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NEW CHARACTERIZATIONS OF T_2 SPACES USING REGULARLY OPEN SETS, CONVERGENCE, AND SUBSPACES

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Abstract

Within this paper, T_2 spaces are further characterized using regularly open sets, convergence, and open, closed, regularly open, and regularly closed subspaces.

1. Introduction

Until recently, the question concerning subspaces for a property P within a topological space has been "Does a space have property P if and only if every subspace has property P? Within this paper, properties for which the above statement is true are called *subspace properties*. The proofs of the converse statement for the subspace theorem cited above are all the same with the property itself only mentioned: "Since the space is a subspace of itself and every subspace has the property, the space has the property." As a result proper subspace properties were

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introduced and investigated giving the properties themselves a new, meaningful role in subspace questions [1].

Definition 1.1. Let (X, T) be a space and let P be a property of topological spaces. If (X, T) has property P when every proper subspace of (X, T) has property P, then P is said to be a proper subspace inherited property [1].

Since singleton set spaces satisfy many topological properties, within the recent paper [1] only spaces with three or more elements were considered. Each of the subspace properties T_0, T_1, T_2, Urysohn, regular, and T_3 proved to be proper subspace inherited properties giving new characterizations for each of those properties.

Theorem 1.1. *A space* (X, T) *has property* P *iff every proper subspace of* (X, T) *has property* P, *where* P *can be each of the properties cited above* [1].

The results above raised the question of whether or not topological properties could be further characterized using only certain types of sets within the space as subspaces. With the important role of open and closed sets in the study of topology, a natural place to start such an investigation would be with open sets and closed sets subspaces, which led to new characterizations of T_0 spaces [2] followed by new characterizations of T_1 spaces [3].

The successes using open set and closed set subspaces for the T_0 and T_1 separation axioms raised questions about other types of sets in a topological space that could be considered for subspace questions leading to the consideration of regularly open sets.

Regularly open sets were introduced in 1937 [8].

Definition 1.2. Let (X, T) be a space and let A be a subset of X. Then A is regularly open if and only if $A = Int(Cl(A))$.

Within the 1937 paper [8], it was shown that the set of regularly open sets of a space (X, T) forms a base for a topology Ts on X coarser than T and the space (X, Ts) was called the *semiregularization space* of (X, T). The space (X, T) is semiregular if and only if the set of regularly open sets of (X, T) is a base for T [8].

The introduction of regularly open sets led to the introduction of regularly closed sets.

Definition 1.3. Let (X, T) be a space and let C be a subset of X. Then C is regularly closed iff one of the following equivalent conditions is satisfied: (1) $X \setminus C$ is regularly open and (2) $C = Cl(Int(C))$ [9].

The investigation of subspace questions for regularly open sets [4] led to the following discoveries, which are used in this paper. For a space (X, T), the regularly open sets of (X, T) equal the regularly open sets of (X, Ts). The semiregularization process generates at most one new topology. Thus a space (X, T) is semiregular if and only if $T = Ts$. For an open set O in a space (X, T), $(Ts)_O = (T_O)s$, which led to the discovery that semiregular is an open set subspace property, but not a proper open set inherited property. Examples are known showing that semiregular is not a closed set subspace property and that for a closed set C in a space (X, T), $(Ts)_C$ need not be $(T_C)s$.

In a follow-up paper [5], regularly open T_i spaces; $i = 0, 1, 2,$ were defined by replacing the word open in the definition of T_i by regularly open; $i = 0, 1, 2,$ respectively. Within the paper [6], regularly open T_0 spaces were further investigated using subspaces and in the paper [7], regularly open T_1 spaces were further investigated using subspaces. In this paper, T_2 spaces are further investigated using convergence and subspaces. As in the initial investigations cited above, [1] and [2], all spaces will have three or more elements. Also, as was the case for the investigation [2], care will be taken for proper subspaces with more than one element to ensure the resulting subspace topology is not the indiscrete topology.

2. Regularly open T_2 Spaces and Convergence

Within the paper [5], it was proven that for a space (X, T) the following are equivalent: (a) (X, T) is T_2, (b) (X, Ts) is T_2, and (c) (X, T) is regularly open T_2. Combining this result with the fact that $((Ts)s = Ts$, gives the next result.

Corollary 2.1. *A space (X, T) is regularly open T_2 iff (X, Ts) is regularly open T_2.*

Hence both T_2 and regularly open T_2 are semiregularization properties, i.e., properties simultaneously shared by both (X, T) and (X, Ts).

Combining a well-known characterization of T_2 spaces with the results above give the next result.

