Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 20, Issue 2, 2026, Pages 31-42
This paper is available online at http://www.frdint.com/
Published online October 29, 2026

ON A NEW CLASS OF DISTRIBUTIONS (PART II)

DIETMAR PFEIFER

Institut für Mathematik
Schwerpunkt Versicherungs-und Finanzmathematik
Carl von Ossietzky Universität Oldenburg
Oldenburg, Deutschland
Germany
e-mail: dietmar.pfeifer@uni-oldenburg.de

This is a follow-up paper to [4], for distributions with bounded support $[0, 1]$. We consider quantile functions of the form

$$Q(u;\ \mathbf{p}) := a \cdot u^b + (1 - a) \cdot \left[1 - (1 - u)^c\right]$$

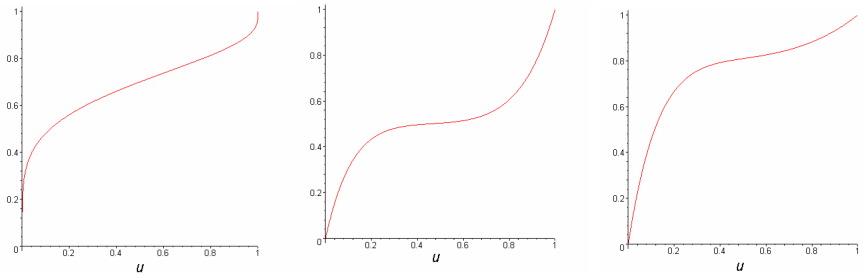
for $0 \le u \le 1;\ 0 < a < 1,\ b, c > 0$, denoting $\mathbf{p} = (a, b, c)$.

Keywords and phrases: quantile functions, beta distribution.

2020 Mathematics Subject Classification: 062E17.

Received September 12, 2026; Accepted October 21, 2026

© 2026 Fundamental Research and Development International



$a = 0.75,\ b = 0.2,\ c = 0.3$ $a = 0.5,\ b = 7,\ c = 9$ $a = 0.2,\ b = 4,\ c = 8$

Examples of quantile functions for different parameter values.

The advantage of these distributions is their easy simulation with standard uniform random numbers and the fact that moments - also for order statistics - can be easily calculated.

In particular, we have, for independent random variables $X_1, ..., X_n$, distributed as X with the above quantile function:

$$E(X;\ \mathbf{p}) = \int_0^1 Q(u;\ \mathbf{p})du = \int_0^1 a \cdot u^b + (1-a) \cdot [1-(1-u)^c\,]du$$

$$= (1-a)\frac{c}{c+1} + \frac{a}{b+1},$$

$$E(X^2;\ \mathbf{p}) = \int_0^1 Q^2(u;\ \mathbf{p})du = \int_0^1 (a \cdot u^b + (1-a) \cdot [1-(1-u)^c\,])^2\, du$$

$$= (1-a)^2\,\frac{c-1}{c+1} + \frac{2a \cdot (1-a)}{b+1} + \frac{a^2}{1+2b} + \frac{(a-1)^2}{1+2c}$$

$$+ 2a \cdot (a-1) \cdot Beta(b+1,\ c+1),$$

$$E(X_{k:n};\ \mathbf{p}) = \frac{1}{Beta(k,\ n+1-k)} \int_0^1 Q(u,\ a,\ b,\ c)u^{k-1}(1-u)^{n-k}\, du$$

$$= \frac{1 - a - \frac{\Gamma(n+1-k-c)\Gamma(n+1)}{\Gamma(n+1+c)\Gamma(n+1-k)}}{Beta(k,\ n+1-k)}$$

$$+ \frac{a \cdot \Gamma(n+1) \cdot \left(\frac{\Gamma(b+k)}{\Gamma(n+b+k)\Gamma(k)} + \frac{\Gamma(n+1-k+c)}{\Gamma(n+1+c)\Gamma(n+1-k)} \right)}{Beta(k,\ n+1-k)}$$

for $1 \le k \le n$ (this follows, e.g., from [2], (5.1), p. 7). Here $Beta(\alpha,\ \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, $\alpha,\ \beta > 0$ denotes Euler's Beta function.

