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NO BETWEEN TOPOLOGICAL PROPERTIES FOR CONSECUTIVE CLASSICAL SEPARATION AXIOMS

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Abstract

In a recent paper, it was reported that there are no topological properties between consecutive classical topological properties weaker than or equal to T_4. In this paper, the classical topological properties are further examined verifying the given statement and extending the statement beyond T_4.

1. Introduction and Preliminaries

The mathematical ancestors in what is today called modern topology did an incredible job, but there remained many natural, unaddressed questions. Either the

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questions did not arise or there were not needed tools and properties to successful resolve the questions. In either case, the continued investigation of topology has revealed new properties and tools that have been used to resolve some of those unaddressed questions. In the paper [3], it was reported that there are no topological properties between consecutive classical topological properties weaker than or equal to T_4. In this paper, the classical separation axioms are further considered verifying the answer of "no" above and the answer of "no" is extended to the separation axioms stronger than or equal to T_4.

2. No Topological Properties Between Consecutive Classical Topological Properties Weaker than or Equal to T_4

In classical topology, the topological properties T_0, T_1, T_2, Urysohn, completely Hausdorff, T_3, $T_{3\frac{1}{2}}$, and T_4 were introduced and thoroughly investigated. From classical topology, it is known T_4 implies $T_{3\frac{1}{2}}$, which in separate branches implies each of T_3 and completely Hausdorff, T_3 implies Urysohn and completely Hausdorff implies Urysohn, Urysohn implies T_2, T_2 implies T_1, and T_1 implies T_0, with none of the implications reversible. A logical, natural question to ask from classical topology is whether there are topological properties between two consecutive classical topological properties in the listing above, where the first property immediately implies the second. In a recent paper [3], the answer of "no" was given. Below the classical separation axioms above are further considered verifying the conclusions above.

The continued investigation of T_0 - identification spaces [4] led to the realization of the important role "not-P", where P is a topological property for which "not-P" exists, would have on the expansion of topology. As an example, the use of "not-P", where "not-P" exists, led to the discovery of the least of all topological properties: $L = (T_0$ or "not-T_0"$) = (P$ or "not-P"$)$, where P is a topological property for which "not-P" exists [5]. The continued use of "not-P", as above, and L opened the gate to a never before imagined, fertile, topological territory expanding, changing, and moving the frontier of topology forward, as illustrated in earlier work and below.

Theorem 2.1. *Let P and Q be topological properties. Then $(P$ is stronger than or equal to $Q)$ iff $((P$ and $Q) = P)$.*

Proof. Since for statements P and Q, $(P$ implies $Q)$ iff $((P$ and $Q)$ iff $P)$, then the statement is true.

Theorem 2.2. *Let P and Q be topological properties such that P is stronger than or equal to Q and "not-P" exists. Then $((P$ or $Q) = Q)$.*

Proof. Since Q = $(Q$ and $L)$ = $((Q$ and $P)$ or $(Q$ and "not-P")) = $(P$ or $(Q$ and "not-P")) = $((P$ or $Q)$ and $(P$ or "not-P")) = $((P$ or $Q)$ and $L)$ = $(P$ or $Q)$, the statement is true.

Theorem 2.3. *Let P, W, and Q be topological properties such that P is stronger than W, which is stronger than or equal to Q, and "not-P" exists. Then W has property Q.*

Proof. If $W = Q$, then W has property Q. Thus consider the case that W is stronger than Q. Since P is stronger than W, then $(W$ and "not-P") exists. Since $(W$ and "not-P") implies $(Q$ and "not-P"), then $(W$ and "not-P") has property $(Q$ and "not-P").

If P is the true immediate successor of Q, as was thought in the paper [3], then there are no topological properties between P and Q. Below, for two consecutive separation axioms in the listing above, it is shown that the stronger one is the immediate successor of the weaker, i.e., there are no topological properties between P and Q.

The introduction of the T_0 separation axiom is credited to Kolmogorff [13]. A space (X, T) is T_0 iff for distinct elements x and y in X, there exists an open set that contains only one of x and y. Thus the two elements can be separated by an open set, but, without choice of which of the two elements is separated. What then would be the minimal strengthening of T_0 ?; being able to separate each of the two elements from the other with an open set.

