Fundamental Journal of Mathematics and Mathematical Sciences
Volume 9, Issue 1, 2018, Pages 47-56
This paper is available online at http://www.frdint.com/
Published online March 23, 2018

NECESSARY AND SUFFICIENT CONDITIONS FOR A SPACE TO HAVE A MAXIMAL, PROPER, DENSE, T_3 SUBSPACE

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Abstract

Within this paper, necessary and sufficient conditions for a space to have a maximal, proper, dense, T_3 subspace are given and subspaces properties of such spaces are further investigated.

1. Introduction and Preliminaries

In 1936 [9], for a space (X, T), an externally generated, strongly (X, T) related space, called the T_0 -identification space of (X, T), was introduced and used to further characterize each of metrizable and pseudometrizable.

Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of

Keywords and phrases: proper subspaces, dense subspaces, maximal T_3 subspaces.

2010 Mathematics Subject Classification: 54B05, 54D10.

Received February 14, 2018; Accepted February 23, 2018

X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, Y))$ is the T_0-identification space of (X, T).

Theorem 1.1. *A space* (X, T) *is pseudometrizable iff* $(X_0, Q(X, T))$ *is metrizable*.

T_0-identification spaces were cleverly created to add T_0 to the externally generated, strongly (X, T) related T_0-identification space of (X, T) [9], which, when combined with the fact that metrizable and (pseudometrizable and T_0) are equivalent, was used to establish the result above.

The characterization of metrizable and pseudometrizable given above using T_0-identification spaces raised the question: What other properties P not necessarily T_0 and (P and T_0), if any, would behave in the same manner as pseudometrizable and metrizable = (pseudometrizable and T_0)?, which led to the introduction and investigation of weakly Po properties in 2015 [1].

Definition 1.2. Let P be topological properties such that $P\text{o} = (P$ and $T_0)$ exists. Then a space (X, T) is weakly Po iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [1].

Since for a space, its T_0-identification space has property T_0, then, for a topological property P for which Po exists, a space is weakly Po iff its T_0-identification space has property Po.

Within the paper [1], it was shown that for a weakly Po property Qo, a space is weakly Qo iff its T_0-identification space is weakly Qo, which led to the introduction and investigation of T_0-identification P properties [2].

Definition 1.3. Let S be a topological property. Then S is a T_0-identification P

property iff both a space and its T_0-identification space simultaneously shares property *S*.

In the introductory weakly Po property paper [1], it was shown that weakly Po is neither T_0 nor "not-T_0", where "not-T_0" is the negation of T_0. The need and use of "not-T_0" revealed "not-T_0" as a useful, powerful topological property and tool, motivating the inclusion of the long-neglected properties "not-*P*", where *P* is a topological property for which "not-*P*" exists, as important properties for investigation and use in the study of topology. As a result, within a short time period, many new, important, fundamental, foundational, never before imagined properties and examples have been discovered, expanding and changing the study of topology forever.

In past studies of weakly Po spaces and properties, for a classical topological property Qo, a special topological property *W* was sought such that for a space with property *W*, its T_0-identification space has property Qo, which then implies the initial space has property *W*, with no certainty that such a topological property *W* exists. As given above, the study of weakly Po spaces and related properties has been a productive study, but, if the past search process continued, the study of weakly Po spaces and properties would continue to be uncertain, tedious, and never ending. To make the process more certain, the question of precisely which topological properties are weakly Po properties arose leading to answers in a 2017 paper [3].

Answer 1.1. Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is a T_0-identification P property that is weakly Po and $Q =$ weakly Qo = (Qo or (Q and "not-T_0")) [3].

Answer 1.2. $\{Q\text{o} \mid Q$ is a T_0-identification P property$\} = \{Q\text{o} \mid Q\text{o}$ is a weakly Po property$\} = \{Q\text{o} \mid Q$ is a topological property and Qo exists$\}$ [3].

Thus, major progress was achieved in the study of topology, weakly Po, and related properties. If Q is a topological property for which both Qo and (Q and "not-T_0") exist, Answer 1.1 quickly and easily gives weakly Qo. If Q is a topological property for which $Q = Q$o, then $Q = Q$o is a weakly Po property, but $Q = Q$o is not a T_0-identification P property or weakly Po. Within the paper [3], a topological property W that can be both T_0 and "not-T_0" was shown to exist that is a T_0-identification P property that is weakly Po such that $W =$ weakly Qo, again making the search process certain, but, just knowing such a W exists, gave little insight into the precise, needed topological property W, raising the question of whether the known information could somehow be used to more precisely determine W for the fixed Qo.

The investigation of that question led to the introduction and investigation of *OXTO* subsets and the corresponding subspace for each space (X, T) [4].

