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NEW CHARACTERIZATIONS OF T_2 AND URYSOHN SPACES USING CLOSED SET, NEARLY OPEN SET, AND NEARLY CLOSED SET SUBSPACES

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Abstract

Within this paper, Urysohn and T_2 spaces are further characterized using semi open set, feebly open set, regular semi open set, and other related set subspaces.

1. Introduction

Within this paper, all spaces are topological spaces. Included within the study of classical topology are subspaces and the study of subspace properties. A topological property P is a subspace property if a space (X, T) has property P iff every subspace of (X, T) has property P. As is proven in classical studies of topology, many of the separation axioms are subspace properties, including the Urysohn and T_2 separation axioms. Thus the question arose: "Could the statement "every subspace" in the definition of subspace properties be replaced by a restricted

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collection of subspaces and the space continue to have the property?" This question has led to many new characterizations of separation axioms, including T_2 spaces [3] and Urysohn spaces [4]. Within a recent paper [5], T_2 spaces were further characterized using long-defined nearly open set subspaces, which are defined below. In this paper, T_2 spaces will be further characterized using nearly closed set subspaces and Urysohn spaces are further characterized using proper closed subspaces, and nearly open and nearly closed set subspaces. Below definitions needed for understanding in this paper are given.

Regularly open sets were introduced in 1937 [10].

Definition 1.2. Let (X, T) be a space and let A be a subset of X. Then A is regularly open, denoted by $A \in RO(X, T)$, if and only if $A = Int(Cl(A))$.

Within the 1937 paper [10] it was shown that the set of regularly open sets of a space (X, T) forms a base for a topology Ts on X coarser than T and the space (X, T_s) was called the semiregularization space of (X, T). The space (X, T) is semiregular if and only if the set of regularly open sets of (X, T) is a base for T [10].

The introduction of regularly open sets led to the introduction of regularly closed sets.

Definition 1.3. Let (X, T) be a space and let C be a subset of X. Then C is regularly closed, denoted by $A \in RC(X, T)$, iff one of the following equivalent conditions is satisfied: (1) $X \setminus C$ is regularly open and (2) $C = Cl(Int(C))$ [11].

In 1963 semi open sets were introduced [8].

Definition 1.4. Let (X, T) be a space and let $A \subseteq X$. Then A is semi open, denoted by $A \in SO(X, T)$, iff there exists an $O \in T$ such that $O \subseteq A \subseteq Cl(O)$.

In 1970, semi open sets were used to define semi closed sets and the semi closure of a set [1].

Definition 1.5. Let (X, T) be space and let $A, B \subseteq X$. Then A is semi closed, denoted by $A \in SC(X, T)$, iff $X \setminus A$ is semi open and the semi closure of B, denoted by $scl(B)$, is the intersection of all semi closed sets containing B.

In 1978 the semi closure operator was used to define feebly open sets, which was used to define feebly closed sets and the feebly closure operator [9].

Definition 1.6. Let (X, T) be a space and let $A, B, C \subseteq X$. Then A is feebly open, denoted by $A \in FO(X, T)$, iff there exists an open set O such that $O \subseteq A \subseteq scl(O)$, B is feebly closed, denoted by $B \in FC(X, T)$, iff $X \setminus B \in FO(X, T)$, and the feebly closure of C, denoted by $fcl(C)$, is the intersection of all feebly closed sets containing C.

In 1978, regular semi open and regularly semi closed sets were introduced [2].

Definition 1.7. Let (X, T) be a space and let $A, B \subseteq X$. Then A is regular semi open, denoted by $A \in RSO(X, T)$, iff there exists a $U \in RO(X, T)$ such that $U \subseteq A \subseteq Cl(U)$ and B is regularly semi closed, denoted by $B \in RSC(X, T)$, iff $X \setminus B$ is regularly semi open.

Within this paper T_2 spaces are further characterized using the nearly closed set subspaces defined above and Urysohn spaces a further characterized using proper closed subspaces, and both the nearly open and nearly closed set subspaces defined above.

2. Further Characterizations Using Nearly Open and Nearly Closed Subspaces

Theorem 2.1. *Let (X, T) be a space, let P be a subspace property, and let $\mathcal{C}$ be a collection of subsets of X that contains X. Then* (a) *(X, T) has property P, iff* (b) *for each element C of $\mathcal{C}$, (C, T_C) has property P.*

Proof. (a) implies (b): Since P is a subspace property, every subspace of (X, T) has property P, which implies (b).

(b) implies (a): Since X is in $\mathcal{C}$ and $(X, T_X) = (X, T)$, (X, T) satisfies property P.