Unfortunately, the corresponding distribution cannot be explicitly calculated in most situations. Here are a few results for some special cases:

$a = 0 : Q(u;\ 0,\ b,\ c) = u^b$: belongs to a standard beta distribution with $\alpha = \frac{1}{b}$, $\beta = 1$.

$a = 1 : Q(u;\ 1,\ b,\ c) = 1 - (1-u)^c$: belongs to a standard beta distribution with $\alpha = 1$, $\beta = \frac{1}{c}$.

$b = c = 2 : Q(u;\ a,\ 2,\ 2) := a \cdot u^2 + (1-a) \cdot \left[1 - (1-u)^2\right]$: belongs to the c.d.f. $F(x;\ a)$ given by

$$F(x;\ a) = \begin{cases} \frac{a - 1 + \sqrt{(a-1)^2 + (2a-1) \cdot x}}{2a-1}, & a \ne \frac{1}{2} \\ x, & a = \frac{1}{2} \end{cases}, \quad 0 \le x \le 1 \qquad \text{with}$$

density $f(x;\ a)$ given by $f(x;\ a) = \frac{1}{2 \cdot \sqrt{(a-1)^2 + (2a-1) \cdot x}}$, $0 \le x \le 1$.

Hence for $a = \frac{1}{2}$, the continuous uniform distribution is obtained.

This is obviously also true for the choice $b = c = 1,\ 0 \le \alpha \le 1$.

Another case of an explicit representation of the c.d.f. $F(x;\, a)$ can be obtained for $b = c = \frac{1}{2}$, namely by solving the pertaining quadratic equation:

$$F(x;\, a) = \frac{x \cdot (a^2 - (1-a)^2) \cdot (x - 2 \cdot (1-a)) + 2a^2 \cdot (1-a)^2}{((1-a)^2 + a^2)^2}$$

$$+ \frac{2a \cdot (1-a)^2 \cdot (x - 1 + a)\sqrt{a^2 - x^2 + 2 \cdot (1-a) \cdot x}}{((1-a)^2 + a^2)^2},$$

$0 < x < 1$. The corresponding density $f(x;\, a)$ is given by

$$f(x;\, a) = \frac{2 \cdot (a^2 - (1-a)^2) \cdot (x - 1 + a) - \dfrac{2a \cdot (1-a) \cdot (x - 1 + a)^2}{\sqrt{a^2 - x^2 + 2 \cdot (1-a) \cdot x}}}{((1-a)^2 + a^2)^2}$$

$$+ \frac{2a \cdot (1-a)\sqrt{a^2 - x^2 + 2 \cdot (1-a) \cdot x}}{((1-a)^2 + a^2)^2},$$

$0 < x < 1$. The following figures show some particular cases for different values of a:

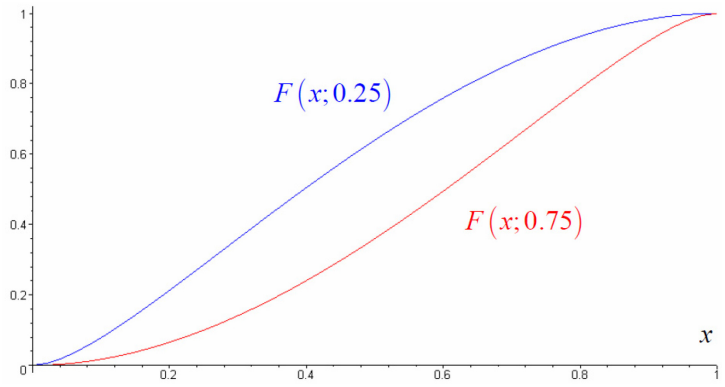


Figure 1.