Definition 1.4. Let (X, T) be a space and for each $x \in X$, let C_x be the T_0-identification space equivalence class containing x. Then Y is an *OXTO* subset of X iff Y contains exactly one element from each equivalence class C_x.

Within the paper [4], it was shown that for a space (X, T), for each *OXTO* subset Y of X, (Y, T_Y) is homeomorphic to $(X_0, Q(X, T))$. Since, as stated earlier, the T_0-identification space of each space is T_0 and T_0 is a topological property, then for each *OXTO* subset Y of X in a space (X, T), (Y, T_Y) is T_0. Also, within the paper [4], it was shown that for each topological property Q for which Qo exists, a space (X, T) is weakly Qo iff for each *OXTO* subset Y of (X, T), (Y, T_Y) has property Qo, which can be, and has been, used to precisely determine weakly Qo [4]. Thus, the study of weakly Po and weakly Po properties has been completely internalized and greatly simplified by the use of *OXTO* subsets and the corresponding subspaces, and the earlier results, replacing many, earlier uncertainties with certainties.

The continued investigation of weakly Po properties, "not-T_0", and *OXTO* subsets and subspaces of a space led to the definition and unexpected results below.

Theorem 1.2. *Let* (X, T) *be a space. Then the following are equivalent*: (a)

(X, T) *is "not-*T_0*"*, (b) (X, T) *has a maximal, proper, dense* T_0 *subspace*, and (c) *for each OXTO subset Y* of X, (Y, T_Y) *is a maximal, proper, dense* T_0 subspace *of* (X, T) [5].

Definition 1.5. Let (X, T) be a space, let Y be a subspace of (X, T), and let Q be a topological property for which Qo exists. Then (Y, T_Y) is a maximal, proper, dense, Qo subspace of (X, T) iff (Y, T_Y) is a proper, dense, Qo subspace of (X, T) such that for each subspace $(Z, T_W))$ of (X, T), where Z properly containing Y, (Z, T_Z) is "not-Qo" [6].

Theorem 1.3. *Let* (X, T) *be a space and let* Q *be a topological property for which* Qo *exists. Then* (a) *for each OXTO subset Y* of X, (Y, T_Y) *is a maximal, proper, dense,* Qo *subspace of* (X, T) *iff* (b) (X, T) *is weakly* Qo *and* "not-T_0" [6].

The results above raised questions about necessary and sufficient conditions, if any, for a space to have a maximal, proper, dense, T_i subspace, $i = 1, 2,$ which were resolved in the paper [7], raising the question of necessary and sufficient conditions, if any, for a space to have a maximal, proper, dense, T_3 subspace.

The regular and T_3 separation axioms were introduced in 1921 [10].

Definition 1.6. A space (X, T) is regular iff for each closed set C and each $x \notin C$, there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$. A regular T_1 space is denoted by T_3.

The work in this paper focuses attention on "not-T_3", whose definition is given below.

Definition 1.7. A space is "not-T_3" iff it is "not-T_1" or "not-regular".

In 2016 [8], it was shown that T_3 is a weakly Po property with regular = weakly (regular)o = weakly T_3. Below Theorem 1.3 is applied to regular and T_3, giving necessary and sufficient conditions in "not-T_0" spaces for a space to have a maximal, proper, dense, T_3 subspace and the question above for maximal, proper, dense, T_3 subspaces is resolved.

2. Weakly T_3 and "not-T_0" Spaces and Necessary and Sufficient Conditions on a Space for the Space to have a Maximal, Proper, Dense, T_3 Subspace

Corollary 2.1. *Let* (X, T) *be a space. Then* (a) *for each OXTO subset Y of* X, (Y, T_Y) *is a maximal, proper, dense,* T_3 *subspace of* (X, T) *iff* (b) (X, T) *is* (*regular* $=$ *weakly* (*regular*)o) *and* "*not-*T_0".

Theorem 2.1. Let (X, T) *be a space and let Y be a proper subset of X. Then the following are equivalent*: (a) (X, T) *has a maximal, proper, dense,* T_3 *subspace* (Y, T_Y), (b) (X, T) *is* "*not-*T_3", (Y, T_Y) *is a proper,* T_3 *subspace of* (X, T), *and for each* $x \in X \setminus Y$, $W = \{x\} \cup Y$, *and* $C_W(x) = \{y \in W \mid Cl_W(\{x\}) = Cl_W(\{y\})\}$, *every T-open set containing x intersects* Y, $C_W(x)$ *contains at most two elements, and if* $C_W(x) = \{x\}$ *and* $\{x\}$ *is not* T_W*-closed,* (W, T_W) *is* "*not-*T_1" *or* ($\{x\}$ *is* T_W*-closed,* (W, T_W) is T_1, *and there exists a* T_W*-closed set C and a* $z \in W \setminus C$ *such that every* T_W*-open set containing* z intersects every T_W*-open set containing C, where* $x \in C$, $z \in Y$, *and for each* T_W*-open set A containing* z, $x \in Cl_W(A)$ *or C is a* T_W-, T_Y*-closed set such that for each* T_W*-open set B containing* C, $x \in Cl_W(B)$.