In 1906, Frechet [7] and Riesz [9] separately and independently introduced the T_1 separation axiom. A space is T_1 iff for distinct elements x and y in X there exist an open set containing x and not y and an open set containing y and not x. Thus T_1 is a minimal strengthening of T_0 and there are no topological property between T_0 and T_1.

What would be a minimal strengthening of T_1 ?

For the two open sets separating x and y for a T_1 space, the two open sets could intersect or not. As shown below, requiring the two open sets not to intersect would be a minimal strengthening of T_1.

In 1914, Hausdorff [8] introduced the Hausdorff or equivalently T_2 separation axiom. A space (X, T) is Hausdorff iff for distinct elements x and y, there exist disjoint open sets one containing x and the other containing y. In the paper [6], it was shown that for spaces (X, T) for which X is finite, T_1 and T_2 are equivalent. Thus, for such a space, T_1 and T_2 are equivalent and there are no topological properties between them. Thus consider the case that (X, T) is T_1 and X is infinite. Let x and y be distinct elements. Since (X, T) is T_1, then singleton sets are closed. Let $z \in X$; z neither x nor y. Then both y and z are in $(X \setminus \{x\})$, which is open, and both x and z are in $(X \setminus \{y\})$, which is open. Thus there exists an open set U containing x and not y and an open set V containing y and not x that are not disjoint. Conversely, if (X, T) is a space such that X is infinite and for distinct elements x and y in X, there exist an open set U containing x and not y, an open set V containing y and not x, and $U \bigcap V \neq \phi$, then (X, T) is T_1. Thus (X, T) is T_1 and not T_2 iff X is infinite and for distinct elements x and y there exist an open set U containing x and not y, an open set V containing y and not x, and $U \bigcap V \neq \phi$. Since there are T_1 spaces that are not T_2, then requiring distinct elements x and y in a space (X, T), where X is infinite, to have disjoint open sets U and V such that $x \in U$ and $y \in V$ is a minimal strengthening of T_1 to obtain T_2. Thus, whether for the space (X, T), X is finite or X is infinite, there are no topological properties between T_1 and T_2.

A space (X, T) is T_2 iff for distinct elements x and y there exists an open set U containing x and $y \notin Cl(U)$. Then $y \in V = (X \setminus Cl(U))$, which is open, and $W = (Cl(U) \bigcap Cl(V))$ can be empty or nonempty. If $W \neq \phi$, then $x \in (U \setminus W)$, which is open, $y \in (V \setminus W)$, which is open, and x and y are separated by two disjoint open sets. Thus a minimal strengthening of T_2 is the separation of two distinct elements by open sets whose closures do not intersect.

A space (X, T) is Urysohn iff for distinct elements x and y in X there exist open sets U and V such that $x \in U$, $y \in V$, and $(Cl(U) \bigcap Cl(V)) = \phi$ [13]. Since there are T_2 spaces that are not Urysohn [13], then Urysohn is a minimal strengthening of T_2 and there are no topological properties between T_2 and Urysohn.

There are two avenues that can be used to possibly strengthen Urysohn; one internally and the other externally.

Consider the case that an attempt is made to strengthen Urysohn internally. Let (X, T) be Urysohn. Then (X, T) is T_1 and singleton sets are closed. Thus being able to separate an element x and a closed set C not containing x by open sets U and V such that $x \in U$, $C \subseteq V$, and $(Cl(U) \bigcap Cl(V)) = \phi$ would simultaneously separate two distinct elements by open sets whose closures are disjoint.

Theorem 2.4. *Let (X, T) be Urysohn and let W be a nonempty subset of X such that for each $x \notin W$, there exist disjoint open sets U and V such that $x \in U$ and $W \subseteq V$. Then W is closed.*

Proof. Suppose W is not closed. Since W is not closed and $Cl(W) = (W \cup D(W))$, where $D(W)$ is the derived set of W, then there exists an $x \in D(W)$ that is not in W, but then every open set containing x intersects every open set containing W, which is a contradiction. Thus W is closed.

