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MUTUALLY ORTHOGONAL LATIN SQUARES OF CONSTRUCTING RESOLVABLE BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract

In 1981, Khare and Federer [13] published a paper on a method of constructing resolvable incomplete block designs for $v = p^2$ treatments, where p being a prime number. Also, (Hinkelmann and Kempthorne [6, Chapter 3]) constructed a design for p being a prime power in incomplete blocks of size k. The method uses an algorithm called a successive diagonalizing method. It is observed that the method only worked for p being a prime number and equally becomes tedious to construct when $v > 16$. As such, this study proposes a new method of construction that captures both prime and prime power for p and also restored the uniqueness of the treatment pairs, that is $\lambda = 1$ for all v. It also mitigates

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the tediousness encountered when $v > 16$.

1. Introduction

At times the situations the experimenters found themselves made them to be totally engulfed in constructing designs in an efficient ways without losing no or much information. This kind of situations arise when the number of experimental units in an experiment is often larger than that can be accommodated in the available blocks of relatively uniform experimental units, in this situation it is often desirable to have resolvable incomplete block designs in which the incomplete blocks can be arranged in complete blocks or replicates. Nowadays, it has been noticed that the levels at which the treatments increase are so high due to a lot of favorable factors that are peculiar to different field of studies while the experimental units that receive the treatments are smaller in numbers. Meanwhile, for the experimenters to be able to rise to these occasions or challenges, the use of resolvable incomplete block designs is inevitable.

The early sources for constructing resolvable incomplete block designs with some files are (Yates [21]) for square lattices, (Habshbabger [7, 8, 9]) for rectangular lattices, (Kempthorn [12]) and (Federer [5]) for prime power lattices, and (David [4]) and (John et al. [10]) on cyclic designs. Remark that there is no total absolute feasibility for constructing a complete file of incomplete block designs for all situations, yet the researchers cut edge of the algebraic structures to attain some simple constructions usable to the experimenters.

Also Morgan et al. [14] presented a paper therein reviewed and extended mathematical knowledge of nested balanced incomplete block designs (NBIBD's); isomorphism and automorphisms were defined for NBIBDs, and methods of construction were outlined. Peter et al. [16] showed the necessary divisibility conditions for the existence of a σ-resolvable BIBD (v, k, λ) are sufficient for large v. Saka and Adeleke [18] developed a new method of construction of nested balanced incomplete block designs in which the resulting design schemes were of the type that harmonizes both the Series-I and Series-II of Rajender et al. [17]. Keerti and Vineeta [11] introduced a new method of construction of a series of Nested

Balanced Incomplete Block Designs (NBIBDs) in which the inner blocks are constructed using Latin square. Saka et al. [19] presented a method of construction of designs which utilizes special matrix structures referred to as Zig-zag, the Zig-zag matrix structures give rise to initial blocks for resolvable nested balanced incomplete block designs (RNBIBDs).

The method presented here requires no generators or tables and it leads to a resolvable balanced incomplete block design for p a prime or prime power, the number of times a pair of varieties (treatments) occur together, a concurrence, in this design is $\lambda = 1$.

2. Mutually Orthogonal Latin Squares (MOLS)

Definition 1. Two Latin squares L_1 and L_2 of the same order, say n, are mutually orthogonal if every ordered pair $(i, j), 1 \leq i, j \leq n,$ appears exactly once when L_1 and L_2 are superimposed on each other.

Example 1. The followings are examples of mutually orthogonal latin square of order 4.

$$L_1 = \begin{array}{|c|c|c|c|}\hline 4&1&2&3\\\hline 1&4&3&2\\\hline 2&3&4&1\\\hline 3&2&1&4\\\hline\end{array}, \quad L_2 = \begin{array}{|c|c|c|c|}\hline 4&1&2&3\\\hline 3&2&1&4\\\hline 1&4&3&2\\\hline 2&3&4&1\\\hline\end{array} \; and \;\; S(L_1, L_2) = \begin{array}{|c|c|c|c|}\hline (4,4)&(1,1)&(2,2)&(3,3)\\\hline (1,3)&(4,2)&(3,1)&(2,4)\\\hline (2,1)&(3,4)&(4,3)&(1,2)\\\hline (3,2)&(2,3)&(1,4)&(4,1)\\\hline\end{array}$$

Definition 2. A set of mutually orthogonal latin squares is a set of two or more latin squares of the same order, all of which are orthogonal to one another.

