Fundamental Journal of Modern Physics
ISSN: 2249-9768
Volume 24, Issue 2, 2025, Pages 65-75
This paper is available online at http://www.frdint.com/
Published online October 10, 2025

NONCOMMUTATIVE-GEOMETRY WORMHOLES IN SPACETIMES WITH A COSMOLOGICAL CONSTANT

PETER K. F. KUHFITTIG

Department of Mathematics
Milwaukee School of Engineering
Milwaukee, Wisconsin 53202-3109
USA
e-mail: kuhfitti@msoe.edu

Abstract

Noncommutative geometry, an offshoot of string theory, replaces point-like particles by smeared objects. Applied to wormhole spacetimes, it has been shown that the noncommutative effects can be implemented by modifying only the energy momentum tensor in the Einstein field equations, while leaving the Einstein tensor unchanged. The implication is that noncommutative-geometry wormholes could be macroscopic. However, it is shown in this paper that such wormholes could not be sufficiently massive to exist on a macroscopic scale. The purpose of this paper is to use $f(Q)$ modified gravity to invoke the cosmological constant, which, in turn, provides the extra degrees of freedom to overcome these obstacles, thereby allowing noncommutative-geometry wormholes to be macroscopic.

Keywords and phrases: Morris-Thorne wormholes, $f(Q)$ gravity, noncommutative geometry.

Received September 19, 2025; Accepted September 29, 2025

1. Introduction

Wormholes are tunnel-like structures in spacetime that link widely separated regions of our Universe or different universes altogether. While wormholes are just as good a prediction of Einstein's theory as black holes, they are subject to severe restrictions from quantum field theory. In particular, holding a wormhole open requires a violation of the null energy condition, calling for the existence of "exotic matter" [1]. This violation is more of a practical than conceptual problem, as illustrated by the Casimir effect [2]: exotic matter can be made in the laboratory. Being a rather small effect, it is not immediately obvious that it is sufficient for supporting a traversable wormhole.

Another area dealing with small effects is noncommutative geometry, an offshoot of string theory, where point-like particles are replaced by smeared objects, to be discussed further below. For now, it is sufficient to note that noncommutative-inspired wormholes can be macroscopic in spite of the small effects. However, as we will see later, such a wormhole cannot be very massive; we will also recall that a Morris-Thorne wormhole is actually a compact stellar object, suggesting that noncommutative-geometry wormholes are likely to be microscopic. The purpose of this paper is to show that by invoking $f(Q)$ modified gravity, the resulting extra degrees of freedom allow a noncommutative-geometry wormhole to be macroscopic.

2. Morris-Thorne Wormholes

Wormholes are handles or tunnels in spacetime that connect widely separated regions of our Universe or different universes altogether. While there had been some forerunners, the first detailed analysis of humanly traversable wormholes was carried out by Morris and Thorne [1]. With the Schwarzschild solution in mind, they proposed the following static and spherically symmetric line element to model a wormhole spacetime:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

using units in which $c = G = 1$. Here $\Phi = \Phi(r)$ is usually referred to as the *redshift function*, which must be finite everywhere to prevent the occurrence of an event horizon. The function $b = b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram. The spherical surface $r = r_0$ is called the *throat* of the wormhole. According to Ref. [1], at the throat, $b = b(r)$ must satisfy the following conditions: $b(r_0) = r_0$, $b(r) < r$ for $r > r_0$, and $b'(r_0) \leq 1$, called the *flare-out condition*. This condition can only be satisfied by violating the null energy condition (NEC), which states that

$$T_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0 \tag{2}$$

for all null vectors k^{α}, where $T_{\alpha\beta}$ is the energy momentum tensor. As noted above, matter that violates the NEC is called "exotic" in Ref. [1]. In particular, for the outgoing null vector $(1, 1, 0, 0)$, the violation becomes $T_{\alpha\beta}k^{\alpha}k^{\beta} = \rho + p_r < 0$. Here, $T^t{}_t = -\rho(r)$ is the energy density, $T^r{}_r = p_r(r)$ is the radial pressure, and $T^\theta{}_\theta = T^\phi{}_\phi = p_t(r)$ is the lateral (transverse) pressure. Another requirement is *asymptotic flatness*: $\lim_{r\to\infty} \Phi(r) = 0$ and $\lim_{r\to\infty} b(r)/r = 0$.