If the property SSC of being T_1 and separating an element and a closed set not containing the element by open sets whose closures are disjoint is stronger than Urysohn, then the property SSC would be a minimal strengthening of Urysohn.

Theorem 2.5. *Let (X, T) be a space. Then (X, T) has property SSC iff it is T_1 and for each open set U and each $x \in U$, there exists an open set V such that $x \in V$ and $Cl(V) \subseteq U$.*

Proof. Suppose (X, T) has property SSC. Then (X, T) is T_1. Let O be open and let $x \in O$. Then $C = (X \setminus O)$ is closed and $x \notin C$. Let U and V be open sets such that $x \in U$, $C \subseteq V$, and $(Cl(U) \cap Cl(V)) = \phi$. Since $C = (X \setminus O) \subseteq V \subseteq Cl(V)$, then $(X \setminus Cl(V)) \subseteq (X \setminus V) \subseteq O$, where $(X \setminus V)$ is closed. Since $(Cl(U) \cap Cl(V)) = \phi$, then $U \subseteq Cl(U) \subseteq (X \setminus Cl(V)) \subseteq O$. Thus U is open, $x \in U$, and $Cl(U) \subseteq O$.

Conversely, suppose (X, T) is T_1 and for each open set U and each $x \in U$, there exists an open set V such that $x \in V$, and $Cl(V) \subseteq U$. Let C be a closed set and let $x \notin C$. Then $U = (X \setminus C)$ is open and $x \in O$. Let V be open such that $x \in V$ and $Cl(V) \subseteq U$. Then V is open and x in V. Let W be open such that $x \in W$ and $Cl(W) \subseteq V$. Then $C \subseteq (X \setminus Cl(V)) \subset (X \setminus V)$ is closed, and $x \in W \subseteq Cl(W) \subseteq V$. Thus $C \subseteq (X \setminus Cl(V))$, which is open, and $(C \setminus Cl(V)) \subseteq (X \setminus V)$, which is closed; and $x \in W$, which is open, and $(Cl(W) \subseteq V)$. Hence

$x \in W$, which is open, $C \subseteq (X \setminus Cl(V))$, which is open, and $Cl(W) \cap Cl(X \setminus Cl(V)) = \phi$.

In 1921, Vietoris [12] defined regular and T_3 spaces. A space is regular iff for each closed set C and each element not in C there exist disjoint open sets, one containing the element and the other containing C. A regular T_1 space is denoted by T_3. Since, from classical topology, a space is regular iff for each open set O and each $x \in O$, there exist open sets U such that $x \in U$ and $Cl(U) \subseteq O$, then, by the result above, T_3 and SSC are equivalent and since T_3 is stronger than Urysohn [13], then T_3 is a minimal strengthening of Urysohn and there are no topological properties between Urysohn and T_3.

Thus consider the case that an attempt is made to strengthen a Urysohn space (X, T) externally. In order to move forward externally, there would have to be a space (Y, S) and a connector between (X, T) and (Y, S). Since the objective is for (X, T) to be Urysohn using properties of (Y, S), then (Y, S) would need to have the Urysohn property and for distinct elements x and y in X, there exists a continuous function $f : (X, T) \to (Y, S)$ such that $f(x) \neq f(y)$.

Theorem 2.6. *Let* (X, T) *be a space. Then* (a) (X, T) *is Urysohn iff* (b) *for each nonempty subset* P *of* X *and each* $x \notin P$, *there exists a continuous function* $f : (X, T) \to (Y, S)$, *where* (Y, S) *satisfies the Urysohn property, such that* $f(x) \notin f(P)$, (c) *for each nonempty closed set* C *in* X *and each* $x \notin C$, *there exists a continuous function* $f : (X, T) \to (Y, S)$, *where* (Y, S) *satisfies the Urysohn property, such that* $f(x) \notin f(C)$, *and* (d) *for distinct elements* x *and* y *in* X, *there exists a continuous function* $f : (X, T) \setminus (Y, S)$, *where* (Y, S) *satisfies the Urysohn property, such that* $f(x) \neq f(y)$.

