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ON AN APPROXIMATION OF THE TOTAL AGGREGATE RISK DISTRIBUTION IN A MODIFIED COLLECTIVE RISK MODEL

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Abstract

We consider a portfolio of n risks $X_1, \ldots, X_n$, $n \in \mathbb{N}$, which are assumed to be independent and identically distributed with a finite expectation $\mu = E(X_k)$, and finite variance $\sigma^2 = Var(X_k)$, $k = 1, \ldots, n$. Moreover, we assume that only a certain random portion p of the contracts will be affected during the insurance period, and that multiple claims are not possible. In order to model this aspect we assume that $J_1, \ldots, J_n$, $n \in \mathbb{N}$, are additional conditionally independent binomially distributed random variables with a random success parameter $p = P(J_k = 1) = 1 - P(J_k = 0)$, $k = 1, \ldots, n$, being

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Beta-distributed with shape parameters $\alpha > 0$ and $\beta > 0$ which are also independent of the risks under consideration.

1. Introduction

We investigate a portfolio of n risks $X_1, \ldots, X_n$, $n \in \mathbb{N}$, being independent and identically distributed as X with a finite expectation $\mu = E(X)$ and finite variance $\sigma^2 = Var(X)$. Additionally, we assume that only a certain random portion p of the contracts will be affected during the insurance period, and that multiple claims are not possible. In order to model this aspect we assume that $J_1, \ldots, J_n$, $n \in \mathbb{N}$, are additional conditionally independent binomially distributed random variables with a random success parameter $p = P(J_k = 1) = 1 - P(J_k = 0)$, $k = 1, \ldots, n$, being Beta-distributed with shape parameters $\alpha > 0$ and $\beta > 0$ which are also independent of the risks under consideration. This model has been investigated recently in Pfeifer [5] (2022). We have, as is well-known,

$$E(p) = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad Var(p) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

(cf. Johnson et al. [1], Chapter 3, p. 217). Then the total aggregate risk S_n is given by $S_n = \sum_{k=1}^{n} J_k \cdot X_k$. Note that the distribution of S_n is stochastically equivalent to the distribution of $\tilde{S}_N = \sum_{k=1}^{N} X_k$ where $N_n = \sum_{k=1}^{n} J_k$ follows a Beta-binomial distribution with parameters

$$E(N_n) = \frac{n\alpha}{\alpha + \beta} \text{ and } Var(N_n) = \frac{n\alpha\beta \cdot (\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)} = n \cdot Var(p) \cdot (\alpha + \beta + n)$$

(cf. Johnson et al. [2], Chapter 6.2.2, p. 253). Note that for all $n \in \mathbb{N}$, N_n and the $\{X_k\}_{k \in \mathbb{N}}$ are independent. It follows that we have

$$E(S_n) = E(N) \cdot \mu = n \cdot E(p) \cdot \mu \text{ and}$$

$$Var(S_n) = E(N) \cdot Var(X) + Var(N) \cdot \{E(X)\}^2$$

$$= n \cdot E(p) \cdot \sigma^2 + n \cdot Var(p) \cdot (\alpha + \beta + n) \cdot \mu^2.$$

Clearly, these moment relations follow from Wald's well-known formula and the Blackwell-Girshick-formula in collective risk theory (cf. Klugman [3], relation (9.9), p.143, or Rotar [6], Chapter 4, Propositions 1 and 2, p. 200).

2. Approximations

If $E(N_n)$ is large enough and $Var(N_n)$ is small enough it seems reasonable to approximate the distribution of S_n by a normal law (cf. Rotar [6], Chapter 4.1.1, p. 228). Note, however, that condition (4.1.6) or (4.1.7) of Theorem 12 in Rotar [6], p. 231 is not satisfied if α and β are constant since in this case,

$$\lim_{n \to \infty} \sqrt{\frac{Var(N_n)}{E(N_n)}} = \lim_{n \to \infty} \sqrt{\frac{\beta \cdot (\alpha + \beta + n)}{(\alpha + \beta) \cdot (\alpha + \beta + 1)}} = \infty.$$

Alternatively, we may assume α and β to be dependent on n, say $\alpha = \gamma n$ and $\beta = \delta n$ with fixed $\gamma, \delta > 0$. In this case,

$$\lim_{n \to \infty} E(N_n) = \infty \text{ and}$$

$$\lim_{n \to \infty} \sqrt{\frac{Var(N_n)}{E(N_n)}} = \lim_{n \to \infty} \sqrt{\frac{\beta \cdot (\alpha + \beta + n)}{(\alpha + \beta) \cdot (\alpha + \beta + 1)}} = \frac{\sqrt{\delta \cdot (\gamma + \delta + 1)}}{\gamma + \delta}.$$

We can observe that for large values of n the corresponding Beta-binomial distribution is close to the binomial distribution with success parameter $p = \frac{\gamma}{\gamma + \delta}$. Likewise, the corresponding Beta-binomial distribution can be approximated by a normal law itself. We present some graphs for a visualization. The red line in Figure 1 to Figure 4 represents the counting density of the Beta-binomial distribution, the blue line represents the counting density of the binomial distribution. Figure 5 to Figure 8 show Quantile-Quantile-Plots for 100 simulated Beta-binomial distributions each, cf. Pfeifer [4] (2019). T denotes the corresponding correlation based test statistic.