For later reference, we now list the Einstein field equations:

$$\rho(r) = \frac{1}{8\pi}\frac{b'}{r^2}, \tag{3}$$

$$p_r(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{4}$$

and

$$p_t(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[\Phi'' - \frac{b'r - b}{2r(r - b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)}\right]. \tag{5}$$

3. Noncommutative Geometry

In this section, we are going to take a brief look at noncommutative geometry, which will play a key role in this paper. This geometric background is based on the realization that coordinates may become noncommutative operators on a D-brane [3, 4]. A critical feature that is consistent with the Heisenberg uncertainty principle is that noncommutativity replaces point-like particles by smeared objects [5, 6, 7]. First proposed by H. S. Snyder [8], this feature eliminates the divergences that normally occur in general relativity. It is shown in Ref. [6] that this objective can be realized by showing that spacetime can be encoded in the commutator $[\mathbf{x}^{\mu}, \mathbf{x}^{\nu}] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix that determines the fundamental cell discretization of spacetime in the same way that Planck's constant $\hbar$ discretizes phase space. Following Refs. [9, 10], we are going to model the smearing using a so-called Lorentzian distribution of minimal length $\sqrt{\gamma}$ instead of the Dirac delta function. So the energy density of a static and spherically symmetric and particle-like gravitational source is given by

$$\rho(r) = \frac{m\sqrt{\gamma}}{\pi^2(r^2 + \gamma)^2}. \tag{6}$$

The usual interpretation is that the gravitational source causes the mass m of a particle to be diffused throughout the region of linear dimension $\sqrt{\gamma}$ due to the uncertainty.

Since Eq. (6) plays such a critical role in this paper, we need to ensure its proper application. According to Ref. [6], it is possible to implement the noncommutative effects in the Einstein field equations $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ by modifying only the energy momentum tensor, while leaving the Einstein tensor $G_{\mu\nu}$ intact. As a consequence, the length scales can be macroscopic, as we will confirm in the next section.

4. The Mass of the Wormhole

Our first task in this section is to confirm that noncommutative-geometry wormholes can be macroscopic. Our discussion begins with Eqs. (3) and (6), which immediately yields the total mass-energy M_γ of a sphere of radius r:

$$M_\gamma = \int_0^r \rho(r')4\pi(r')^2 dr' = \frac{2m}{\pi}\left[\tan^{-1}\frac{r}{\sqrt{\gamma}} - \frac{r\sqrt{\gamma}}{r^2 + \gamma}\right]. \tag{7}$$

Similarly, for the shape function $b = b(r)$, we have from Eq. (3),

$$b(r) = \frac{4m\sqrt{\gamma}}{\pi}\left[\frac{1}{\sqrt{\gamma}}\tan^{-1}\frac{r}{\sqrt{\gamma}} - \frac{r}{r^2 + \gamma}\right.$$

$$\left. - \frac{1}{\sqrt{\gamma}}\tan^{-1}\frac{r_0}{\sqrt{\gamma}} + \frac{r_0}{r_0^2 + \gamma}\right] + r_0. \tag{8}$$

Observe that $b(r_0) = r_0$ and $\lim_{r\to\infty} b(r)/r = 0$. To ensure asymptotic flatness, we retain the assumption $\lim_{r\to\infty} \Phi(r) = 0$. Following Ref. [11], we let $B = b/\sqrt{\gamma}$ be the form of the shape function, even though $B(r_0) \neq 0$. The reason is that B can be expressed as a function of $r/\sqrt{\gamma}$ by a simple algebraic rearrangement:

$$\frac{1}{\sqrt{\gamma}}b(r) = B\left(\frac{r}{\sqrt{\gamma}}\right) = \frac{4m}{\pi}\frac{1}{r}\left[\frac{r}{\sqrt{\gamma}}\tan^{-1}\frac{r}{\sqrt{\gamma}} - \frac{\left(\frac{r}{\sqrt{\gamma}}\right)^2}{\left(\frac{r}{\sqrt{\gamma}}\right)^2+1}\right.$$

$$\left. - \frac{r}{\sqrt{\gamma}}\tan^{-1}\frac{r_0}{\sqrt{\gamma}} + \frac{r}{\sqrt{\gamma}}\frac{\frac{r_0}{\sqrt{\gamma}}}{\left(\frac{r_0}{\sqrt{\gamma}}\right)^2+1}\right] + \frac{r_0}{\sqrt{\gamma}}; \qquad (9)$$

observe that

$$B\left(\frac{r_0}{\sqrt{\gamma}}\right) = \frac{r_0}{\sqrt{\gamma}}, \qquad (10)$$

the analogue of $b(r_0) = r_0$. So the throat radius can be macroscopic. The line element can be written as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{B\left(r/\sqrt{\gamma}\right)}{r/\sqrt{\gamma}}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (11)$$

In view of Eq. (9), this line element represents a wormhole with throat radius $r_0/\sqrt{\gamma}$, while retaining asymptotic flatness [11].

