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ON AN ELEMENTARY APPROXIMATE CONSTRUCTION OF THE REGULAR NONAGON WITH RULER AND COMPASS

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Abstract

In this note, we present a simple elementary approximate construction of the regular nonagon with ruler and compass. The underlying idea goes back to my paper "On an elementary approximate construction of the regular heptagon with ruler and compass" [1].

1. Introduction

Approximate constructions of regular polygons date back to ancient times, cf. Scriba and Schreiber [2] or Johnson and Pimpinelli [3]. Exact constructions of regular polygons with n vertices with ruler and compass, however, are, according to Gauss, only possible if $n = 2^{(m-1)} \cdot p_1 \cdot \ \ldots \ \cdot p_k$,

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where $m, k \in \mathbb{N}$ and the p_i are pairwise different prime numbers of the form $p_i = 2^j + 1$ with integer $j \in \mathbb{N}$, cf. Scriba and Schreiber [2], p. 425. Generally the vertices of such polygons in the complex plane are solutions of the circle division equation

$$(x-1)\cdot\sum_{k=0}^{n-1} x^k = x^n - 1 = 0.$$

In the previous paper "On an elementary approximate construction of the regular heptagon with ruler and compass" (Pfeifer (2024) [1]) it was shown that an approximate solution of this equation for $n = 7$ is given by

$$x_0 = -\frac{2}{9} + \frac{i}{9}\sqrt{77}$$

with the error estimate

$$x_0^7 = \left(-\frac{2}{9} + \frac{i}{9}\sqrt{77}\right)^7 = \frac{4{,}782{,}958}{4{,}782{,}969} - \frac{1{,}169}{4{,}782{,}969}\sqrt{77}i$$

$$= 0.999997\ldots - 0.002144\ldots j.$$

In the case of $n = 9$ an approximate solution is given by

$$x_0 = \frac{36}{47} + \frac{1}{47}\sqrt{913}i$$

(first counterclockwise vertex) with the error estimate

$$x_0^9 = \left(\frac{36}{47} + \frac{1}{47}\sqrt{913}i\right)^9 = \frac{1{,}119{,}129{,}642{,}999{,}108}{1{,}119{,}130{,}473{,}102{,}767}$$

$$+ \frac{45{,}111{,}373{,}825}{1{,}119{,}130{,}473{,}102{,}767}\sqrt{913}i$$

$$= 0.999999258\ldots + 0.001217981\ldots i.$$

The subsequent figure shows the geometric construction:

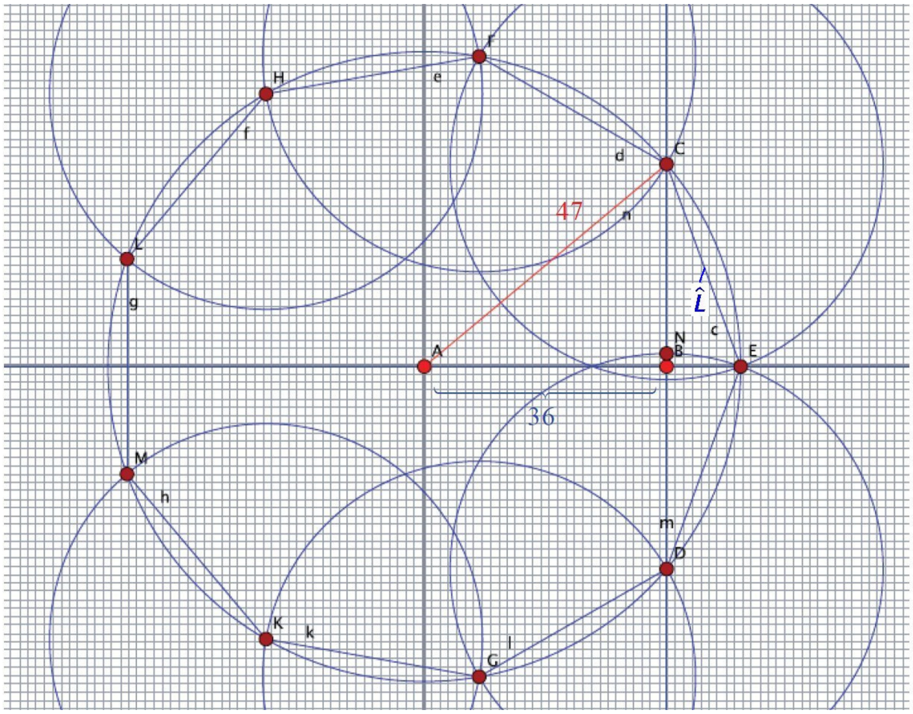

Figure 1.

The length $\hat{L}$ of the first counterclockwise chord hence is given by $\hat{L} = \frac{\sqrt{1034}}{47} = 0.684167...$ while the exact chord length L is given by $L = 2\sin\left(\frac{\pi}{9}\right) = 0.684040...$. The relative error hence amounts to 0.0859...%.

For a judgement of the goodness of approximation imagine that the underlying circle has a radius of 47 m, then the difference between the true and the approximate chord length would be about 6 mm!

