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NECESSARY AND SUFFICIENT CONDITIONS FOR A SPACE TO HAVE A MAXIMAL, PROPER, DENSE, T_i SUBSPACE, $i = 1, 2$

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Abstract

Within this paper, necessary and sufficient conditions for a space to have a maximal, proper, dense, T_i subspace, $i = 1, 2$, are given and the properties of such subspaces are investigated.

1. Introduction and Preliminaries

In 1936 [10], for a space (X, T), an externally generated, strongly (X, T) related space, called the T_0-identification space of (X, T), was introduced and used to further characterize each of metrizable and pseudometrizable.

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Definition 1.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \rightarrow X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, Y))$ is the T_0-identification space of (X, T).

Theorem 1.1. *A space (X, T) is pseudometrizable iff $(X_0, Q(X, Y))$ is metrizable.*

T_0-identification spaces were cleverly created to add T_0 to the externally generated, strongly (X, T) related T_0-identification space of (X, T) [10], which, when combined with the fact that metrizable and (pseudometrizable and T_0) are equivalent, was used to establish the result above.

The characterization of metrizable given above raised the following questions: What property, if any, together with T_i would be equivalent to T_{i+1}, $i = 0, 1$, respectively?, which led to the introduction and investigation of the R_0 and R_1 separation axioms.

The R_0 separation axiom was introduced in 1943 [9].

Definition 1.2. A space (X, T) is R_0 iff for each closed set C and each $x \notin C$, $C \bigcap Cl(\{x\}) = \phi$.

In 1961 [1], the R_0 separation axiom was rediscovered and used to resolve the question above for T_1 and the R_1 separation axiom was introduced and used to resolve the question above for T_2.

Definition 1.3. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 1.2. *A space is T_i iff it is $(R_{i-1}$ and $T_{i-1})$; $i = 1, 2$, respectively* [1].

In the 1961 paper [1], it was shown that R_1 implies R_0, which was used with the result above to show a space is T_2 iff it is $(R_1$ and $T_0)$. Thus the questions of whether T_0-identification spaces could be used to further characterize R_i and T_{i+1}, $i = 0, 1$, as in the case of pseudometrizable and metrizable given above, arose.

In 1975 [8], T_0-identification spaces were used to further characterize R_1 and T_2.

Theorem 1.3. *A space* (X, T) *is* R_1 *iff its* T_0*-identification space* $(X_0, Q(X, T))$ *is* T_2 [8].

Within the paper [2], the metrizable and T_2 properties were generalized to weakly Po properties.

Definition 1.4. Let P be topological properties such that $P\text{o} = (P$ and $T_0)$ exists. Then a space (X, T) is weakly Po iff its T_0-identification space $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property [2].

Since for a space, its T_0-identification space has property T_0, then, for a topological property P for which Po exists, a space is weakly Po iff its T_0-identification space has property Po.

Within the paper [2], it was shown that for a weakly Po property Qo, a space is weakly Qo iff its T_0-identification space is weakly Qo, which led to the introduction and investigation of T_0-identification P properties [4].

Definition 1.5. Let S be a topological property. Then S is a T_0-identification P property iff both a space and its T_0-identification space simultaneously shares property S.

In the introductory weakly Po property paper [2], it was shown that R_0 = weakly (R_0)o = weakly T_1 and that weakly Po is neither T_0 nor "not-T_0", where "not-T_0" is the negation of T_0. The need and use of "not-T_0" revealed "not-T_0" as a useful topological property and tool, motivating the inclusion of the long-neglected properties "not-*P*", where *P* is a topological property for which "not-*P*" exists, as important properties for investigation and use in the study of topology. As a result, within a short time period, many new, important, fundamental, foundational, never before imagined properties have been discovered, expanding and changing the study of topology forever.

In past studies of weakly Po spaces and properties, for a classical topological property Qo, a special topological property *W* was sought such that for a space with property *W*, its T_0-identification space has property Qo, which then implied the initial space has property *W*, with no certainty that such a topological property *W* exists. As given above, the study of weakly Po spaces and related properties has been a productive study, but, if the past search process continued, the study of weakly Po spaces and properties would continue to be uncertain, tedious, and never ending. To make the process more certain, the question of precisely which topological properties are weakly Po properties arose leading to answers in a 2017 paper [4].