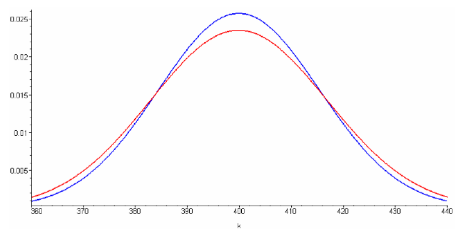

Figure 1.

$n = 1000,\ \alpha = 2n,\ \beta = 3n.$

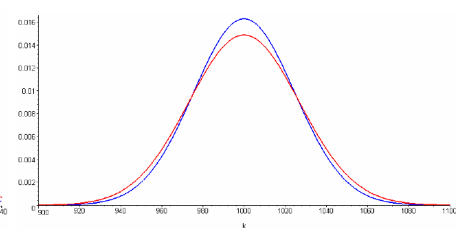

Figure 2.

$n = 2500,\ \alpha = 2n,\ \beta = 3n.$

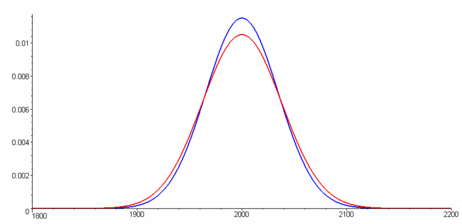

Figure 3.

$n = 5000,\ \alpha = 2n,\ \beta = 3n.$

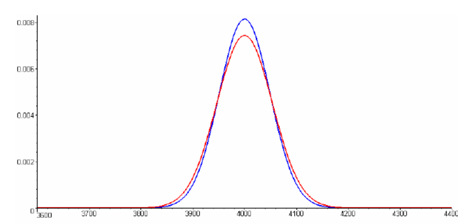

Figure 4.

$n = 10000,\ \alpha = 2n,\ \beta = 3n.$

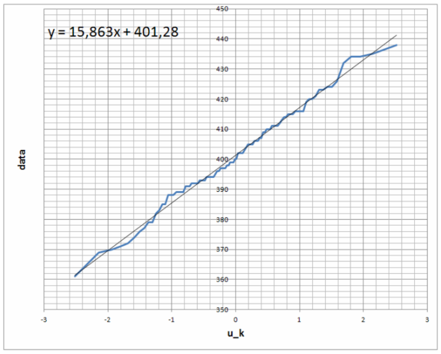

Figure 5.

$n = 1000$, $\alpha = 2n$, $\beta = 3n$,

$T = 5.6477$, $p = 87.07\%$.

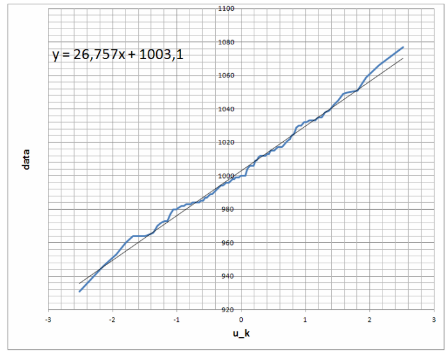

Figure 6.

$n = 2500$, $\alpha = 2n$, $\beta = 3n$,

$T = 5.5583$, $p = 82.89\%$.

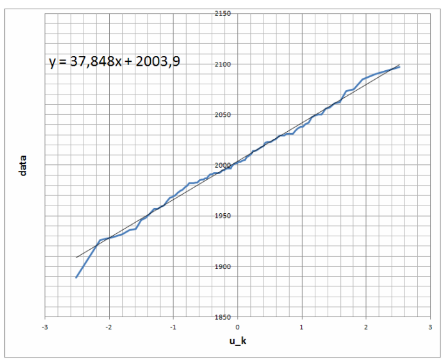

Figure 7.

$n = 5000$, $\alpha = 2n$, $\beta = 3n$,

$T = 5.4882$, $p = 79.07\%$.

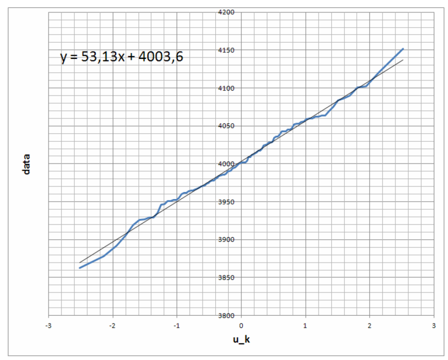

Figure 8.

$n = 10000$, $\alpha = 2n$, $\beta = 3n$,

$T = 5.8477$, $p = 93.73\%$.

Since the Binomial distribution fulfils the conditions of Theorem 12 in Rotar [6], p. 231, and the approximation of the Beta-binomial distribution by a normal law under the conditions discussed above seems acceptable it seems reasonable to approximate the distribution of the total aggregate risk S_n itself by a normal law.