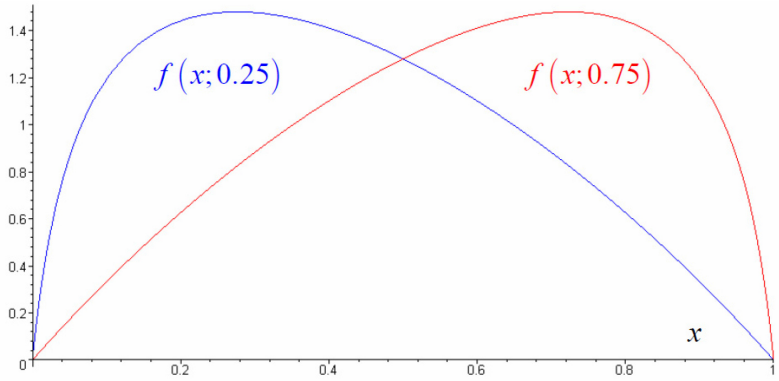


Figure 2.

Also, in some cases the quantile function (or c.d.f., resp.) of a standard beta distribution can be closely approximated by our approach, in particular, if $b, c < 1$. Note that the density $g(x;\, \alpha, \beta)$ of the standard beta distribution is given by

$$g(x;\, \alpha, \beta) = \frac{x^{\alpha-1} \cdot (1-x)^{\beta-1}}{\beta(\alpha, \beta)}, \quad 0 < x < 1; \quad \alpha, \beta > 0$$

(cf. [1], Chapter 25, p. 210).

A good approximation can be obtained by equating the first moments

of the beta distribution with those of our approach. For this purpose, let Y be standard beta distributed with parameters $\alpha, \beta > 0$ and X be distributed as above. From the equations

$$E(Y) = \frac{\alpha}{\alpha+\beta} = E(X) = (1-a)\frac{c}{c+1} + \frac{a}{b+1}$$

and

$$E(Y^2) = \frac{\alpha \cdot (\alpha+1)}{(\alpha+\beta)\cdot(\alpha+\beta+1)} = E(X^2)$$

we obtain the relations

$$\alpha = \frac{E(X)\cdot(E(X)-E(X^2))}{E(X^2)-\{E(X)\}^2}, \quad \beta = \frac{(E(X)-E(X^2))\cdot(1-E(X))}{E(X^2)-\{E(X)\}^2}. \qquad (*)$$

Example 1. Consider the above case $a = 0.75$, $b = 0.2$, $c = 0.3$. From (*) we get $\alpha = 6.3026$ and $\beta = 2.9294$. The c.d.f. F derived from our quantile function is quite close to the c.d.f. $\widetilde{F}$ of a standard beta distribution with the above parameters, as can be seen from the following graph, obtained by numerical inversion:

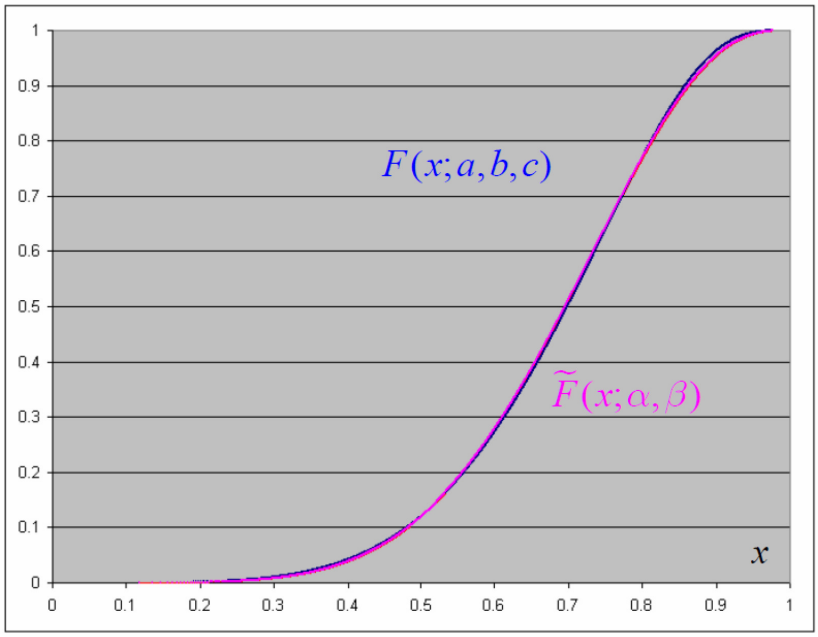


Figure 3. Comparison of F and $\widetilde{F}$.