Answer 1.1. Let Q be a topological property for which both Qo and (Q and "not-T_0") exist. Then Q is a T_0-identification P property that is weakly Po and $Q =$ weakly Qo $= (Q$o or (Q and "not-T_0")) [4].

Answer 1.2. $\{Q\text{o} \mid Q$ is a T_0-identification P property$\}$ $= \{Q\text{o} \mid Q\text{o}$ is a weakly Po property$\}$ $= \{Q\text{o} \mid Q$ is a topological property and Qo exists$\}$ [4].

Thus, major progress was achieved in the study of weakly Po and related properties. If Q is a topological property for which both Qo and (Q and "not-T_0") exist, Answer 1.1 quickly and easily gives weakly Qo. If Q is a topological property for which $Q = Q$o, then $Q = Q$o is a weakly Po property, but $Q = Q$o is not a T_0-identification P property or weakly Po. Within the paper [4], a topological property *W* that can be both T_0 and "not-T_0" was shown to exist that is a T_0-identification P property that is weakly Po such that $W =$ weakly Qo, again making the search process certain, but, just knowing such a *W* exists, gave little insight into the precise, needed topological property *W*, raising the question of whether the known information could somehow be used to more precisely determine *W* for the fixed Qo.

The investigation of that question led to the introduction and investigation of *OXTO* subsets and the corresponding subspace for each space (X, T) [5].

Definition 1.6. Let (X, T) be a space and for each $x \in X$, let C_x be the T_0-identification space equivalence class containing *x*. Then *Y* is an *OXTO* is a subset of *X* iff *Y* contains exactly one element from each equivalence class C_x.

Within the paper [5], it was shown that for each *OXTO* subset *Y* of X, (Y, T_Y) is homeomorphic to $(X_0, Q(X, T))$. Since, as stated earlier, the T_0-identification space of each space is T_0 and T_0 is a topological property, then for each *OXTO* subset *Y* of *X* in a space (X, T), (Y, T_Y) is T_0. Also, within the paper [5], it was shown that for each topological property *Q* for which Qo exists, a space (X, T) is weakly Qo iff for each *OXTO* subset *Y* of X, (Y, T_Y) has property Qo, which can be, and has been, used to precisely determine weakly Qo [5]. Thus, the study of weakly Po and weakly Po properties has been completely internalized and greatly simplified by the use of *OXTO* subsets and the corresponding subspaces and the earlier results, replacing many, earlier uncertainties with certainties.

As given above, the behavior of R_0 and R_1 in T_0 spaces is long-known raising questions about their behavior in “not-T_0” spaces. The investigation of those questions again revealed “not-T_0” as a strong, useful topological tool whose use continued to reveal additional never before imagined properties in the study of topology, including the results below.

Theorem 1.4. *Let* (X, T) *be a space*. *Then the following are equivalent*: (a) (X, T) is “*not*-T_0”, (b) (X, T) *has a maximal*, *proper*, *dense* T_0 *subspace*, *and* (c) *for each OXTO subset Y of* X, (Y, T_Y) *is a maximal*, *proper*, *dense* T_0 *subspace of* (X, T) [6].

Theorem 1.5. *Let* (X, T) be a "not-T_0" *space. Then* (X, T) *is* R_0 *iff for each OXTO subset Y of* X, (Y, T_Y) *is a maximal, proper, dense* T_1 *subspace of* (X, T) [6].

Theorem 1.6. *Let* (X, T) *be a* "*not*-T_0" *space. Then* (X, T) *is* R_1 *iff for each OXTO subset Y of* X, (Y, T_Y) *is a maximal, proper, dense* T_2 *subspace of* (X, T) [6].

The results above raised the following question: Can the results above be extended to all weakly Qo and "not-T_0" properties?, which was resolved in the paper [7].

Definition 1.7. Let (X, T) be a space, let Y be a subspace of (X, T), and let Q be a topological property for which Qo exists. Then (Y, T_Y) is a maximal, proper, dense, Qo subspace of (X, T) iff (Y, T_Y) is a proper, dense, Qo subspace of (X, T) such that for each subspace (Z, T_W) of (X, T), where Z properly containing Y, (Z, T_Z) is not Qo.

Theorem 1.7. *Let* (X, T) *be a space and let Q be a topological property for which* Qo *exists. Then* (a) *for each OXTO subset Y of* X, (Y, T_Y) *is a maximal, proper, dense,* Qo *subspace of* (X, T) *iff* (b) (X, T) *is weakly* Qo *and* "*not*-T_0".

