample made from a good heat-conductive material, main stream will still be directed along its axis. Attempt to account for lateral heat-flows may well complicate the mathematical part of the problem hiding physical mechanism of the phenomenon. In 1D approximation, when the heat-flow is directed along the Ox axis the temperature T of the heat-conducting medium has to be a function of coordinate x and time $t : T = T(x, t)$. The requested temperature-field will be determined by the 1D heat equation

$$\rho(T(x,\, t))\, c(T(x,\, t)) \frac{\partial T(x,\, t)}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T(x,\, t)) \frac{\partial T(x,\, t)}{\partial x}\right). \tag{1}$$

This is nothing but the heat balance equation written for a heat-conductor with no internal heat-sources. Here $\lambda = \lambda(T)$, $\rho = \rho(T)$ and $c = c(T)$ are the thermal conductivity, density and specific heat of the heat-conductor material, respectively. In general, these parameters depend on the temperature and, therefore, coordinate and time.

The second assumption is neglecting the mentioned temperature-dependencies. Otherwise, the basic equation (1) is nonlinear, what makes very difficult to obtain its analytical solutions or even requires numerical solutions. Fortunately, in most of the cases for good heat-conductors (metals) these temperature-dependences are weak except the region of low temperatures. Thus, the heat equation takes the linear form

$$\frac{\partial T(x,\, t)}{\partial t} = \alpha \frac{\partial^2 T(x,\, t)}{\partial x^2}, \tag{2}$$

where the material thermal diffusivity α is introduced by the ratio

$$\alpha = \frac{\lambda}{\rho c}. \tag{3}$$

The parameter α can be considered as a given constant. The initial condition for equation (2) is to specify the initial (at the moment $t = 0$) temperature-distribution $T(x,\, 0)$ in the heat-conductor. It must be supplemented by two boundary conditions at heat-conductor ends $x = 0$ and $x = L$.

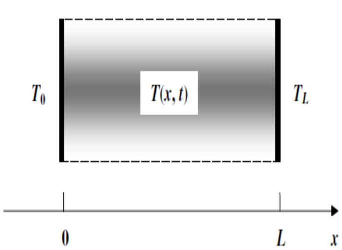

Figure 1. Schematic representation of bar-shaped heat-conductor.

Let L denotes the length of the bar-shaped heat-conductor (Figure 1), ends of which are located at the points $x = 0$ and $x = L$ and freely exchange heat with adjacent media at temperatures T_0 and T_L, respectively. We introduce the heat-transfer coefficient Λ, the same for both ends, i.e., for simplicity we assume that both thermal contacts to the heat-conductor are identical. This is the third assumption of the model greatly simplifying calculations, as well as results analysis. In fact, thermal resistances of these two thermal contacts should differ from each other because they are placed in different media at different temperatures. However, the thermal resistance of a high-quality thermal contact should be small if compared with that of the heat-conductor itself. As we shall see below, in this case the parameter Λ characterizing contacts falls out of the final relations. The boundary conditions can be written as the requirement of the heat-flux continuity at the heat-conductor ends

$$-\lambda \frac{\partial T(0, t)}{\partial x} = -\Lambda(T(0, t) - T_0), \tag{4}$$

$$-\lambda \frac{\partial T(L, t)}{\partial x} = -\Lambda(T_L - T(L, t)). \tag{5}$$

Here we have introduced a length parameter

$$l = \frac{\lambda}{\Lambda} \tag{6}$$

characterizing the heat-transfer at the heat-conductor ends. It is a virtual thickness of the boundary layer between the heat-conductor and adjacent media. As a result, we obtain the so-called mixed boundary conditions

$$-\frac{\partial T(0, t)}{\partial x} + \frac{T(0, t)}{l} = \frac{T_0}{l}, \tag{7}$$

$$\frac{\partial T(L, t)}{\partial x} + \frac{T(L, t)}{l} = \frac{T_L}{l}. \tag{8}$$

They are heterogeneous, because in addition to the unknown function $T(x, t)$ and its partial derivatives with respect to the coordinate $\partial T(x, t)/\partial x$ in their left parts they contain non-zero constants also in their right parts.

The solution procedure of a linear 1D heat equation with homogeneous boundary conditions is well known. For this reason, it is advisable to transform the function $T(x, t)$ to the new unknown function $\Theta(x, t)$ determined by the heat equation with homogeneous boundary conditions. This can be done adding the linear term

$$T(x, t) = T_0 + \frac{(T_L - T_0)(x + l)}{L + 2l} + \Theta(x, t). \tag{9}$$

Applying the substitution (9) in boundary conditions (7) and (8), we can see that they are transformed to the homogeneous forms

$$-\frac{\partial\Theta(0, t)}{\partial x} + \frac{\Theta(0, t)}{l} = 0, \tag{10}$$

$$\frac{\partial\Theta(L, t)}{\partial x} + \frac{\Theta(L, t)}{l} = 0 \tag{11}$$

while the initial condition takes the following form:

$$\Theta(x, 0) = T(x, 0) - T_0 - \frac{(T_L - T_0)(x + l)}{L + 2l}. \tag{12}$$

As for the new unknown function, it will satisfy the equation of same form

$$\frac{\partial\Theta(x, t)}{\partial t} = \alpha\frac{\partial^2\Theta(x, t)}{\partial x^2} \tag{13}$$

since the linear heat equation contains only the first-order partial derivative with respect to time and second-order partial derivative with respect coordinate: both of these operations vanish the term distinguishing $T(x, t)$ from $\Theta(x, t)$