The following graph shows the corresponding densities f and $\widetilde{f}$:

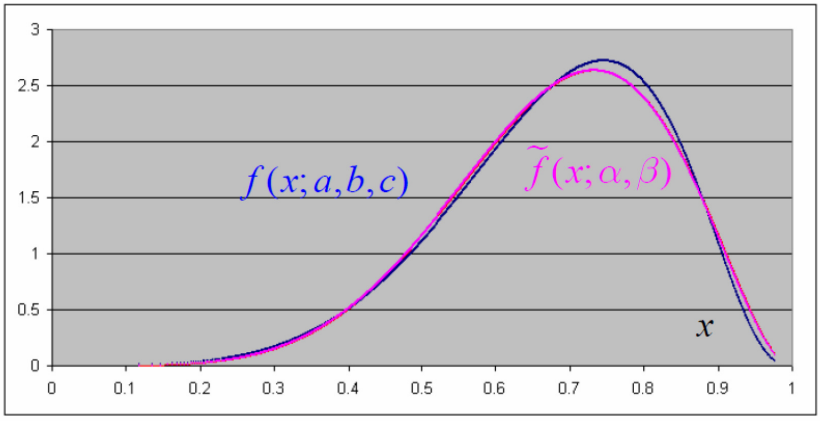


Figure 4. Comparison of f and $\widetilde{f}$.

Example 2. Consider the case $a = 0.25$, $b = 0.4$, $c = 0.3$. From (*)

we get $\alpha = 1.9867$ and $\beta = 3.6630$. The c.d.f. F derived from our quantile function is also quite close to the c.d.f. $\widetilde{F}$ of a standard beta distribution with the above parameter, as can be seen from the following graph, obtained by numerical inversion:

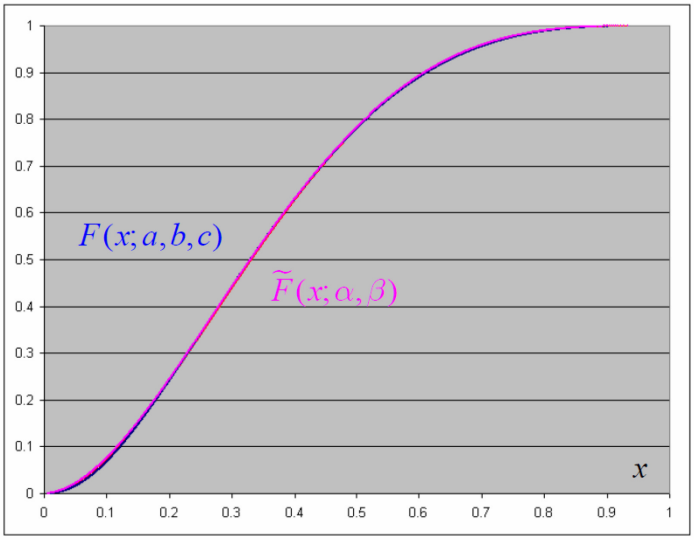


Figure 5. Comparison of F and $\widetilde{F}$.

The following graph shows the corresponding densities f and $\widetilde{f}$:

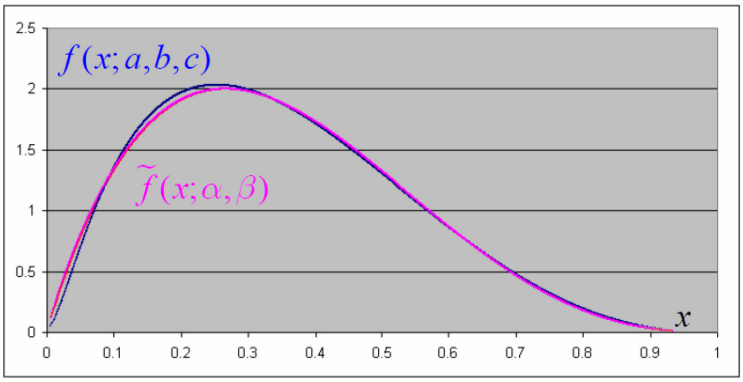


Figure 6. Comparison of f and $\widetilde{f}$.