Proof. (a) implies (b): Since (Y, T_Y) is a maximal, proper, T_3 subspace of (X, T) and T_3 is a subspace property, then (X, T) is "not-T_3" and (Y, T_Y) is a proper, T_3 subspace of (X, T). If $\{x\}$ is T_W-open, then there exists a *T*-open set *O* containing *x* such that $O \cap Y = \phi$ and (Y, T_Y) is not dense in (X, T). Thus $\{x\}$ is not T_W-open. Suppose $C_W(x)$ contains three or more elements. Let x, a, and b be distinct elements of $C_W(x)$. Then $Cl_W(\{a\}) = Cl_W(\{x\}) = Cl_W(\{b\})$. Thus $Cl_Y(\{a\}) = Cl_Y(\{b\})$ and (Y, T_Y) is "not-T_0" and hence "not-T_1", which is a contradiction. Suppose $C_W(x) = \{x, a\}$, $a \neq x$. Then $Cl_W(\{x\}) = Cl_W(\{a\})$ and (W, T_W) is "not-T_0", which implies (W, T_W) is "not-T_1" and thus, "not-T_3". Suppose $C_W(x) = \{x\}$ and $\{x\}$ is not T_W-closed. Then (there exists a $y \in Y$ such that $y \in Cl_W(\{x\})$ and $x \notin Cl_W(\{y\})$ or $y \notin Cl_W(\{x\})$ and $x \in Cl_W(\{y\})$ or for all $y \in Y$, $x \notin Cl_W(\{y\})$ and $y \notin Cl_W(\{x\})$. In either of the first two cases, (W, T_W) is "not-T_1" and, thus, (W, T_W) is "not-T_3". Thus, consider the third case. Since for each $y \in Y$, $y \notin Cl_W(\{x\})$, then $Cl_W(\{x\}) = \{x\}$. Thus $Y = W \setminus \{x\}$ is T_W-, T_Y-open and each T_Y-open set is T_W-open. Since (Y, T_Y) is T_1, $x \notin Cl_W(\{y\})$ for all $y \in Y$, and $y \notin Cl_W(\{x\})$, for all $y \in Y$, then (W, T_W) is T_1. Since (W, T_W) is "not-T_3" and T_1, then (W, T_W) is "not-regular" and there exists a T_W-closed set C and a $z \notin C$ such that every T_W-open set containing z intersects every T_W-open set containing C. Then $C \subset Y$ and $z \in Y$ or $\{x\} \subset C$ and $z \in Y$ or $C \subseteq Y$ and $z = x$. Consider the case that $C \subset Y$ and $z \in Y$. Since (Y, T_Y) is T_3, there exist disjoint T_W-, T_Y-open sets A and B such that $z \in A$ and $C \subseteq B$, which is a contradiction. Thus $x \in C$ and $z \in Y$ or $C \subseteq Y$ and $z = x$. Consider the case that $x \in C$ and $z \in Y$. Then $C = \{x\}$ or $\{x\}$ is a proper subset of C. Consider the case that $C = \{x\}$. Let $A \in T_W$ such that $z \in A$. Then $x \in Cl_W(A)$ since, otherwise, $x \in (B = W \setminus Cl_W(A)) \in T_W$, $z \in A \in T_W$, and $A \cap B = \phi$, which is a contradiction. Consider the case that $\{x\}$ is a proper subset of C. Let $D = C \cap Y$, which is T_Y-closed. Let A and B be disjoint T_W-, T_Y-open sets such that $z \in A$ and $D \subseteq B$. Suppose there exist disjoint T_W-open sets E and F such that $z \in E$ and $x \in F$. Then $z \in (A \cap E) \in T_W$, $C \subseteq (B \cup F) \in T_W$, and $(A \cap E) \cap (B \cup F) = \phi$, which is a contradiction. Thus every T_W-open set containing z intersects every T_W-open set containing C and for each T_W-open set G containing z, $x \in Cl_W(G)$. Consider the case that $z = x$ and $C \subseteq Y$. Let A be a T_W-open set containing C.

Since every T_W -open set containing x intersects every T_W -open set containing C, then $x \in Cl_W(A)$.

Clearly (b) implies (c).