Example 2. The followings are the four distinct latin squares of order $n = 5$.

$$L_1 = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 2&3&4&5&1\\\hline 3&4&5&1&2\\\hline 4&5&1&2&3\\\hline 5&1&2&3&4\\\hline\end{array}, \; L_2 = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 3&4&5&1&2\\\hline 4&5&1&2&3\\\hline 5&1&2&3&4\\\hline 2&3&4&5&1\\\hline\end{array}, \; L_3 = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 4&5&1&2&3\\\hline 5&1&2&3&4\\\hline 2&3&4&5&1\\\hline 3&4&5&1&2\\\hline\end{array}, \; L_4 = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 5&1&2&3&4\\\hline 2&3&4&5&1\\\hline 3&4&5&1&2\\\hline 4&5&1&2&3\\\hline\end{array}$$

Definition 3. A set $t \geq 2$ MOLS of order n is called a complete set if $t = n - 1$, or a set of $n - 1$ MOLS of order n is called a complete set of MOLSs.

3. Constructions of Resolvable BIBD Using Successive Diagonalizing Method

The successive diagonalizing method is a method for constructing resolvable balanced incomplete block (BIB) designs for $v = p^2$, p being a prime number, in $b = p(p+1)$ blocks of size p, for the number of replicates $r = p+1$, and for $\lambda = 1$. This method is formalized below in Algorithm 1 and is exemplified in Examples 3 and 4.

Algorithm 1. The steps constructing BIB designs with parameters $v = p^2$, $k = p$, $b = p(p+1) = p^2 + p$, $r = p+1$ and $\lambda = 1$ of Algorithm for Successive Diagonalizing Method. For detail on the Algorithm see (Khare and Federer [13]).

Example 3. The steps of Algorithm for Successive Diagonalizing Method for $p^2 = 9 = 3^2$ are:

$$L_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline\end{array}, \quad L_2 = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 & 9 \\ \hline\end{array}, \quad L_2 = \begin{array}{|c|c|c|}\hline 1 & 5 & 9 \\ \hline 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \hline\end{array}, \quad L_4 = \begin{array}{|c|c|c|}\hline 1 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \hline 3 & 5 & 7 \\ \hline\end{array}$$

Example 4. The steps of Algorithm for Successive Diagonalizing Method for $p^2 = 16 = 4^2$ are:

$$L_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 \\ \hline\end{array} \quad L_2 = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 9 & 13 \\ \hline 2 & 6 & 10 & 14 \\ \hline 3 & 7 & 11 & 15 \\ \hline 4 & 8 & 12 & 16 \\ \hline\end{array} \quad L_3 = \begin{array}{|c|c|c|c|}\hline 1 & 6 & 11 & 16 \\ \hline 2 & 7 & 12 & 13 \\ \hline 3 & 8 & 9 & 14 \\ \hline 4 & 5 & 10 & 15 \\ \hline\end{array}$$

$$L_4 = \begin{array}{|c|c|c|c|}\hline 1 & 7 & 9 & 15 \\ \hline 2 & 8 & 10 & 16 \\ \hline 3 & 5 & 11 & 13 \\ \hline 4 & 6 & 12 & 14 \\ \hline\end{array} \quad L_5 = \begin{array}{|c|c|c|c|}\hline 1 & 8 & 11 & 14 \\ \hline 2 & 5 & 12 & 15 \\ \hline 3 & 6 & 9 & 16 \\ \hline 4 & 7 & 10 & 13 \\ \hline\end{array}$$

Remark 1. Because of the repeated application of step 3 in the algorithm, this method has been referred to as successive diagonalizing method (Khare and Federer [13]). However, it is observed that the method only worked for p being a prime number but not for prime power as it can be seen from Example 4 that the design is not balanced (i.e., $\lambda \neq 1$), see example 3.8 of Hinkelmann and Kempthorne [6] where treatments are denoted by ordered pairs (x, y) with $x, y = 1, 2, \ldots, p^2$, $x \neq y$, that is, pairs of treatment $(3, 11)$ appear together in square $-2(L_2)$ and square $-4(L_4)$, pairs of treatment $(4, 12)$ appear together in square $-2(L_2)$ and square $-4(L_4)$ again pairs of treatment $(2, 12)$ appear together in square $-3(L_3)$ and square $-5(L_5)$, pairs of treatment $(1, 11)$ appear together in square $-3(L_3)$ and square $-5(L_5)$ and so on. This is a pointer to the fact the design is not balanced. Also the method becomes tedious to construct when $v > 16$.