Example 3. Consider the case $a = 0.75$, $b = 0.5$, $c = 0.5$ (cf. p. 3 above). From (*) we get $\alpha = 2.0361$ and $\beta = 1.4544$. The c.d.f F derived from our quantile function is also quite close to the c.d.f. $\widetilde{F}$ of a standard beta distribution with the above parameter, as can be seen from the following graph, obtained by numerical inversion:

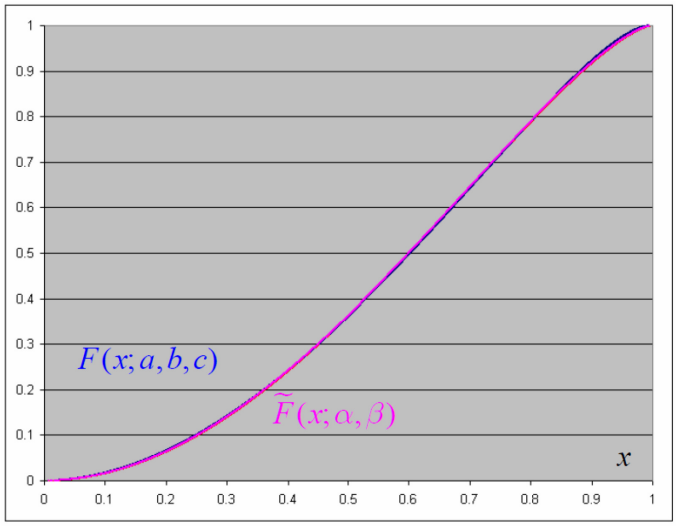


Figure 7. Comparison of F and $\widetilde{F}$.

The following graph shows the corresponding densities f and $\widetilde{f}$:

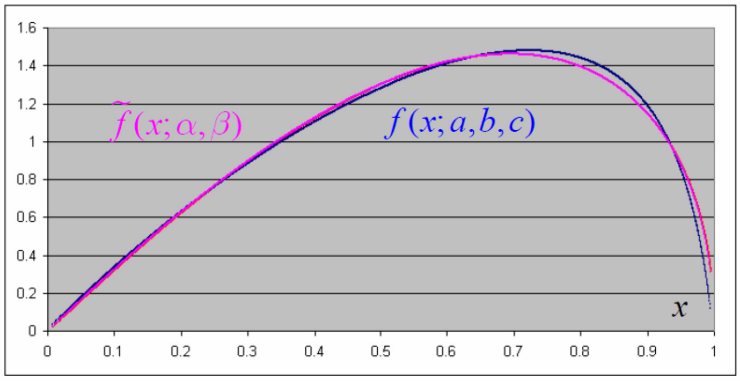


Figure 8. Comparison of f and $\widetilde{f}$.

If $b = c = \frac{1}{2}$, we can find the following close estimates for the parameters of the approximate beta distribution, depending on a:

$$\alpha = \alpha(a) = 1 + 2.3584a - 2.1931a^2 + 2.5142a^3 - 1.6795a^4, \qquad (**)$$

$$\beta = \beta(a) = 1 + 2.3584(1-a) - 2.1931(1-a)^2 + 2.5142(1-a)^3 - 1.6795(1-a)^4$$

for $0 \le a \le 1$. The following figure shows the functional relationship between α and $\alpha(a)$ and $\beta(a)$.