Corollary 2.2. *Let* (X, T) *be a space. Then the following are equivalent*: (*a*) (X, T) *is regularly open* T_2 *iff each net in X Ts-converges to at most one element.*

This result above motivated the following definition.

Definition 2.1. Let (X, T) be a space, let $\{x_a\}_{a \text{ in } A}$ be a net in *X*, and let *x* be in *X*. Then the net $\{x_a\}_{a \text{ in } A}$ regularly open converges to *x* iff it is eventually in each regularly open set containing *x*.

Corollary 2.3. *Let* (X, T) *be a space, let* $\{x_a\}_{a \text{ in } A}$ *be a net in X, and let x be in X. Then the net regularly open converges to x iff it Ts-converges to x.*

Corollary 2.4. *Let* (X, T) *be a space and let x be an element in X. If a net in X T-converves to x, then the net regularly open converges to x.*

The following example shows the converse of Corollary 2.4 is false.

Example 2.1. Let *X* be an infinite set, let *T* be the finite complement topology on *X*, and let x, y be in *X*, *x* not y. Then *Ts* is the indiscrete topology on *X*. For each natural number *n*, let $x_n = x$. Then the net $\{x_n\}$ regularly open converges to *y*, but does not *T*-converge to *y*.

Corollary 2.5. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_2, (b) *every net in X regularly open converges to at most one element, and* (c) *every net in X Ts-converges to at most one element of X.*

The results above raised the following question: "If (X, T) is T_2, and thus regularly open T_2, must $T = Ts$?" The following example shows the answer is "no."

Example 2.2. Let *X* denote the real numbers, let *R* denote the set of rational numbers, let *T* be the usual absolute metric topology on *X*, and let $S = T$ union $\{(a, b)$ intersection *R*: *a* and *b* are real numbers, $a < b\}$. Then *S* is a subbase for a topology *W* on X, (X, W) is T_2, and *W* is not *Ws*.

3. T_2 Spaces and Subspaces

Theorem 3.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_2, (b) *for each subset Y of* X, $(Y, (Ts)_Y)$ *is* T_2, *and* (c) *for each proper subset Y of* X, $(Y, (Ts)_Y)$ *is* T_2.

The proof is straightforward using the results above and is omitted.

Theorem 3.2. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_2, (b) *for each T-open set* O, (O, T_O) *is* T_2, (c) *for each T-open set* O, $(O, (T_O)s)$ *is* T_2, (d) *for each T-open set* O, $(O, (Ts)_O)$ *is* T_2, (e) *for each proper T-open set* O, (O, T_O) *is* T_2 *and for each z in X, there exists a proper T-open set containing z, and* (f) *for each proper T-open set* O, $(O, (Ts)_O)$ *is* T_2 *and for each z in* X, *there exists a proper T-open set containing z.*

Proof. By the results above (a) implies (b), (b) implies (c), and (c) implies (d).

(d) implies (e): Since *X* is *T*-open, $(X, (Ts)_X) = (X, Ts)$ is T_2, which implies (X, T) is T_2. Then by the results above, for each proper *T*-open set O, (O, T_O) is T_2. Let *z* be in *X*. Since *X* contains three or more elements, let *y* be in X, *y* not *z*. Since there exist disjoint open sets, one containing *z* and the other containing *y*, there exists a proper *T*-open set containing *z*.

(e) implies (f): Let *O* be a proper *T*-open set in *X*. Since (O, T_O) is T_2, $(O, (T_O)s)$ is T_2 and since $(T_O)s = (Ts)_O$, $(O, (Ts)_O)$ is T_2.

(f) implies (a): Let *x* and *y* be in X, *x* not *y*. Then there exist proper *T*-open sets, one containing *x* and the other *y*. Consider the case that there is a proper *T*-open *U* containing both *x* and *y*. Then $(U, (Ts)_U)$ is T_2. Let *V* and *W* be disjoint $(Ts)_U$-open sets such that *x* is in *V* and *y* is in *W*. Since *Ts* is weaker than *T*, then *V* and *W* are disjoint T_U-open sets, which are *T*-open in *X*. Thus consider the case that no proper *T*-open set contains both *x* and *y*.

Let *U* and *V* be proper *T*-open sets with *x* in *U* and *y* in *V*. If *U* and *V* are disjoint, then *U* and *V* are disjoint *T*-open sets containing *x* and *y*, respectively. Thus consider the case that *U* and *V* are not disjoint. Let *z* be in both *U* and *V*. Then

$(U, (Ts)_U) = (U, (T_U)s)$ is T_2. Hence (U, T_U) is T_2. Let A and B be disjoint T_U-open sets such that x is in A and z is in B. Thus A and B are disjoint T-open sets with x in A and z in B. In similar manner, let C and D be disjoint T-open sets with z in C and y in D. Then E, the intersection of U and A, F, the intersection of V and D, and G, the intersection of B and C, are mutually disjoint T-open sets with x in E, y in F, and z in G. Hence neither x nor y is in $Cl_T(\{z\})$ and both x and y are in $X \setminus Cl_T(\{z\})$, which is a proper T-open set, which, in this case, is a contradiction.