The solution of the equation (13) with initial condition (12) and the homogeneous boundary conditions (10) and (11) is known (see, e.g., [2]). It is given by the following infinite series:

$$\Theta(x, t) = \sum_{m=1}^{m=\infty} c_m \varphi(\beta_m, x) \exp\left(-\frac{\alpha\beta_m^2 t}{L^2}\right). \tag{14}$$

Here

$$\varphi(\beta_m, x) = \beta_m \cos\frac{\beta_m x}{L} + \frac{L}{l}\sin\frac{\beta_m x}{L} \tag{15}$$

form a complete set of orthogonal functions containing parameters β_m, the positive roots of the transcendental equation

$$\cos\beta_m = 1 - \frac{2}{1 + \beta_m^2 l^2 / L^2}. \tag{16}$$

Parameters β_m are numbered in ascending order of their numerical values. Parameters c_m denote coefficients of this series expansion

$$c_m = \frac{\int_0^L d\xi\, \varphi(\beta_m, \xi)\Theta(\xi, 0)}{\int_0^L d\xi \varphi^2(\beta_m, \xi)} = \frac{2}{\beta_m^2 + L^2 / l^2 + 2L/l}$$

$$\times \left(\int_0^L d\xi \left(\beta_m \cos\frac{\beta_m \xi}{L} + \frac{L}{l}\sin\frac{\beta_m \xi}{L} \right) \frac{T(\xi, 0) - T_0}{L} - \frac{(T_L - T_0)L}{\beta_m l} \right). \tag{17}$$

Thus, the solution of the original problem is

$$T(x, t) = T_0 + \frac{(T_L - T_0)(x + l)}{L + 2l} + \sum_{m=1}^{m=\infty} \frac{2}{\beta_m^2 + L^2 / l^2 + 2\, L/l}$$

$$\times \left(\int_0^L d\xi \left(\beta_m \cos\frac{\beta_m \xi}{L} + \frac{L}{l}\sin\frac{\beta_m \xi}{L} \right) \frac{T(\xi, 0) - T_0}{L} - \frac{(T_L - T_0)L}{\beta_m l} \right)$$

$$\times \left(\beta_m \cos\frac{\beta_m x}{L} + \frac{L}{l}\sin\frac{\beta_m x}{L} \right) \exp\left(-\frac{\alpha\beta_m^2 t}{L^2} \right). \tag{18}$$

Now, to obtain the final expression one requires computation the integral containing $T(x, 0)$ function and determining the β_m values for the heat-conductor under the consideration.

Terms standing separately from the series describe the monotone, namely, linear variations in the static temperature-field when moving from point $x = 0$ to point $x = L$. Series describes oscillating coordinate-dependence, which however vanishes over the time. Therefore, after a sufficiently long period, $t >> L^2 / \alpha\beta_m^2$, in the heat-conductor a steady temperature-field distribution will be achieved.

3. First Stage: Initial Heating

In the first stage of the heat-flow process, temperature T_L' of the medium adjacent to the $x = L$ end of the heat-conductor is higher than that of the $x = 0$ end

$T_0 : T_0 < T'_L$. Let at $t = 0$ the heat-conductor is in thermal equilibrium with the medium at lower temperature. Then, the initial condition takes the form: $T(x, 0) = T_0$. This assumption greatly simplifies solving the heat equation

$$T(x, t) = T_0 + \frac{(T'_L - T_0)(x + l)}{L + 2l}$$

$$- \frac{2L(T'_L - T_0)}{l} \sum_{m=1}^{m=\infty} \frac{\cos \frac{\beta_m x}{L} + \frac{L}{\beta_m l} \sin \frac{\beta_m x}{L}}{\beta_m^2 + L^2 / l^2 + 2\, L/l} \exp\left(- \frac{\alpha \beta_m^2 t}{L^2} \right). \tag{19}$$

After a sufficiently long period, the non-monotonic terms of this solution vanish and the temperature-distribution in the heat-conductor achieves its steady-state corresponding to the linear increase in temperature with increasing coordinate

$$T(x, \infty) = T_0 + \frac{(T'_L - T_0)(x + l)}{L + 2l}. \tag{20}$$

The temperatures achieved on the heat-conductor ends, respectively, are

$$T(0, \infty) = T_0 + \frac{T'_L - T_0}{2 + L/l} < \frac{T_0 + T'_L}{2}, \tag{21}$$

$$T(L, \infty) = T_1 - \frac{T'_L - T_0}{2 + L/l} > \frac{T_0 + T'_L}{2}. \tag{22}$$

Thus, at the point $x = 0$ temperature will be higher than that of the adjacent media T_0, but mandatory lower the average temperature $(T_0 + T'_L)/2$ of two media adjacent the heat-conductor ends. As for temperature at the point $x = L$, it will be lower than that of the adjacent media T'_L, but mandatory higher the average temperature $(T_0 + T'_L)/2$ of two media. Consequently, $T_0 < T(0, \infty) < (T_0 + T'_L)/2 < T(L, \infty) < T'_L$.