4. Main Results

Based on the difficulties encounter in the construction of designs when p is a prime power and also when $v < 16$, this new proposed method is presented in Algorithm 2.

Algorithm 2. The steps constructing BIB designs with parameters $v = p^2, k = p, b = p(p+1) = p^2 + p, r = p+1$ and $\lambda = 1$ of Algorithm for MOLS of Resolvable BIBD for both p being a prime number or prime power are:

1. Write the number $1, 2, \ldots, p^2$ consecutively in a square array of p rows and p columns beginning in the left-hand order corner of the first row and subsequently continuing at the beginning of each row. This is square 1 with rows being the blocks.

2. Transpose the rows and columns of the square I to obtain square 2 with rows being the blocks.

3. Obtain the set of MOLS of order n (considered order).

4. Number the elements in the set of MOLS of order n from $1, 2, \ldots, n^2 = p^2$.

5. Let the letters of square I of the set of MOLS of order n form the square III.

6. Superimpose square II on I to obtain square IV.

7. Continuing the superimposition of squares, i.e., III, IV, ..., $n-1$ on square I

until square $k+1$ is obtained.

Design 1. The steps of Algorithm for Successive MOLS Superimposition Method (SMSM) for $p^2 = 3^2$ are:

$$L_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}, \quad L_2 = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 & 9 \\ \hline \end{array}, \quad L_2 = \begin{array}{|c|c|c|}\hline 1 & 5 & 9 \\ \hline 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \hline \end{array}, \quad L_4 = \begin{array}{|c|c|c|}\hline 1 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \hline 3 & 5 & 7 \\ \hline \end{array}$$

Design 2. The steps of Algorithm for Successive MOLS Superimposition Method (SMSM) for $p^2 = 4^2$, where $P = 2^2$ is prime power are:

$$L_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 \\ \hline \end{array} \quad L_2 = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 9 & 13 \\ \hline 2 & 6 & 10 & 14 \\ \hline 3 & 7 & 11 & 15 \\ \hline 4 & 8 & 12 & 16 \\ \hline \end{array} \quad L_3 = \begin{array}{|c|c|c|c|}\hline 1 & 6 & 11 & 16 \\ \hline 2 & 7 & 12 & 15 \\ \hline 3 & 8 & 9 & 14 \\ \hline 4 & 7 & 10 & 13 \\ \hline \end{array}$$

$$L_4 = \begin{array}{|c|c|c|c|}\hline 1 & 7 & 12 & 14 \\ \hline 2 & 8 & 11 & 13 \\ \hline 3 & 5 & 10 & 16 \\ \hline 4 & 6 & 9 & 15 \\ \hline \end{array} \quad L_5 = \begin{array}{|c|c|c|c|}\hline 1 & 8 & 10 & 15 \\ \hline 2 & 7 & 9 & 16 \\ \hline 3 & 6 & 12 & 13 \\ \hline 4 & 5 & 11 & 143 \\ \hline \end{array}$$

Design 3. The steps of Algorithm for Successive MOLS Superimposition Method (SMSM) for $p^2 = 5^2$ are:

$$L_1 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & 18 & 19 & 20 \\ \hline 21 & 22 & 23 & 24 & 25 \\ \hline \end{array} \quad L_2 = \begin{array}{|c|c|c|c|c|}\hline 1 & 6 & 11 & 16 & 21 \\ \hline 2 & 7 & 12 & 17 & 22 \\ \hline 3 & 8 & 13 & 18 & 23 \\ \hline 4 & 9 & 14 & 19 & 24 \\ \hline 5 & 10 & 15 & 20 & 25 \\ \hline \end{array} \quad L_3 = \begin{array}{|c|c|c|c|c|}\hline 1 & 10 & 14 & 18 & 22 \\ \hline 2 & 6 & 15 & 19 & 23 \\ \hline 3 & 7 & 11 & 20 & 24 \\ \hline 4 & 8 & 12 & 16 & 25 \\ \hline 5 & 9 & 13 & 17 & 21 \\ \hline \end{array}$$