Theorem 1.7 can be applied to each of R_0 = weakly T_1 and R_1 = weakly T_2 improving the earlier results given above. However, simple examples can be given of spaces that are T_0 and "not-R_i", $i = 0, 1,$ which have a maximal, proper, dense T_i subspace, $i = 1, 2,$ raising the questions: What condition, if any, on a space will be equivalent to the space having a maximal, proper, dense, T_i subspace; $i = 1, 2$? In the work below, those questions are resolved.

2. Necessary and Sufficient Conditions on a Space to Answer the T_1 Question

Theorem 2.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *has a maximal, proper, dense,* T_1 *subspace* (Y, T_Y), (b) (X, T) *is "not-*T_1*" and there exists a proper* T_1 *subspace* (Z, T_Z) *of* (X, T) *such that for each* $x \in X \setminus Z$, *there exists a* $y \in Z$ *such that* $x \in Cl(\{y\})$, *and* (c) *there exists a proper* T_1 *subspace* (Z, T_Z) *of* (X, T) *such that for each* $x \in X \setminus Z$, *there exists a* $y \in Z$ *such that* $x \in Cl(\{y\})$.

Proof. (a) implies (b): Since T_1 is a subspace property and (Y, T_Y) is a maximal, proper, T_1 subspace of (X, T), then (X, T) is "not-T_1". Let $Z = Y$. Then (Z, T_Z) is a proper T_1 subspace of (X, T). Suppose there exists a $x \in X \setminus Z$ such that $x \notin Cl(\{y\})$ for each $y \in Z$. Let $W = Z \cup \{x\}$. Since $(Z, T_Z) = (Y, T_Y)$ is maximal T_1, then there exist distinct elements *u* and *v* in *W* such that every open set containing *u* contains *v* or every open set containing *v* contains *u*, say every open set containing *u* contains *v*. Since $Z = Y$ and (Y, T_Y) is T_1, then $u \notin Z$. Thus $u = x$, but, then $x \in X \setminus Cl(\{v\})$, which is open and does not contain *v*, which is a contradiction. Thus, for each $x \in X \setminus Z$, there exists a $y \in Z$ such that $x \in Cl(\{y\})$.

Clearly (b) implies (c).

(c) implies (a): Let (Z, T_Z) be a proper T_1 subspace of (X, T) such that for each $x \in X \setminus Z$, there exists a $y \in Z$ such that $x \in Cl(\{y\})$. Let $x \in X \setminus Z$. Let $y \in Z$ such that $x \in Cl(\{y\})$. Then $x \in Cl(\{y\}) \subseteq Cl(Z)$. Hence, $Cl(Z) = X$ and (Z, T_Z) is dense in (X, T). Let *W* be a subset of *X* that properly contains *Z*. Let $u \in W \setminus Z$. Let $v \in Z$ such that $u \in Cl(\{v\})$. Then every open set containing *u* contains *v* and (W, T_W) is "not-T_1". Hence, (Z, T_Z) is a maximal, proper, dense, T_1 subspace of (X, T).

Theorem 2.2. *Let* (X, T) *be a space that has a maximal, proper, dense,* T_1 *subspace and let Y be a subset of X. Then* (Y, T_Y) *is a maximal, proper, dense,* T_1 *subspace of* (X, T) *iff* (Y, T_Y) *is a proper,* T_1 *subspace of* (X, T) *such that for each* $x \in X \setminus Y$, *there exists a* $y \in Y$ *such that* $x \in Cl(\{y\})$.

Proof. If (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (X, T), then, by the argument above (a) implies (b), (Y, T_Y) is a proper T_1 subspace of (X, T) such that for each $x \in X \setminus Y$, there exists a $y \in Y$ such that $x \in Cl(\{y\})$.

If (Y, T_Y) is a proper T_1 subspace of (X, T) such that for each $x \in X \setminus Y$, there exists a $y \in Y$ such that $x \in Cl(\{y\})$, then, by the argument above (c) implies (a), (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (X, T).

Thus, the condition on a space that is equivalent to the space having a maximal, proper, dense, T_1 subspace immediately gives a maximal, proper, dense, T_1 subspace of the space.