Since both T_2 and Urysohn are subspace properties and for a space (X, T), X

is in each of $SO(X, T)$, $SC(X, T)$, $RO(X, T)$, $RC(X, T)$, $FO(X, T)$, $FC(X, T)$, $RSO(X, T)$, and $RSC(X, T)$, then, in Theorem 2.1, property P can be replaced by T_2 or Urysohn and C can be replaced by each of the nearly open or nearly closed sets given immediately above this statement. For T_2, the results above are new for $SC(C, T)$, $RC(X, T)$, $FC(X, T)$, and $RSC(X, T)$. For Urysohn, the results above are new except for $RO(X, T)$.

As observed above, and is true in classical studies, the proofs of the converse statement for subspace property theorems are the same, with the property itself only mentioned: "Since every subspace has property P and the space is a subspace of itself, then the space has property P." In 2014 [6], the feeling that the property itself should have a more meaningful role in the proof of the converse statement proofs led to the introduction and investigation of proper subspace inherited properties.

Definition 2.1. Let (X, T) be a space and let P be a topological property. If (X, T) has property P when every proper subspace of (X, T) has property P, then P is called a proper subspace inherited property.

In the 2014 paper [6], T_2 and Urysohn were included in those properties proven to be proper subspace inherited properties. In a recent paper [5], T_2 was further characterized using the nearly open sets above. Below T_2 is characterized using the proper nearly closed sets given above and Urysohn is further characterized using proper closed sets, proper nearly open, and proper nearly closed sets given above.

3. New Characterizations Using Proper Nearly Open and Nearly Closed Sets

When using proper subspaces, care must be taken to ensure there are proper subspaces and that the property under consideration can be used to extend the property from a proper subset of interest to the full set X. Thus, as was true above, the property itself has a more meaningful role when dealing with proper subsets of interest. Since one element spaces automatically satisfy many properties, within this paper, as in the earlier cited papers, the spaces considered here have three or more elements. Also, the results in the 2014 paper [3] in which T_2 was characterized using proper regularly open and proper regularly closed are extremely useful in the work below as does the fact that for a space (X, T), $RO(X, T) = \{Int_T(Cl_T((O)) : O \in T\}$ [7].

Theorem 3.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_2, (b) *for each* $C \in SC(X, T)$, (C, T_C) *is* T_2 *and for each z in X, there exists a proper T-open set O such that* $z \in O$ *and* $Cl_T(O)$ *is a proper subset of X*, (c) *for each* $C \in FC(X, T)$, (C, T_C) *is* T_2 *and for each z in X, there exists a proper regularly open set O such that* $z \in O$ *and* $Cl_T(O)$ *is a proper subset of X*, *and* (d) *for each* $C \in RSC(X, T)$, (C, T_C) *is* T_2 *and for each z in X, there exists a proper regularly open set O such that z is in O and* $Cl_T(O)$ *is a proper subset of X*.

Proof. By the results above and the fact that *X* has three or more elements, (a) implies (b).

(b) implies (c): Since $FC(X, T) \subseteq SC(X, T)$, then for each $C \in FC(X, T)$, (C, T_C) is T_2. Let *z* be in *X*. Let *O* be a proper *T*-open set such that *z* is in *O* and $Cl_T(O)$ is a proper subset of *X*. Then $z \in U = Int_T(Cl_T(O))$, which is a proper regularly open set and $Cl_T(U) = Cl_T(O)$ is a proper subset of *X*.

(c) implies (d): Since $RC(X, T) \subseteq FC(X, T)$, then for each $C \in RC(X, T)$, (C, T_C) is T_2 and for each *z* in *X*, there exists a proper regularly open set *O* such that *z* is in *O* and $Cl_T(O)$ is a proper subset of *X*, which implies (X, T) is T_2 [3]. Then (d) follows immediately from the results above.

(d) implies (a): Since $RC(X, T) \subseteq RSC(X, T)$, then for each $C \in RC(C, T)$, (C, T_C) is T_2 and for each *z* in *X*, there exists a proper regularly open set *O* such that *z* is in *O* and $Cl_T(O)$ is a proper subset of *X*, which implies (X, T) is T_2 [3].