3. A Case Study

In this section, we consider a portfolio with $n = 50.000$ insurance contracts and assume for simplicity that the sums insured follow a lognormal distribution with expectation $\mu_{st} > 0$ and standard deviation $\sigma_{st} > 0$. Further, we assume that the individual loss realized is a beta distributed multiple of the individual sum insured, independent of the sums insured. The corresponding beta parameters are α_{st} and β_{st}. The expectation of the loss factor then is $\mu_{factor} = E(p) = \dfrac{\alpha_{st}}{\alpha_{st} + \beta_{st}}$ and the standard deviation is $\sigma_{factor} = \sqrt{\dfrac{\alpha_{st} \cdot \beta_{st}}{(\alpha_{st} + \beta_{st})^2(\alpha_{st} + \beta_{st} + 1)}}$ while the expectation of the realized loss is $\mu = \mu_{factor} \cdot \mu_{st}$ and its standard deviation $\sigma = \sqrt{(\sigma_{st}^2 + \mu_{st}^2) \cdot (\sigma_{factor}^2 + \mu_{factor}^2) - \mu^2}$. Figure 9 to Figure 13 show Quantile-Quantile-Plots for 100 simulated aggregated losses, cf. Pfeifer [4] (2019).

Parameters	α	β	$E(N_n)$	α_{st}	β_{st}	μ_{factor}	σ_{factor}	μ_{st}	σ_{st}	μ	σ
Fig. 9	70	2.730	1.250	2	398	0.5%	0.35%	100.000	10.000	500	357.50
Fig. 10	90	4.410	1.000	2	198	1.0%	0.70%	100.000	10.000	1.000	712.36
Fig. 11	70	2.730	1.250	2	398	0.5%	0.35%	100.000	20.000	500	372.86
Fig. 12	70	$2.263,\overline{3}$	1.500	2	198	1.0%	0.70%	100.000	20.000	1.000	743.13
Fig. 13	70	$2.263,\overline{3}$	1.500	2	198	1.0%	0.70%	100.000	50.000	1.000	930.41

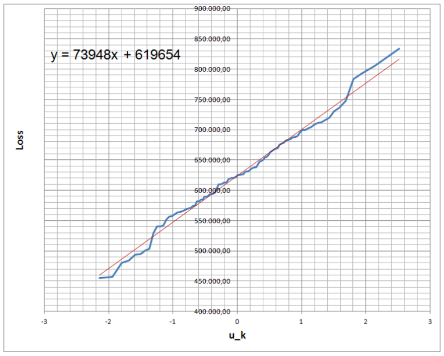

Figure 9.

$T = 5.2584, \;\; p = 63.54\%.$

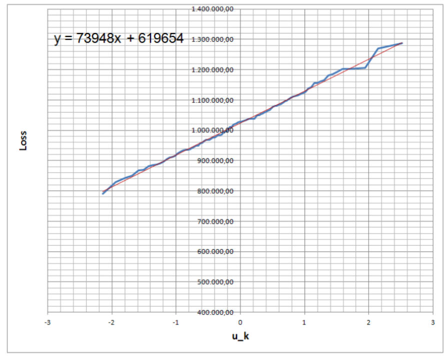

Figure 10.

$T = 6.4951, \;\; p = 99.77\%.$

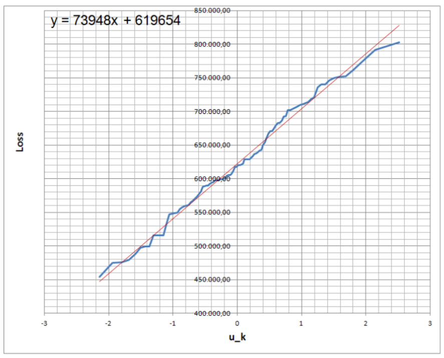

Figure 11.

$T = 5.5421, \;\; p = 82.05\%.$

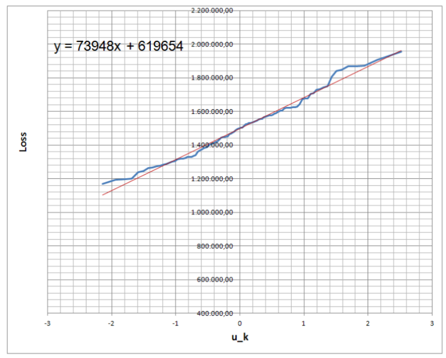

Figure 12.

$T = 5.0173, \;\; p = 44.47\%.$

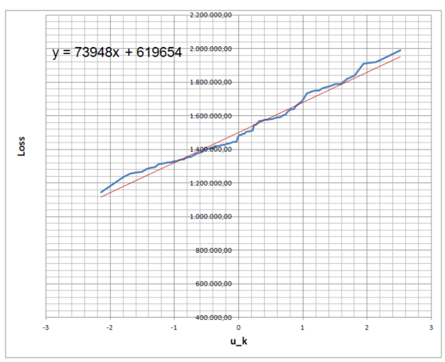

Figure 13. $T = 4.5438, \;\; p = 13.74\%.$

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