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# ON AN ELEMENTARY APPROXIMATE CONSTRUCTION OF THE REGULAR HEPTAGON WITH RULER AND COMPASS

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**Abstract**

In this note, we present a simple elementary approximate construction of the regular heptagon with ruler and compass. The original idea goes back to calculations which I have made during school time in 1967 at the age of 14 years.

## 1. Introduction

Approximate constructions of regular polygons date back to ancient times, cf. Scriba and Schreiber [6] or Johnson and Pimpinelli [5]. Exact constructions of regular polygons with $n$ vertices with ruler and compass, however, are, according to Gauss, only possible if $n = 2^{(m-1)} \cdot p_1 \cdot \ \ldots \ \cdot p_k$,

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where $m, k \in \mathbb{N}$ and the $p_i$ are pairwise different prime numbers of the form $p_i = 2^j + 1$ with integer $j \in \mathbb{N}$, cf. Scriba and Schreiber [6], p. 405. Generally the vertices of such polygons in the complex plane are solutions of the circle division equation

$$(x-1)\cdot\sum_{k=0}^{n-1} x^k = x^n - 1 = 0.$$

In the case of a regular heptagon the corresponding equation

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

can be transformed into a more simple cubic equation

$$z^3 + z^2 - 2z - 1 = 0$$

after division by $x^3$ and by the substitution $z = x + \frac{1}{x}$ (cf. Geretschlaeger [2]). With a further substitution $z = y - \frac{1}{3}$ this leads to

$$f(y) := y^3 - \frac{7}{3}y - \frac{7}{27} = 0,$$

which corresponds to the so called *Casus irreducibilis* for cubic equations.

A first simple approximative solution can be obtained by a cancellation of the term $y^3$ giving $-\frac{7}{3}y - \frac{7}{27} = 0$ with the solution $y_0 = -\frac{1}{9}$ and $f(y_0) = -\frac{1}{729} = -0.001371\ldots$, i.e., $z = -\frac{4}{9}$, which leads to the two complex vertices

$$x_{1,2} = -\frac{2}{9} \pm \frac{i}{9}\sqrt{77}.$$

The accuracy of this solution can be seen here:

$$x_{1,2}^7 = \left(-\frac{2}{9} \pm \frac{i}{9}\sqrt{77}\right)^7 = \frac{4{,}782{,}958}{4{,}782{,}969} \mp \frac{1.169}{4{,}782{,}969}\sqrt{77}i$$

$$= 0.999997... \mp 0.002144...j.$$

The "first" counterclockwise vertex $E$ can be constructed by angle bisection. For the corresponding angle $\alpha$ we have $2\cos^2(\alpha) - 1 = \cos(2\alpha)$ and hence

$$\cos(\alpha) = \sqrt{\frac{1+\cos(2\alpha)}{2}} = \sqrt{\frac{1-\frac{2}{9}}{2}} = \frac{1}{6}\sqrt{14}$$

and $\sin(\alpha) = \sqrt{1-\cos^2(\alpha)} = \frac{1}{6}\sqrt{22}$. The length $\hat{L}$ of the first counterclockwise chord hence is given by

$$\hat{L} = \sqrt{\sin^2(\alpha) + (1-\cos(\alpha))^2} = \frac{1}{3}\sqrt{18-3\sqrt{14}} = 0.867629...$$

while the exact value is $L = 2\sin\left(\frac{\pi}{7}\right) = 0.867767...$ . The relative error thus amounts to $-0.015905...\%$.

A further significant improvement can be achieved by putting $y_0 = -\frac{1}{9} + h$ with $f(y_0) = -\frac{1}{729} - \frac{62}{27}h - \frac{1}{3}h^2 + h^3$. Neglecting the term $h^3$ and solving correspondingly $-\frac{1}{729} - \frac{62}{27}h - \frac{1}{3}h^2 = 0$, we obtain $h = -\frac{31}{9} + \frac{1}{27}\sqrt{8646}$ as the only admissible solution or $y_0 = -\frac{32}{9} + \frac{1}{27}\sqrt{8646}$ with $f(y_0) = -0.213229...\cdot 10^{-9}$ and hence

$$x_{1,2} = -\frac{35}{18} + \frac{1}{54}\sqrt{8646} \pm \frac{1}{54}\sqrt{210\sqrt{8646} - 16755}i$$