Theorem 3.2. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is Urysohn*, (b) *for each proper T-closed set* C, (C, T_C) *is Urysohn and for distinct elements x and y in X, there exists a T-open set O such that x is in O and Y is not in* $Cl_T(O)$, (c) *for each proper T-closed set* C, (C, T_C) *is Urysohn and* (X, T) *is* T_2, (d) *for each proper regularly closed set* C, (C, T_C) *is Urysohn and for each z in X, there exists an open set O such that Z is in O and* $Cl_T(O)$ *is a* *proper subset of X*, (e) *for each proper regularly closed set* C, (C, T_C) *is Urysohn and for each z in X*, *there exists a regularly open set O such that z is in O and* $Cl_T(O)$ *is a proper subset of X*, *and* (f) *for each proper regularly closed set* C, (C, T_C) *is Urysohn and* (X, T) *is* T_2.

Proof. (a) implies (b): By the results above, for each proper *T*-closed set C, (C, T_C) is Urysohn. Let *x* and *y* be distinct elements of *X*. Since (X, T) is Urysohn, let *O* and *U* be in *T* such that *x* is in O, y is in *U*, and $Cl_T(O) \cap Cl_T(U) = \Phi$. Then *O* satisfies the desired property.

Clearly (b) implies (c). Since each regularly closed set is *T*-closed, then (c) implies (d), and by arguments similar to those above, (d) implies (e) and (e) implies (f).

(f) implies (a): Let *x* and *y* be distinct elements of *X*. Let *z* be an element of *X* distinct from *x* and *y*. Since (X, T) is T_2, let A, B, and *D* be pairwise disjoint open sets such that *x* is in A, y, is in *B*, and *z* is in *D*. Then $E = Int(Cl(D))$ is regularly open and $A \cup B \subseteq X \setminus E = C$, which is proper regularly closed. Let *F* and *G* be T_C-open sets such that *x* is in F, y is in *G*, and the T_C-closures of *F* and *G* are disjoint. Let $H = F \cap A$ and let $J = G \cap B$, which are disjoint *T*-open with *x* in *H* and *y* in *J*. Since the T_C-closure of *H* is a subset of the T_C-closure of *F* and *C* is *T*-closed, then both the T_C-closures of *H* and *F* are *T*-closed. Similarly, the T_C-closure of *G* is *T*-closed and a subset of the T_C-closure of *G*. Thus *H* and *J* are *T*-open sets such that *x* is in H, y is in *J*, and $Cl_T(H) \cap Cl_T(J) = \Phi$. Hence (X, T) is Urysohn.

Theorem 3.3. *Let* (X, T) *be a space*. *Then the following are equivalent*: (a) (X, T) *is Urysohn*, (b) *for each* $C \in SC(X, T)$, (C, T_C) *is Urysohn and for each z in X*, *there exists a proper T-open set O such that* $z \in O$ *and* $Cl_T(O)$ *is a proper subset of X*, (c) *for each* $C \in FC(X, T)$, (C, T_C) *is Urysohn and for each z in X*, *there exists a proper regularly open set O such that* $z \in O$ *and* $Cl_T(O)$ *is a proper subset of X*, *and* (d) *for each* $C \in RSC(X, T)$, (C, T_C) *is Urysohn and for each z in X*, *there exists a proper regularly open set O such that z is in O and* $Cl_T(O)$ *is a* *proper subset of X, and* (e) *for each* $C \in SC(X, T)$, (C, T_C) *is Urysohn and* (X, T) is T_2.

The proof is straightforward using Theorem 3.2 and an argument similar to that of Theorem 3.1 and is omitted.

References

[1] N. Biswas, On characterizations of semi continuous functions, Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur. 8(48) (1970), 399-402.

[2] D. Cameron, Properties of *S*-closed spaces, Proc. Amer. Math. Soc. 72 (1978), 581-586.

[3] C. Dorsett, New characterizations of T_2 spaces using regularly open sets, convergence, and subspaces, Fundamental J. Math. Math. Sci. 1(1) (2014), 31-40.

[4] C. Dorsett, New characterizations of Urysohn spaces using regularly open sets and subspaces, J. Ultra Scientists of Phys. Sci. 26(3) (2014).

[5] C. Dorsett, New characterizations of T_2 spaces using semi open set, feebly open set, regular semi open set, and related subspaces, accepted.

[6] C. Dorsett, Proper subspace inherited properties, new characterizations of classical topological properties, and related new properties, Questions and Answers in General Topology 32 (2014), 43-51.

[7] C. Dorsett, Regular open sets and *r*-topological properties, Nat. Acad. Sci. Letters 10(1) (1987), 17-21.

[8] N. Levine, Semi open sets and semi continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.

[9] S. Maheshwari and U. Tapi, Note on some applications of feebly open sets, Madhya Bharati J. Un. Saugar, 1978.

[10] M. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 374-481.

[11] S. Willard, General Topology, Addison-Wesley, 1970.