Therefore (X, T) is T_2.

Theorem 3.3. *Let (X, T) be a space. Then the following are equivalent*: (a) *(X, T) is T_2,* (b) *for each Ts-open set O, (O, T_O) is T_2,* (c) *for each Ts-open set O, $(O, (T_O)s)$ is T_2,* (d) *for each Ts-open set O, $(O, (Ts)_O)$ is T_2,* (e) *for each proper Ts-open set O, (O, T_O) is T_2 and for each z in X, there exists a proper Ts-open set containing z, and* (f) *for each proper Ts-open set O, $(O, (Ts)_O)$ is T_2 and for each z in X, there exists a proper Ts-open set containing z.*

The proof is straightforward and omitted.

Within the paper, the fact that for a space (X, T) and for each open set O, $Cl_T(O) = Cl_T(Int_T(O))$ is regularly closed was used. Below, this result will be used.

Theorem 3.4. *Let (X, T) be a space. Then the following are equivalent*: (a) *(X, T) is T_2,* (b) *for each regularly open set O, (O, T_O) is T_2,* (c) *for each regularly open set O, $(O, (T_O)s)$ is T_2,* (d) *for each regularly open set O, $(O, (Ts)_O)$ is T_2,* (e) *for each proper regularly open set O, (O, T_O) is T_2 and for each z in X, there exists a proper Ts-open set containing z,* (f) *for each proper regularly open set O, (O, T_O) is T_2 and for each z in X, there exists a proper regularly open set containing z,* (g) *for each proper regularly open set O, $(O, (T_O)s)$ is T_2 and for each z in X, there exists proper regularly open set containing z, and* (h) *for each proper regularly open set O, $(O, (Ts)_O)$ is T_2 and for each z in X, there exists a proper Ts-open set containing z.*

Proof. From the results above, (a) implies (b), (b) implies (c), and (c) implies (d).

(d) implies (e): Since X is regularly open, $(X, (Ts)_X) = (X, Ts)$ is T_2. Thus for each regularly open set O, (O, T_O) is T_2. Let z be in X. Let x be in X, x not y. Since there exist disjoint Ts-open sets, one containing x and the other containing z, there exists a proper Ts-open set containing z and thus a proper regularly open set containing z.

By the results above (e) implies (f), (f) implies (g), and (g) implies (h).

(h) implies (a): Let x and y be in X, x not y. Consider the case that there exists a proper regularly open set U containing both x and y. Then $(U, (Ts)_U)$ is T_2 and there exist disjoint $(Ts)_U$-open sets, which are T-open, one containing x and the other containing y.

Thus consider the case that there is no proper regularly open set containing both x and y. Let U and V be proper regularly open sets with x in U and y in V. If U and V are disjoint, then U and V are disjoint T-open sets with x in U and y in V. Thus consider the case that there exists a z in both U and V. Then x is not z and z is not y. Let A and B be disjoint regularly open sets such that x is in A, a subset of U, and z is in B, a subset of U. Similarly, let C and D be disjoint regularly open sets such that z is in C, a subset of V, and y is in D, a subset of V. Then A, B intersection C, and D are mutually disjoint Ts-open sets, which are T-open, and A and D are subsets of $X \setminus Cl_T(B$ intersection $C)$, which is a proper regularly open set and a contradiction.

Therefore (X, T) is T_2.

A set with two or more elements with the indiscrete topology is trivially proper T-open set T_2, but not T_2. The following example shows that T_2 is not a proper T-open set or proper Ts-open set or proper T-closed set or proper Ts-closed set inherited property.

Example 3.3. Let N denote the set of natural numbers and let T be the topology on N containing the empty set and all subset of N containing l. Then (N, T) is not T_l. There are no non empty proper Ts-open sets and for each proper T-closed set C, T_C is the discrete topology on C and $(T_C)s$ is the indiscrete topology on C.

Theorem 3.5. *Let* (X, T) *be a space. Then the following are equivalent*: (a)

(X, T) *is* T_2, (b) *for each T-closed set* C, (C, T_C) *is* T_2, (c) *for each x and y in* X, *x not y, there exists a T-open set O containing x with y not in the* $Cl_T(O)$, *and* (d) *for each proper T-closed set* C, (C, T_C) *is* T_2 *and for each z in X, there exists a T-open set U such that z is in U and* $Cl_T(U)$ *is a proper subset of X.*

Proof. By the results above (a) implies (b).