Theorem 2.3. *Let* (X, T) *be a space and let Y be a subset of X. Then* (Y, T_Y) *is a maximal, proper, dense,* T_1 *subspace of* (X, T) *iff for each subset Z of X that properly contains* Y, (Y, T_Y) *is a maximal, proper, dense,* T_1 *subspace of* (Z, T_Z).

Proof. Suppose (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (X, T). Then (Y, T_Y) is a proper, T_1 subspace of (X, T) such that for each $x \in X \setminus Y$, there exists a $y \in Y$ such that $x \in Cl(\{y\})$. Since Z properly contains Y, then (Y, T_Y) is a proper subspace of (Z, T_Z). Let u and v be distinct elements of Y. Let U and V be open sets in X such that $u \in U$ and $v \notin U$ and $v \in V$ and $u \notin V$. Then $A = U \cap Z$ and $B = V \cap Z$ are open sets in Z such that $u \in A$ and $v \notin A$ and $v \in B$ and $u \notin B$. Thus (Y, T_Y) is a T_1 subspace of (Z, T_Z). Let $z \in Z \setminus Y$. Then $z \in X \setminus Y$ and there exists a $y \in Y$ such that $z \in Cl(\{y\})$. Since $Cl_Z(\{y\}) = Z \cap Cl(\{y\})$, then $z \in Cl_Z(\{y\})$. Thus, by the results above, (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (Z, T_Z).

Conversely, suppose that for each subset Z of X that properly contains Y, (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (Z, T_Z). Since X is a subset of X that properly contains Y, then (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (X, T).

Corollary 2.1. *Let (X, T) be a space and let Y be a subset of X such that (Y, T_Y) is a maximal, proper, dense, T_1 subspace of (X, T). Then for each subset Z of X that properly contains Y, (Z, T_Z) is "not-T_1" and for each subset W of Y, (W, T_W) is T_1.*

3. Necessary and Sufficient Conditions on a Space to Answer the T_2 Question

Theorem 3.1. *Let (X, T) be a space. Then the following are equivalent*: (a) *(X, T) has a maximal, proper, dense, T_2 subspace (Y, T_Y)*, (b) *(X, T) is "not-T_2" and there exists a proper, T_2 subspace (Z, T_Z) of (X, T) such that for each $x \in X \setminus Z$, there exists a $z \in Z$ such that every open set U containing x intersects every open set V containing z and $x \in Cl(U \cap V \cap Z)$, and* (c) *there exists a proper, T_2 subspace (Z, T_Z) of (X, T) such that for each $x \in X \setminus Z$, there exists a $z \in Z$ such that every open set U containing x intersects every open set V containing z and $x \in Cl(U \cap V \cap Z)$.*

Proof. (a) implies (b): Since T_2 is a subspace property and (Y, T_Y) is a maximal, T_2 subspace of (X, T), then (X, T) is "not-T_2". Let $Z = Y$. Then $(Z, T_Z) = (Y, T_Y)$ is a proper, T_2 subspace of (X, T). Let $x \in X \setminus Y$ and let $W = Y \cup \{x\}$. Then (W, T_W) is "not-T_2". Let u and v be distinct elements of W such that every T_W-open set containing u intersects every T_W-open set containing v. Then one of u and v is x, for suppose not. Then u and v are in Y and there exist disjoint T_Y-open sets A and B such that $u \in A$ and $v \in B$. Let C and D be T-open sets such that $A = C \cap Y$ and $B = D \cap Y$. Then $C \cap W$ and $D \cap W$ are T_W-open sets such that $u \in (C \cap W)$ and $v \in (D \cap W)$ and $(C \cap W) \cap (D \cap W) \neq \phi$. Since $(C \cap Y)$ and $(D \cap Y)$ are disjoint, then $((C \cap W) \cap (D \cap W)) = \{x\}$. Then $(C \cap D)$ is T-open, $((C \cap D) \cap Y) = \phi$, and $x \in (C \cap D)$, which implies $x \notin Cl(y)$, which is a contradiction. Thus one of u and v is x, say $u = x$. Then every T_W-open set containing x intersects every T_W-open set containing v. Let E and F be T-open sets such that $x \in E$ and $v \in F$. Then $(E \cap W)$ and $(F \cap W)$ are T_W-open sets containing x and v, respectively and $((E \cap W) \cap (F \cap W)) \neq \phi$. If $((E \cap W) \cap (F \cap W)) = \{x\}$, then $((E \cap Y) \cap (F \cap Y)) = \phi$ and $x \in (E \cap F)$, which is T-open and $((E \cap F) \cap Y) = \phi$, which contradicts $Cl(Y) = X$. Thus $((E \cap F) \cap Y) \neq \phi$. Then $x \in Cl((E \cap F) \cap Y)$, for suppose not. Then $x \in ((E \cap F) \setminus Cl((E \cap F) \cap Y)$, which is T-open and misses Y, which contradicts $x \in Cl(Y)$.