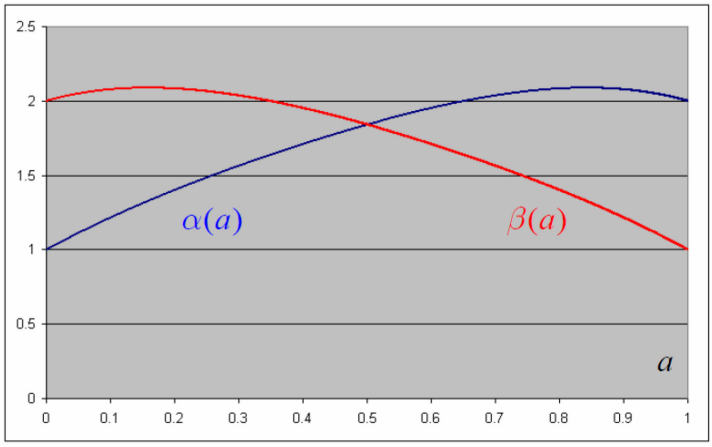


Figure 9. Functional dependence of $\alpha(a)$ and $\beta(a)$.

The following graph shows the difference $d(u; a) := \left(u; a, \frac{1}{2}, \frac{1}{2}\right) - \tilde{Q}(u; \alpha(a), \beta(a))$ of the approximate quantile function Q and the exact quantile function $\tilde{Q}$ with the parameters $\alpha(a)$ and $\beta(a)$ according to (**), for different values of a.

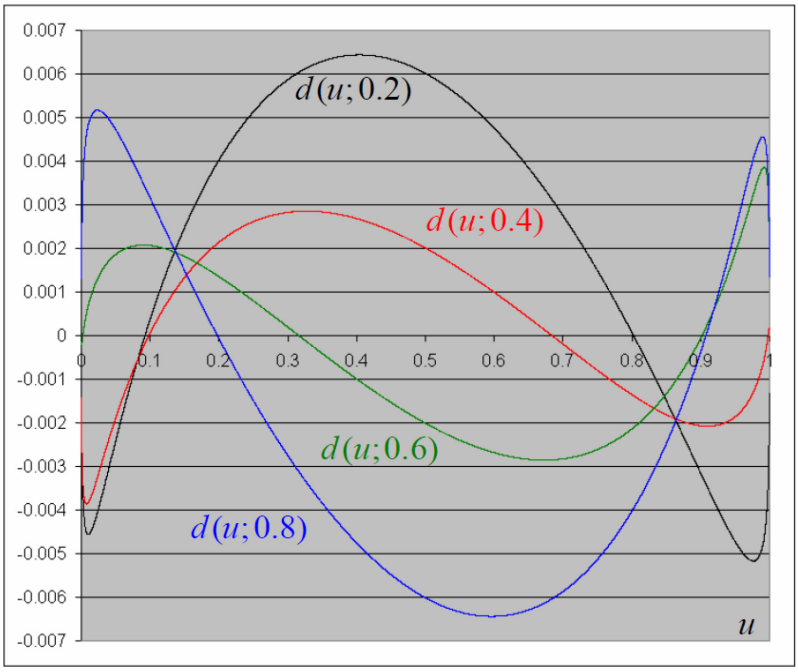


Figure 10. Graphical representation of $d(u; a)$ for different values of $a \in [0, 1]$.

Conversely, if α of the beta distribution is given, then we should have approximately

$$a = 0.7297\alpha^4 - 4.1522\alpha^3 + 8.923\alpha^2 - 7.9947\alpha + 2.518,$$

$$\beta = -1.6795a^4 + 4.204a^3 - 4.7278a^2 + 1.1972a + 2.0066$$

for $1 \le \alpha \le 2$. For a further discussion of distributions concentrated on $[0, 1]$ see [3].

References

[1] N. L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 2, 2nd ed., Wiley, N.Y., 1995.

[2] N. Balakrishnan and C. R. Rao, (eds.) Handbook of Statistics 16: Order Statistics: Theory and Methods, Elsevier, Amsterdam, 1998.

[3] S. Kotz and J. R. van Dorp, Beyond Beta. Other Continuous Families of Distributions with Bounded Support and Applications, World Scientific Publishing Co. Pt. Ltd., Singapore, 2004.

[4] D. Pfeifer, On a new class of distributions, Fundamental J. Math. Math. Sci. 20(2) (2026), 19-30.