(b) implies (c): Since *X* is *T*-closed, (X, T) is T_2. Let *x* and *y* be in X, *x* not *y*. Let *U* and *V* be disjoint *T*-open sets with *x* in *U* and *y* in *V*. Then *x* is in *U* and *y* is not in $Cl_T(U)$.

Clearly (c) implies (d).

(d) implies (e): Suppose (X, T) is not T_2. Let *x* and *y* be in X, *x* not *y*, such that every *T*-open set containing *x* intersects every *T*-open set containing *y*. Let *U* and *V* be *T*-open sets such that *x* is in *U* and $Cl_T(U)$ is a proper subset of *X* and *y* is in *V* and $Cl_T(V)$ is a proper subset of *X*. Then *W*, the intersection of *U* and *V*, is a nonempty *T*-open set and $C = Cl_T(W)$ is a proper *T*-closed set of *X*. If *A* is *T*-open containing *x*, then the intersection of *A* and *W* is nonempty and *x* is in $C = Cl_T(W)$. Similarly *y* is in *C*. Let *D* and *E* be disjoint T_C-open sets with *x* in *D* and *y* in *E*. Let *F* and *G* be *T*-open sets such that *D* is the intersection of *C* and *F* and *D* is the intersection of *C* and *G*. Then the intersection of *F* and *U* and the intersection of *G* and *V* are disjoint *T*-open sets the first containing *x* and the second containing *y*, which is a contradiction. Thus (X, T) is T_2.

Theorem 3.6. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_2, (b) *for each Ts-closed set* C, $(C, (Ts)_C)$ *is* T_2, (c) *for each x and y in* X, *x not y, there exists a Ts-open set O containing x and y not in* $Cl_{Ts}(O)$, *and* (d) *for each proper Ts-closed set* C, $(C, (Ts)_C)$ *is* T_2 *and for each z in X, there exists a Ts-open set U such that z is in U and* $Cl_{Ts}(U)$ *is a proper subset of X.*

The proof is straightforward and omitted.

Theorem 3.7. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *is* T_2, (b) *for each regularly closed set* C, (C, T_C) *is* T_2, (c) *for x and y in* X, *x not y, there exists a regularly open set O such that x is in O and y is not in*

$Cl_T(O)$, *and* (d) *for each proper regularly closed set* C, (C, T_C) *is* T_2 *and for each z in X, there exists a regularly open set O such that z is in O and* $Cl_T(O)$ *is a proper subset of X.*

Proof. By the results above (a) implies (b).

(b) implies (c): Since *X* is regularly closed, (X, T) is T_2. Let *x* and *y* be in X, *x* not *y*. Let *U* be *T*-open such that *x* is in *U* and *y* is not in $Cl_T(U)$. Then *x* is in $Int_T(Cl_T(U))$, which is regularly open, and *y* is not in $Cl_T(U) = Cl_T(Int_T(Cl_T(U)))$.

(c) implies (d): Since Ts is a subset of T, (X, T) is T_2 and for each regularly closed set C, (C, T_C) is T_2. Since (X, T) is T_2, (X, Ts) is T_2. Let *z* be in *X*. Let *x* be in X, *x* not *z*. Then there exists a regularly open set *O* such that *z* is in *O* and *x* is not in $Cl_T(O)$.

(d) implies (a): Suppose (X, T) is not T_2. Then (X, Ts) is not T_2. Let *x* and *y* be in X, *x* not *y*, such that each *Ts*-open set containing *x* intersects every *Ts*-open set containing *y*. Let *U* and *V* be regularly open sets such that *x* is in U, *y* is not in $Cl_T(U)$, *y* is in *V*, and *x* is not in $Cl_T(V)$. Then *W*, the intersection of *U* and *V*, is *Ts*-open and *x* and *y* are in $Cl_{Ts}(W)$. Since T_s is a subset of T, *W* is *T*-open and $Cl_{Ts}(W) = Cl_T(W)$, which is regularly closed and a proper subset of *X*. Let *A* and *B* be disjoint $(Ts)_W$-open sets such hat *x* is in *A* and *y* is in *B*. Let *C* and *D* be *Ts*-open sets such that *A* is the intersection of *W* and *C* and *B* is the intersection of *W* and *B*. Then *x* is in *E*, the intersection of *U* and *C*, which is *Ts*-open, *y* is in *F*, the intersection of *V* and *D*, which is *Ts*-open, and *E* and *F* are disjoint, which is a contradiction. Thus (X, T) is T_2.

For each of the theorems above, the statement "(X, Ts) is T_2" could be added as an equivalent statement.

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