The steady heat-flux in the heat-conductor equals to

$$- \lambda \frac{\partial T(x, \infty)}{\partial x} = - \frac{\lambda (T'_L - T_0)}{L + 2l}. \tag{23}$$

The minus sign indicates that the heat-flow is opposite to the Ox axis direction, i.e., heat releases at the $x = 0$ end.

4. Second Stage: Instantaneous Cooling

In the second stage of the heat-flow process, temperature T_L'' of the medium adjacent to the $x = L$ end of the heat-conductor becomes lower than that of the $x = 0$ end $T_0 : T_0 > T_L''$. Initial moment of this stage, i.e., moment of the instantaneous cooling the bar end, should be counted after the steady heat-flow is achieved in the first stage. Therefore, the initial condition is obtained from equation (20) replacing $t = \infty \to t = 0$:

$$T(x, 0) = T_0 + \frac{(T_L' - T_0)(x + l)}{L + 2l}. \tag{24}$$

It leads to the following temperature-field distribution in the heat-conductor:

$$T(x, t) = T_0 - \frac{(T_0 - T_L'')(x + l)}{L + 2l}$$
$$+ \frac{2L(T_L' - T_L'')}{l} \sum_{m=1}^{m=\infty} \frac{\cos \frac{\beta_m x}{L} + \frac{L}{\beta_m l} \sin \frac{\beta_m x}{L}}{\beta_m^2 + L^2 / l^2 + 2\, 2L/l} \exp\left(- \frac{\alpha \beta_m^2 t}{L^2} \right). \tag{25}$$

This implies that the heat-flux at the point $x = 0$ at certain moment of the time $t = \tau$ will be

$$- \lambda \frac{\partial T(0, \tau)}{\partial x} = \frac{\lambda(T_0 - T_L'')}{L + 2l} - \frac{2L\lambda(T_L' - T_L'')}{l^2} \sum_{m=1}^{m=\infty} \frac{\exp\left(\frac{- \alpha \beta_m^2 \tau}{L^2} \right)}{\beta_m^2 + L^2 / l^2 + 2\, L/l}. \tag{26}$$

Here are two components positive and negative. Thanks to the latter in the initial period the total heat-flux can remain negative.

It is natural to assume that $L / l >> 1$, i.e., neglect the thermal resistance of a thermal contact to the heat-conductor in comparison with that of the heat conductor itself. In addition, we can note that terms of the series vanish very rapidly with increasing index m. Therefore, this series can be approximated by its first term. Further, note that according to the equation (16), in these conditions $\beta_1 \approx \pi$. Then, for a period

$$\tau \approx \frac{L^2}{\pi^2 \alpha} \ln 2\left(1 + \frac{T_L' - T_0}{T_0 - T_L''} \right) \tag{27}$$

the ratio of the output and input heat-fluxes at the $x = 0$ end will be greater than 1.

Since $T_0 < T_L'$ and $T_0 > T_L''$, the argument of the logarithmic function in the expression (27) exceeds 1 and, consequently, this estimate is always meaningful.

5. Conclusion

Thus, the magnitude of the heat-flow direction inertia effect is determined by the thermal characteristics of the heat-conductor material, its geometry, as well as the conditions of heat exchange with the environment. Namely, the inertia time is proportional of the squared heat-conductor length L, that is its size in the temperature-gradient direction, and inversely proportional of the material thermal diffusivity $\alpha : \tau \sim L^2 / \alpha$. Its dependence on temperature differences between adjacent media is relatively weak, logarithmic. It is natural that as the average temperature-gradient at stages of heating and cooling increases the inertia time, respectively, increases or decreases.

Make numerical estimate for the carbon steel bar of length $L \approx 10\,\text{cm}$ with the room temperature thermal diffusivity of $\alpha \approx 0.14\,\text{cm}^2/\text{s}$. Let the ambient temperature is equal to room temperature $T_0 \approx 298\,\text{K}$ and temperatures of adjacent media on the steps of heating and cooling equal, respectively, temperatures of water boiling $T_L' \approx 373\,\text{K}$ and freezing $T_L'' \approx 273\,\text{K}$. Using the formula (27), we get $\tau \approx 150\,\text{s}$, which is in agreement with the order of magnitude of thermal inertia periods observed experimentally in such heat-conductors.

If the above analyzed phenomenon is called the direct effect, one can easily imagine the reverse effect of the heat-flow direction inertia, when the initially cooled heat-conductor end is heated instantly, but for a finite period the cold end still absorbs the heat. Obviously, both the direct and reverse effects of the heat-flow direction inertia occur regularly in such technological processes as surface hardening, hot rolling, tubes broaching, etc. For this reason, it seems appropriate that these effects be accounted in elaboration of heat treatment technologies for metals and alloys.

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