(c) implies (a): Since every T-open set containing x intersects Y, then (Y, T_Y) is dense in (X, T) and (Y, T_Y) is a proper, dense, T_3 subspace of (X, T). If $C_W(x) = \{x, a\}$, $a \neq x$, then $Cl_W(\{x\} = Cl_W(\{a\})$ and (W, T_W) is "not-T_0" and thus "not-T_1" and "not-T_3" and (Y, T_Y) is a maximal, proper, dense, T_3 subspace of (X, T). If $C_W(x) = \{x\}$ and $\{x\}$ is not T_W -closed, then (W, T_W) is "not-T_1" and thus "not-T_3" and (Y, T_Y) is a maximal, proper, dense, T_3 subspace of (X, T). If $C_W(x) = \{x\}$ and $\{x\}$ is T_W -closed, then (W, T_W) is T_1 and there exists a T_W -closed set C and a $z \notin C$ such that every T_W -open set containing z intersects every T_W -open set containing C. Thus (W, T_W) is "not-regular", which implies (W, T_W) is "not-T_3" and (Y, T_Y) is a maximal, proper, dense, T_3 subspace of (X, T).

3. An Additional Characterization and Properties of Such Spaces

Theorem 3.1. *Let (X, T) be a space and let Y be a proper subset of X. Then* (a) (Y, T_Y) *is a maximal, proper, dense,* T_3 *subspace of* (X, T) *iff* (b) *for each subset Z of X that properly contains Y,* (Z, T_Z) *is "not-*T_3*" and each subspace of* (Y, T_Y) *is* T_3.

Proof. (a) implies (b): Since (Y, T_Y) is T_3 and T_3 is a subspace property, then each subspace of (Y, T_Y) is T_3. Since (Y, T_Y) is a maximal, proper, dense, T_3 subspace of (X, T), then, by definition, for each subset Z of X that properly contains Y, (Z, T_Z) is "not-T_3".

(b) implies (a): Since Y is a subspace of itself, then (Y, T_Y) is a proper, T_3 subspace of (X, T) and since for each subset Z of X that properly contains Y,

(Z, T_Z) is "not-T_3", then, by definition, (Y, T_Y) is a maximal, proper, dense, T_3 subspace of (X, T).

Definition 3.1. A space (X, T) has the maximal, proper, dense, T_3 subspace property iff (X, T) has a maximal, proper, dense, T_3 subspace of (Y, T_Y).

Theorem 3.2. *The maximal, proper, dense, T_3 subspace property is a topological property.*

Proof. Let (X, T) have the maximal, proper, dense, T_3 subspace property and let $f : (X, T) \to (Z, S)$ be a homeomorphism. Let (Y, T_Y) be a maximal, proper, dense, T_3 subspace of (X, T). Then $f(Y)$ is a proper subset of *Z* and since T_3 is a topological property and $f_{f(Y)} : (Y, T_Y) \to (f(Y), S_{f(Y)})$ is a homeomorphism, then $(f(Y), S_{f(Y)})$ is a proper, T_3 subspace of (Z, S). Let $z \in Z \setminus f(Y)$ and let $U \in S$ such that $z \in U$. Let *x* be the unique element of *X* such that $f(x) = z$. Then $x \notin Y$ and $x \in f^{-1}(U) \in T$ and, since (Y, T_Y) is dense in (X, T), $f^{-1}(U) \cap Y \neq \phi$. Thus $U \cap f(Y) \neq \phi$ and $(f(Y), S_{f(Y)})$ is dense in (Z, S). Suppose there exists a subset *W* of *Z* that properly contains $f(Y)$ that is T_3. Then $f^{-1}(W)$ is a subset of *X* that properly contains *Y* and $(f^{-1}(W), T_{f^{-1}(W)})$ is T_3, which is a contradiction. Thus for each subset *W* of *Z* that properly contains $f(Y)$, (W, S_W) is "not-T_3". Thus $f(Y), (S_{f(Y)})$ is a maximal, proper, dense, T_3 subspace of (Z, S).

Theorem 3.3. *The maximal, proper, dense, T_3 subspace property is not a subspace property.*

Proof. Let (X, T) have the maximal, proper, dense, T_3 subspace property with maximal, proper, dense, T_3 subspace (Y, T_Y) having two or more elements. Let a and *b* be distinct elements of *Y*, let $Z = \{a\}$, and let $W = \{a, b\}$. Then (Z, T_Z) is a proper, T_3 subspace of (W, T_W), where (W, T_W) is T_3 and "not-"not-T_3" ". Thus the maximal, proper, dense, T_3 subspace property is not a subspace property.

The following example shows that for a space (X, T), a maximal, proper, dense, T_3 subspace of (X, T) need not be unique.

Example 3.1. Let $X = \{a, b\}$, where a and b are distinct, and let T be the indiscrete topology on X. Then each singleton set subspace of (X, T) is a maximal, proper, dense, T_3 subspace of (X, T) and maximal, proper, dense, T_3 subspaces are not unique.

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