Proof. (a) implies (b): Let P be a nonempty subset of X and let $x \notin P$. Then the identity function $f : (X, T) \to (X, T)$ is continuous, (X, T) has the Urysohn property and $f(x) \neq f(P)$.

Clearly (b) implies (c) and (c) implies (d).

(d) implies (a): Let x and y be distinct elements in X. Let $f:(X, T) \to (Y, S)$, where (Y, S) satisfies the Urysohn property, such that $f(x) \neq f(y)$. Since (Y, S) satisfies the Urysohn property, there exist open sets U and V in Y such that $f(x) \in U$, $f(y) \in V$, and $(Cl(U) \cap Cl(V)) = \phi$. Then $x \in f^{-1}(U)$, which is open in X, $y \in f^{-1}(V)$, which is open in X, $f^{-1}(U) \subseteq f^{-1}(Cl(U))$, which is closed in X, $f^{-1}(V) \subseteq f^{-1}(Cl(V))$, which is closed in X, and, since $(Cl(U) \cap Cl(V)) = \phi$, $(Cl(f^{-1}(U)) \cap Cl(f^{-1}(V))) = \phi$. Thus (X, T) is Urysohn.

Focusing attention on Theorem 2.6(d), how can Urysohn possible be strengthened? The space (X, T) would have to satisfy the Urysohn property and a continuous function that would separate distinct elements of X as required would be required, leaving only a specific choice for the space (Y, S). If, in fact, the objective can be accomplished, to make the process as simple as possible, the choice for (Y, S) would need to be a familiar space immediately known to satisfy the Urysohn property and each of $f(x)$ and $f(y)$ assigned distinct values.

Completely Hausdorff spaces were introduced in classical topology and shown to exist and be stronger than Urysohn [13]. A space (X, T) is completely Hausdorff iff for distinct elements x and y in X there exists a continuous function $f:(X, T) \to (I, U)$, where $I = [0, 1]$ and U is the usual relative metric topology on I, such that $f(x) = 0$ and $f(y) = 1$. Since (I, U) is a familiar space immediately known to satisfy the Urysohn property and completely Hausdorff is stronger than Urysohn, then completely Hausdorff is a minimal strengthening of Urysohn and there are no topological properties between Urysohn and completely Hausdorff.

Since completely Hausdorff was obtained from Urysohn externally, then to strengthen completely Hausdorff the external process would need to be continued. Using the same process as above, a space (X, T) is completely Hausdorff iff for distinct elements x and y in X there exists a continuous function $f : (X, T) \to (I, U)$ such that $f(x) = 0$ and $f(y) = 1$. To possible strengthen completely Hausdorff, at least one of the two distinct elements would need to be expanded to a set containing more than one element.

Theorem 2.7. *Let (X, T) be completely Hausdorff. If P is a set such that for each $x \notin P$ there exists a continuous function $f : (X, T) \to (I, U)$ such that $f(x) = 0$ and $f(P) = 1$, then P is closed.*

Proof. Suppose P is not closed. Let $x \in D(P)$ such that $x \notin P$. Let $g : (X, T) \to (I, U)$ such that $g(x) = 0$ and $g(P) = 1$. Then $x \in g^{-1}([0, \frac{1}{3}))$, which is open in X, and $P \subseteq g^{-1}((\frac{2}{3}, 1])$, which is open in X, which is a contradiction. Thus P is closed.

Thus to possibly strengthen completely Hausdorff one of the two distinct points would have to be replaced by closed set. Also, in the given T_1 setting, separating a closed set and a point not in the closed set as required would simultaneously separate two distinct points in the required manner and the replacement of one of the two distinct elements by a closed set is a possible minimal strengthening of completely Hausdorff.