Clearly (b) implies (c).

(c) implies (a): Let $Y = Z$. Then (Y, T_Y) is a proper, T_2 subspace of (X, T). Let W be a subset of X that properly contains Y. Let $x \in W \setminus Y$. Then $x \in X \setminus Y$. Let $z \in Y$ such that every T-open set U containing x intersects every T-open set V containing z and $x \in Cl(U \cap V \cap Y)$. Thus $x \in Cl(Y)$. Hence $Cl(Y) = X$ and (Y, T_Y) is dense in (X, T). Then $U \cap W$ and $V \cap W$ are T_W-open sets containing x and z, respectively. Since $U \cap V \cap Y \neq \phi$, then $U \cap V \cap W = (U \cap W) \cap ((V \cap W)) \neq \phi$. Thus (W, T_W) is "not-T_2". Hence, (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (X, T).

Theorem 3.2. *Let (X, T) be a space and let Y be a subset of X. Then (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (X, T) iff (Y, T_Y) is a proper T_2 subspace of (X, T) such that for each $x \in X \setminus Y$, there exists a $z \in Y$ such that every open set U containing x intersects every open set V containing z and $x \in Cl(U \cap V \cap Y)$.*

Proof. If (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (X, T), then, by the argument above (a) implies (b), (Y, T_Y) is a proper T_2 subspace of (X, T) such that for each $x \in X \setminus Y$, there exists a $z \in Y$ such that every open set U containing x intersects every open set V containing z and $x \in Cl(U \cap V \cap Y)$.

If (Y, T_Y) is a proper T_2 subspace of (X, T) such that for each $x \in X \setminus Y$, there exists a $z \in Y$ such that every open set U containing x intersects every open set V containing z and $x \in Cl(U \cap V \cap Y)$, then, by the argument above (c) implies (a), (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (X, T).

Thus, the condition on a space that is equivalent to the space having a maximal, proper, dense, T_2 subspace immediately gives a maximal, proper, dense, T_2 subspace of the space.

Theorem 3.3. *Let (X, T) be a space and let Y be a subset of X. Then (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (X, T) iff for each subset Z of X that properly contains Y, (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (Z, T_Z).*

Proof. Let (Y, T_Y) be a maximal, proper, dense, T_2 subspace of (X, T) and let Z be a subset of X that properly contains Y. Let $x \in Z \setminus Y$. Then $x \in X \setminus Y$ and there exists a $z \in Y$ such that every open set U containing x intersects every open set V containing z and $x \in Cl(U \cap V \cap Y)$. Then $U \cap Z$ and $V \cap Z$ are T_Z-open sets containing x and z, respectively and since $((U \cap W) \cap (V \cap W) \cap Y) = (U \cap V \cap Y) = \phi$, $((U \cap W) \cap (V \cap W)) \neq \phi$ and, since $x \in Cl(U \cap V \cap Y)$, then $x \in Cl_Z((U \cap W) \cap (V \cap W) \cap Y)$.

The converse follows immediately since X is a subset of X that properly contains Y.

Corollary 3.1. *Let (X, T) be a space and let Y be a subset of X such that (Y, T_Y) is a maximal, proper, dense, T_2 subspace of (X, T). Then for each subset*

Z of X that properly contains Y, (Z, T_Z) *is "not-*T_2*" and for each subset W of* Y, (W, T_W) *is* T_2.

The results above raise questions about the uniqueness of maximal, proper, dense, T_i subspaces, $i = 1, 2,$ which are resolved by the following example.

Example 3.1. Let $X = \{a, b\}$, where a and b are distinct, and let T be the indiscrete topology on X. Then each singleton set subspace of (X, T) is a maximal, proper, dense, T_i subspace of (X, T) and maximal, proper, dense, T_i, $i = 1, 2$ are not unique.

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