Next, let us obtain an estimate of the mass of the wormhole. From Eq. (6),

$$m(r) = \int_{r_0}^{r}\rho(r')4\pi(r')^2dr'$$

$$= \frac{2m}{\pi}\left[\tan^{-1}\frac{r}{\sqrt{\gamma}} - \frac{r\sqrt{\gamma}}{r^2+\gamma} - \tan^{-1}\frac{r_0}{\sqrt{\gamma}} + \frac{r_0\sqrt{\gamma}}{r_0^2+\gamma}\right]. \qquad (12)$$

Sine m in Eq. (6) represents the mass of a particle, we conclude that the mass $m(r)$ cannot be very large. It has been shown, however, that Morris-Thorne wormholes are actually compact stellar objects [12]. The implication is that noncommutative-inspired wormholes are likely to be microscopic after all. To retain our macroscopic scale, we will turn to $f(Q)$ modified gravity and its consequences. That is the topic of the next section.

5. Invoking the Cosmological Constant

Attempts to overcome the theoretical and practical problems confronting Morris-Thorne wormholes have relied heavily on various modified gravitational theories. A recently proposed modified theory, called $f(Q)$ gravity, is due to Jimenez et al. [13]. Here Q is the non-metricity scalar from the field of differential geometry. The action for this gravitational theory is

$$S = \int \frac{1}{2} f(Q)\sqrt{-g}d^4x + \int \mathcal{L}_m \sqrt{-g}dx^4, \tag{13}$$

where $f(Q)$ is an arbitrary function of Q, $\mathcal{L}_m$ is the Lagrangian density of matter, and g is the determinant of the metric tensor $g_{\mu\nu}$. Even though it is a fairly new theory, numerous applications have already been found; see, for example, [13, 14, 15, 16, 17, 18, 19, 20, 21]. Since we are primarily interested in wormholes, our focus is necessarily more narrow. Accordingly, we are going to follow Ref. [14], in part because it uses a simple but commonly employed form of $f(Q)$: $f(Q) = \alpha Q + \beta$ [14], where α and β are free parameters. Since $f'(Q) =$ a constant and $f''(Q) = 0$, this produces the Einstein field equations with a cosmological constant [13]. The corresponding field equations are [14]:

$$\rho = \frac{\alpha b'}{r^2} + \frac{\beta}{2}, \tag{14}$$

$$p_r = \frac{1}{r^3}\left[2\alpha r(r-b)\Phi' - \alpha b\right] - \frac{\beta}{2}, \tag{15}$$

and

$$p_t = \frac{1}{2r^3}\left[\alpha(r\Phi'+1)(-rb' + 2r(r-b)\Phi' + b)\right] + \frac{\alpha(r-b)\Phi''}{r} - \frac{\beta}{2}. \tag{16}$$

Eq. (14) can now be combined with Eq. (6):

$$\frac{m\sqrt{\gamma}}{\pi^2(r^2+\gamma)^2} = \frac{\alpha b'(r)}{r^2} + \frac{\beta}{2}. \tag{17}$$

Solving for $b'(r)$, we get

$$b'(r) = \frac{1}{\alpha}\frac{mr^2\sqrt{\gamma}}{\pi^2(r^2+\gamma)^2} - \frac{1}{2}\frac{\beta}{\alpha}r^2 \tag{18}$$

and

$$b(r) = \frac{1}{\alpha}\int_{r_0}^{r}\left[\frac{m(r')^2\sqrt{\gamma}}{\pi^2[(r')^2+\gamma]^2} - \frac{1}{2}\frac{\beta}{\alpha}(r')^2\right]dr' + r_0. \tag{19}$$

To retain asymptotic flatness, we let $\beta = 0$, yielding

$$b(r) = \frac{1}{\alpha}\frac{m\sqrt{\gamma}}{\pi^2}\left[\frac{\tan^{-1}\frac{r}{\sqrt{\gamma}}}{2\sqrt{\gamma}} - \frac{r}{2(r^2+\gamma)} - \frac{\tan^{-1}\frac{r_0}{\sqrt{\gamma}}}{2\sqrt{\gamma}} + \frac{r_0}{2(r_0^2+\gamma)}\right] + r_0. \tag{20}$$

Observe that $b(r_0) = r_0$, as required. Thanks to the free parameter α

from $f(Q)$ gravity, the mass of the wormhole, $m(r) = \int_{r_0}^{r} \rho(r')4\pi(r')^2 dr'$ $= \frac{1}{2} b(r)$ from Eq. (3), can now be macroscopic. This is our main conclusion.