In 1925, Urysohn [11] introduced completely regular and $T_{3\frac{1}{2}}$ spaces. A space (X, T) is completely regular iff for each closed set C and each $x \notin C$ there exists a continuous function $f : (X, T) \to (I, U)$ such that $f(x) = 0$ and $f(C) = 1$. A completely regular T_1 space is denoted by $T_{3\frac{1}{2}}$. Since completely regular and $T_{3\frac{1}{2}}$ spaces exist and $T_{3\frac{1}{2}}$ is stronger than completely Hausdorff, then $T_{3\frac{1}{2}}$ is a minimal strengthening of completely Hausdorff and there are no topological properties between completely Hausdorff and $T_{3\frac{1}{2}}$.

T_3 spaces can possibly be strengthened both externally and internally. Consider the case an attempt is made to strengthen T_3 externally.

Theorem 2.8. *Let* (X, T) *be a space. Then* (X, T) *is* T_3 *iff it is* T_1 *and for each closed set* C *and each* $x \notin C$ *there is a continuous function* $f : (X, T) \to (Y, S)$, *where* (Y, S) *satisfies the* T_3 *property*, $f(C)$ *is closed*, *and* $f(x) \notin f(C)$.

Proof. Suppose (X, T) is T_3. Then (X, T) is T_1. Let C be a closed set in X and let $x \notin C$. Then the identity function $f : (X, T) \to (X, T)$ satisfies the stated requirements.

Conversely, suppose (X, T) is T_1 and for each closed set C in X and each $x \notin C$ there exists a continuous function $f : (X, T) \to (Y, S)$, where (Y, S) satisfies the T_3 property, $f(C)$ is closed in Y, and $f(x) \notin f(C)$. Let C be closed in X and let $x \notin C$. Let $f : (X, T) \to (Y, S)$ such that f is continuous, (Y, S) satisfies the T_3 property, $f(C)$ is closed in Y, and $f(x) \notin f(C)$. Let U and V be disjoint open sets in Y such that $f(x) \in U$ and $f(C) \subseteq V$. Then $x \in f^{-1}(U)$, $C \subseteq f^{-1}(V)$, and $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in X.

Then, by the discussion above, $T_{3\frac{1}{2}}$ is a minimal strengthening of T_3 and there are no topological properties between T_3 and $T_{3\frac{1}{2}}$.

How could $T_{3\frac{1}{2}}$ be possibly minimally strengthened? Instead of separating an element from a closed set as required for $T_{3\frac{1}{2}}$, separate a set possibly containing more than one element from a closed set as required by $T_{3\frac{1}{2}}$.

Theorem 2.9. *Let* (X, T) *be* $T_{3\frac{1}{2}}$ *and let* P *be a subset of* X *such that for each closed set* C *such that* $(C \cap P) = \phi$ *there exists a continuous function* $f : (X, T) \to (I, U)$ *such that* $f(C)$ *is closed,* $f(P) = 0,$ *and* $f(C) = 1.$ *Then* P *is closed.*

Proof. Suppose P is not closed. Let $x \in (D(P) \setminus P)$. Then $\{x\}$ is closed and $(\{x\} \cap P) = \phi$. Let $g : (X, T) \to (I, U)$ such that $g(\{x\})$ is closed in I, $g(P) = 0,$ and $g(\{x\}) = 1.$ Then $P \subseteq g^{-1}([0, \frac{1}{3})),$ which is open in X and $x \in g^{-1}((\frac{2}{3}, 1]),$ which is open in $X,$ which is a contradiction.

For $T_{3\frac{1}{2}}$ replacing the singleton set by a closed set, as given above, would simultaneously separate a closed set from an element not in the closed set as required by $T_{3\frac{1}{2}}$ and replacing the singleton set by a closed set could possibly be a strengthening of $T_{3\frac{1}{2}}$.

In 1923, Tietze [10] introduced normal and T_4 spaces. A space (X, T) is normal iff for disjoint closed sets C and D there exist disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. A normal T_1 space is denoted by T_4. In 1925, Urysohn [11] gave the following characterization of normal: A space (X, T) is normal iff for disjoint closed sets C and D in X there exists a continuous function $f : (X, T) \to (I, U)$ such that $f(C) = 0$ and $f(D) = 1.$ Since normal T_1 is T_4, T_4 exists and is stronger than completely regular, then T_4 is a minimal strengthening of $T_{3\frac{1}{2}}$ and there are no separation axioms between $T_{3\frac{1}{2}}$ and T_4.