A further significant improvement can be achieved by putting $z = x + \frac{1}{x}$ so that the second factor in the above circle division becomes

$$z^4 + z^3 - 2z^2 - 2z + 1 = (z+1)\cdot(z^3 - 3z + 1).$$

Note that $z_0 := \frac{72}{47}$ is a very good solution to the equation $f(z) := z^3 - 3z + 1 = 0$ with $z_0^3 - 3z_0 + 1 = -0.0007031...$ being even the best rational approximation with a denominator between 1 and 100! If we insert $z_1 := z_0 + h$ for z in the last equation and neglect the term h^3 (cf. Pfeifer (2024) [1]), the resulting quadratic equation gives the solutions

$$h = -\frac{2{,}975}{6{,}768} \pm \frac{\sqrt{8{,}857{,}633}}{6{,}768} \text{ or } z_1 = z_0 - \frac{2{,}975}{6{,}768} \pm \frac{\sqrt{8{,}857{,}633}}{6{,}768}.$$

With the choice $z_1 = z_0 - \frac{2{,}975}{6{,}768} + \frac{\sqrt{8{,}857{,}633}}{6{,}768}$ we obtain $f(z_1) = 0.5272\cdot 10^{-11}$. Solving finally $z_1 = x_1 + \frac{1}{x_1}$ for x_1, we get

$$\begin{aligned} x_1 &= \frac{7{,}393}{13{,}536} + \frac{\sqrt{8{,}857{,}633}}{13{,}536} \\ &\quad \pm \frac{i}{13{,}536}\sqrt{119{,}709{,}214 - 14{,}786\sqrt{8{,}857{,}633}} \\ &= 0.76604444... - 0.64278760...i \end{aligned}$$

with $x_1^9 = 1.00000000... - 0.91236215...\cdot 10^{-11} i$.

Note that the prime number decomposition of $8{,}857{,}633$ is given by

$$8{,}857{,}633 = 941 \cdot 9{,}413,$$

which allows for an improved construction with ruler and compass on the basis of the Pythagorean height theorem. The subsequent figure shows the steps of construction:

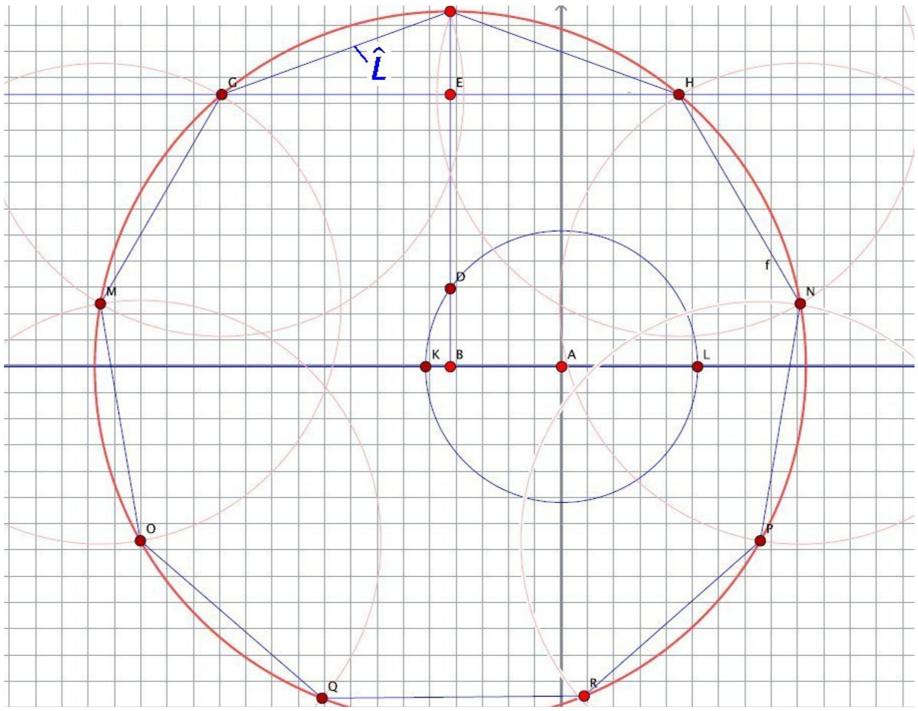

Figure 2.

Here the lengths of the individual line segments are given by

$$\overline{\text{AL}} = 5177 = \frac{941 + 9413}{2}, \quad \overline{\text{BL}} = 9413, \quad \overline{\text{KB}} = 941,$$

$$\overline{\text{BD}} = \sqrt{941 \cdot 9413}, \quad \overline{\text{BE}} = \overline{\text{BD}} + 7393.$$

The radius of the relevant circle around B is given by 13,536 graphical units.

The resulting chord length $\hat{L}$ for the approximate nonagon is hence given by $\hat{L} = \sqrt{(1 - a)^2 + 1 - a^2} = 0.684040286650...$ where $a = \frac{7{,}393}{13{,}536} + \frac{\sqrt{8{,}857{,}633}}{13{,}536}$ while the exact chord length L is given by $L =$

$2\sin\left(\frac{\pi}{9}\right) = 0.684040286651\ldots$. The relative error thus amounts to only $0.13924 \cdot 10^{-9}\%$.

For a judgement of the goodness of approximation imagine that the underlying circle has a radius of 13,536 km, then the difference between the true and the approximate chord length would be less than 0.02 mm!

References

[1] D. Pfeifer, On an elementary approximate construction of the regular heptagon with ruler and compass, Fundamental J. Math. Math. Sci. 18(2) (2024), 87-93.

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[3] A. Johnson and A. Pimpinelli, Pegs and Ropes: Geometry at Stonehenge, Nat. Prec. (2008), https://doi.org/10.1038/npre.2008.2153.1