6. Summary

Noncommutative geometry replaces point-like particles by smeared objects, which is consistent with the Heisenberg uncertainty principle. Although a small effect, wormholes supported by a noncommutative-geometry background can still have a macroscopic throat size. To explain this outcome, we recall from Section 3 that the noncommutative effects can be implemented by modifying only the energy momentum tensor in the Einstein field equations, while leaving the Einstein tensor intact. It is shown in this paper that such wormholes cannot have a large enough mass to exist on a macroscopic scale. However, Morris-Thorne wormholes are likely to be compact stellar objects, akin to neutron stars, and so would normally be quite massive. Using $f(Q)$ modified gravity to invoke the cosmological constant, it is shown that the resulting extra degrees of freedom enable us to overcome these obstacles, thereby allowing the wormholes to be sufficiently massive despite the noncommutative-geometry background.

References

[1] M. S. Morris and K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, American Journal of Physics 56 (1988), 395-412.

[2] H. G. B. Casimir, On the attraction between two perfectly conducting plates, Proceedings of the Royal Netherlands Academy of Arts and Sciences 51 (1948), 793-795.

[3] E. Witten, Bound states of strings and p-branes, Nuclear Physics B 460 (1996), 335-350.

[4] N. Seiberg and E. Witten, String theory and noncommutative geometry, Journal of High Energy Physics 9909 (1999), ID: 032.

[5] A. Smailagic and E. Spallucci, Feynman path integral on the non-commutative plane, Journal of Physics A 36 (2003), L467-L471.

[6] P. Nicolini, A. Smailagic and E. Spallucci, Noncommutative geometry inspired Schwarzschild black hole, Physics Letters B 632 (2006), 547-551.

[7] P. Nicolini and E. Spallucci, Noncommutative geometry-inspired dirty black holes, Classical and Quantum Gravity 27 (2010), ID: 015010.

[8] H. S. Snyder, Quantized space-time, Physical Review 71 (1947), ID: 38.

[9] K. Nozari and S. H. Mehdipour, Hawking radiation as quantum tunneling for a noncommutative Schwarzschild black hole, Classical and Quantum Gravity 25 (2008), ID: 175015.

[10] J. Liang and B. Liu, Thermodynamics of noncommutative geometry inspired BTZ black hole based on Lorentzian smeared mass distribution, Europhysics Letters 100 (2012), ID: 30001.

[11] P. K. F. Kuhfittig, Macroscopic noncommutative-geometry wormholes as emergent phenomena, Letters in High Energy Physics 2023 (2023), ID: 399.

[12] P. K. F. Kuhfittig, A note on wormholes as compact stellar objects, Fundamental J. Modern Physics 17(1) (2022), 63-70.

[13] J. B. Jimenez, L. Heisenberg and T. Koivisto, Coincident general relativity, Physical Review D 98 (2018), ID: 044048.

[14] Z. Hassan, S. Ghosh, P. K. Sahoo and K. Bamba, Casimir wormholes in modified symmetric teleparallel gravity, European Physical Journal C 82 (2022), ID: 1116.

[15] Z. Hassan, S. Mandal and P. K. Sahoo, Traversable wormhole geometries in $f(Q)$ gravity, Fortschritte der Physik 69 (2021), ID: 2100023.

[16] G. Mustafa, Z. Hassan, P. H. R. S. Moraes and P. K. Sahoo, Wormhole solutions in symmetric teleparallel gravity, Physics Letters B 821 (2021), ID: 136612.

[17] U. K. Sharma, Shweta and A. K. Mishra, Traversable wormhole solutions with non-exotic fluid in framework of $f(Q)$ gravity, International Journal of Geometric Methods in Modern Physics 19 (2021), ID: 2250019.

[18] A. Banerjee, A. Pradhan, T. Tangphati and F. Rahaman, Wormhole geometries in $f(Q)$ gravity and the energy conditions, European Physical Journal C 81 (2021), ID: 1031.

[19] G. Mustafa, Z. Hassan and P. K. Sahoo, Traversable wormholes inspired by noncommutative geometries in $f(Q)$ gravity with conformal symmetry, Annals of Physics 437 (2022), ID: 168751.

[20] F. Parsaei, S. Rastgoo and P. K. Sahoo, Wormhole in $f(Q)$ gravity, European Physical Journal Plus 137 (2022), ID: 1083.

[21] Z. Hassan, G. Mustafa, Joao R. L. Santos and P. K. Sahoo, Embedding procedure and wormhole solutions in $f(Q)$ gravity, Europhysics Letters 139 (2022), ID: 39001.