The attempt to strengthen T_3 internally gives T_4 as a possible strengthening of T_3, but, as given above, T_4 is not a minimal strengthening of T_3.

Thus the results in the paper [3] have been verified.

3. Classical Separation Axioms Extended with No Between Topological Properties and Applications

In 1950, Alexandroff and Urysohn [1] introduced perfectly normal: A space (X, T) is perfectly normal iff it is T_1 and for disjoint closed sets C and D in X there exists a continuous function $f : (X, T) \to (I, U)$ such that $C = f^{-1}(0)$ and $D = f^{-1}(1)$. Is perfectly normal a minimal strengthening of T_4 ?

In their paper [1], Alexandroff and Urysohn gave the answer: A space is perfectly normal iff it is T_4 and every closed set is a G_δ. A subset of a topological space is a G_δ iff the subset is a countable intersection of open sets.

Since there are T_4 spaces that are not perfectly normal, then perfectly normal is a minimal strengthening of T_4 and there are no topological properties between T_4 and perfectly normal.

In 1977 [2], perfectly normal was generalized to perfectly Hausdorff: A space is perfectly Hausdorff iff for distinct elements x and y in X there exists a continuous function $f : (X, T) \to (I, U)$ such that $\{x\} = f^{-1}(0)$ and $\{y\} = f^{-1}(1)$.

Since a space is perfectly Hausdorff iff it is completely Hausdorff and each singleton set in X is a G_δ and perfectly Hausdorff is stronger than completely Hausdorff, then perfectly Hausdorff is a minimal strengthening of completely Hausdorff and there are no topological properties between completely Hausdorff and perfectly Hausdorff.

Clearly, if (X, T) be perfectly Hausdorff, P be a subset of X and for each $x \notin P$, there exists a continuous function $f : (X, T) \to (I, U)$ such that $\{x\} = f^{-1}(0)$ and $P = f^{-1}(1)$, then P is closed in X. Thus perfectly normal is a possible minimal strengthening of perfectly Hausdorff and since perfectly normal is stronger than perfectly Hausdorff [2], then perfectly normal is a minimal strengthening of perfectly Hausdorff and there are no topological properties between perfectly Hausdorff and perfectly normal.

Thus, after the needed investigation above, the proof in the paper [3] concerning no between topological properties is correct. Also, in one branch perfectly normal implies perfectly Hausdorff, which implies completely Hausdorff, which implies Urysohn, and in another branch perfectly normal implies T_4, which implies $T_{3\frac{1}{2}}$, which in one branch implies completely Hausdorff, and in another implies T_3, which implies Urysohn, which implies T_2, which implies T_1, which implies T_0 with none of the implications reversible, and the results in [3] are extended to include perfectly normal, and perfectly Hausdorff. Other applications of the work above include the following.

Corollary 3.1. *Let P, W, and Q be three consecutive separation axioms in the listing given immediately above. Then W is "isolated" in the sense that W is the only topological property between P and Q.*

Theorem 3.1. *Let Q be a separation axiom in the listing immediately above weaker than perfectly normal, let $A_Q = \{W \mid W$ is a topological property stronger than $Q\}$, and let P be the immediate predecessor of Q in the listing above. Then for each $W \in A_Q$, $W = (W$ and $T_1)$ and A_Q has least element P.*

Proof. Since P is stronger than Q, then $P \in A_Q$. Since there are no topological properties between P and Q, then A_Q has least element P. If $W \in A_Q$, then W implies $P = (P$ and $T_1)$ and $W = (W$ and $T_1)$.

Theorem 3.2. *Let Q be a separation axiom in the listing immediately above stronger than T_0, let $B_Q = \{W \mid W$ is a topological property weaker than Q and stronger than or equal to $T_0\}$, and let Z be the immediate successor of Q in the listing above. Then B_Q has strongest element Z.*

Proof. Since Q is greater than T_0, then Z is greater than or equal to T_0 and

$Z \in B_Q$ and since there are no topological properties between Q and Z, then B_Q has strongest element Z.

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