with $x_{1,2}^7 = 0.9999999993 - 0.6710439934 \cdot 10^{-8} i.$

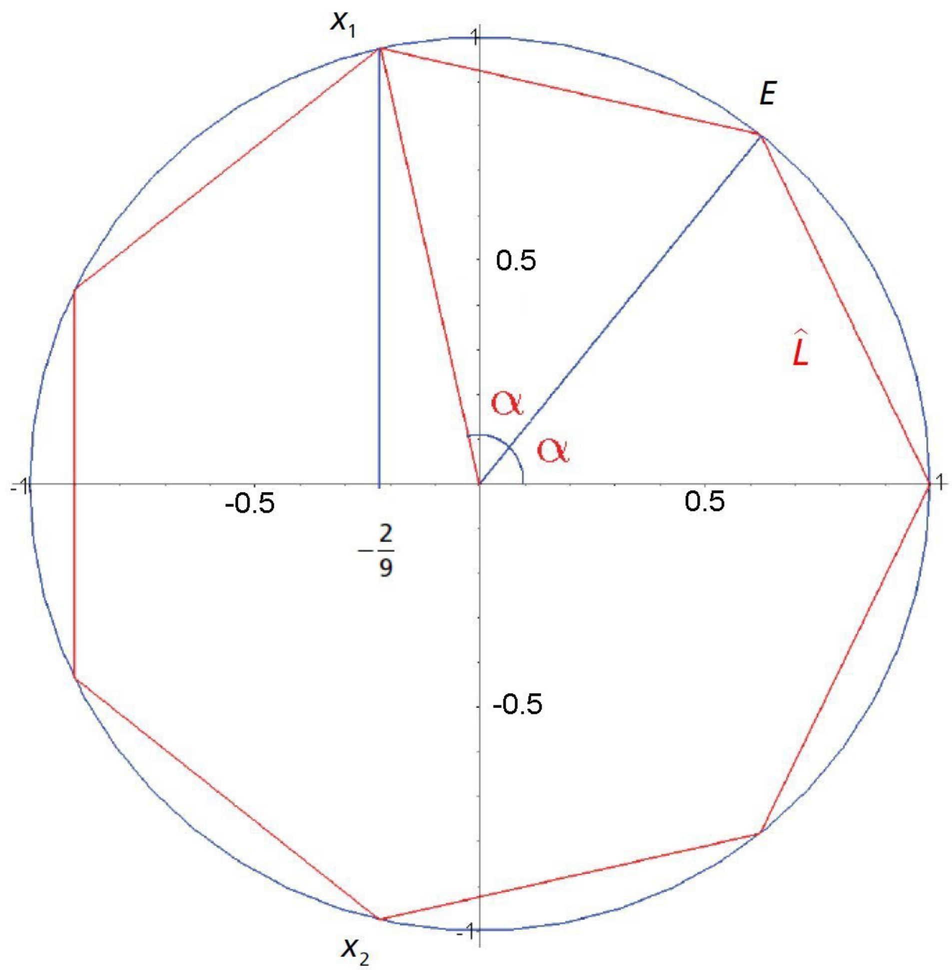


**Figure 1.**

The new counterclockwise first chord length now is

$$\hat{L} = \sqrt{\sin^2(\alpha) + (1 - \cos(\alpha))^2} = \frac{1}{3}\sqrt{18 - \sqrt{3\sqrt{8646} - 153}}$$

$$= 0.867767478213\ldots$$

in comparison with

$$L = 2\sin\left(\frac{\pi}{7}\right) = 0.867767478235...$$

with a relative error of only $-0.247268...\cdot 10^{-8}\%$. Observe that the corresponding approximate vertex $x_1$ can still be constructed with ruler and compass since its real part is given by

$$\Re(x_1) = -\frac{35}{18} + \frac{1}{54}\sqrt{8646} = -0.22252093390$$

$$\approx -0.22252093395 = \cos\left(\frac{4\pi}{7}\right)$$

which is the correct value. The following figure shows the corresponding construction.

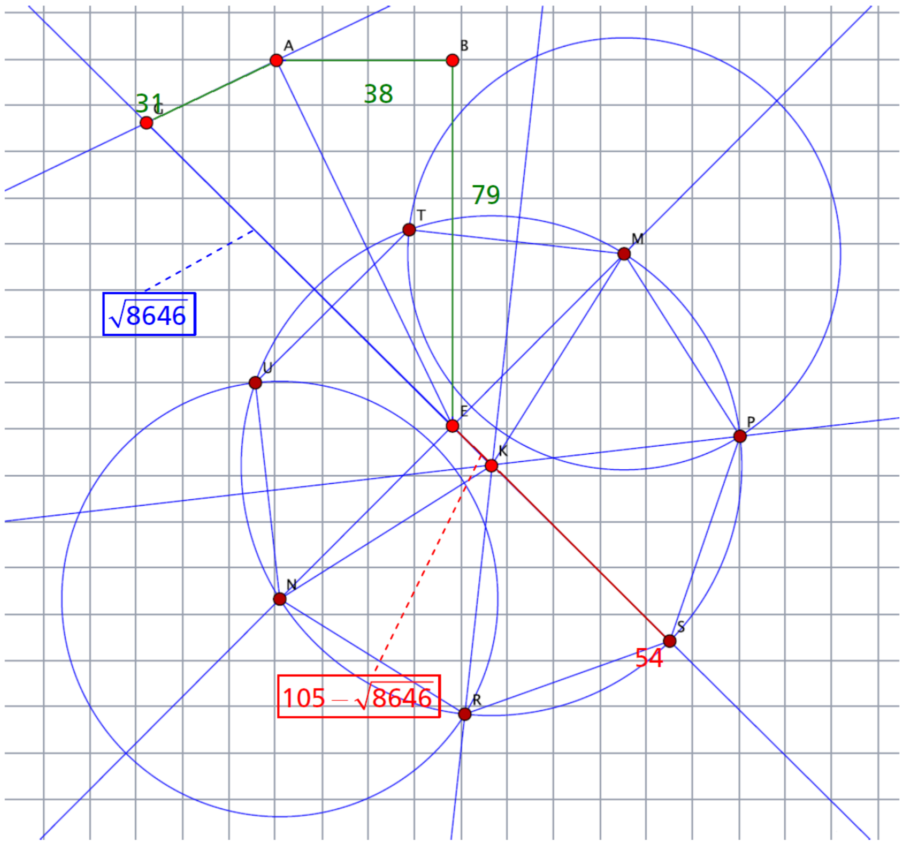


**Figure 2.**

Please observe that there are several possibilities to represent the number 8646 as sum of three squares, e.g.,

$$8646 = 31^2 + 38^2 + 79^2 = 31^2 + 31^2 + 82^2 = 14^2 + 23^2 + 89^2$$

$$= 14^2 + 35^2 + 85^2 = 5^2 + 61^2 + 70^2.$$

For a judgment of the goodness of approximation imagine that the underlying circle has a radius of 54,000 km, then the difference between the true and the approximate chord length would be less than 1 mm.

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