$$L_4 = \begin{array}{|c|c|c|c|c|}\hline 1 & 7 & 13 & 19 & 25 \\ \hline 2 & 8 & 14 & 20 & 21 \\ \hline 3 & 9 & 15 & 16 & 22 \\ \hline 4 & 10 & 11 & 17 & 23 \\ \hline 5 & 6 & 12 & 18 & 24 \\ \hline \end{array} \quad L_5 = \begin{array}{|c|c|c|c|c|}\hline 1 & 8 & 12 & 20 & 24 \\ \hline 2 & 9 & 13 & 16 & 25 \\ \hline 3 & 10 & 14 & 17 & 21 \\ \hline 4 & 6 & 15 & 18 & 22 \\ \hline 5 & 7 & 11 & 19 & 23 \\ \hline \end{array} \quad L_6 = \begin{array}{|c|c|c|c|c|}\hline 1 & 7 & 15 & 19 & 23 \\ \hline 2 & 8 & 11 & 20 & 24 \\ \hline 3 & 9 & 12 & 16 & 25 \\ \hline 4 & 10 & 13 & 17 & 21 \\ \hline 5 & 6 & 14 & 18 & 22 \\ \hline \end{array}$$

Design 4. The steps of Algorithm for Successive MOLS Superimposition Method (SMSM) for $p^2 = 8^2$ where $P = 2^3$ is prime power are:

$L_1 =$

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

$L_2 =$

1	9	17	25	33	41	49	57
2	10	18	26	34	42	50	58
3	11	19	27	35	43	51	59
4	12	20	28	36	44	52	60
5	13	21	29	37	45	53	61
6	14	22	30	38	46	54	62
7	15	23	31	39	47	55	63
8	16	24	32	40	48	56	64

$L_3 =$

1	16	19	26	37	44	55	62
2	11	24	25	38	47	52	61
3	10	17	32	39	46	53	60
4	13	22	31	40	41	50	57
5	12	23	30	33	48	51	58
6	15	20	29	34	43	56	57
7	14	21	28	35	42	49	64
8	9	18	27	36	48	54	63

$L_4 =$

1	11	21	31	34	48	54	60
2	16	22	28	33	43	53	63
3	9	23	29	40	42	52	62
4	14	24	26	39	45	51	57
5	15	17	27	38	44	50	64
6	12	18	32	37	47	49	59
7	13	19	25	36	46	56	58
8	10	20	30	35	41	55	61

$L_5 =$

1	10	23	28	38	45	56	59
2	9	26	31	37	46	51	64
3	16	21	30	36	47	50	57
4	15	18	25	35	48	63	62
5	14	19	32	34	41	55	63
6	13	24	27	33	42	55	60
7	12	17	26	40	43	54	61
8	11	22	29	39	49	49	58

$L_6 =$

1	13	18	30	39	43	52	64
2	14	17	29	36	48	55	59
3	15	24	28	37	41	54	58
4	16	23	27	34	46	49	61
5	9	22	26	35	47	56	60
6	10	21	25	40	44	51	63
7	11	20	32	33	45	50	62
8	12	19	31	38	42	53	57

$L_7 =$

1	12	24	29	35	46	50	63
2	15	19	30	40	45	49	60
3	14	18	31	33	44	56	61
4	9	21	32	38	43	55	58
5	16	20	25	39	42	54	59
6	11	23	26	36	41	53	64
7	10	22	27	37	48	52	57
8	13	17	28	34	47	51	62

$L_8 =$

1	15	22	32	36	42	51	61
2	12	21	27	39	41	56	62
3	13	20	26	38	48	49	63
4	10	19	29	33	47	54	64
5	11	18	28	40	46	55	57
6	16	17	31	35	45	52	58
7	9	24	30	34	44	53	59
8	14	23	25	37	43	50	60

$L_9 =$

1	14	20	27	40	47	53	58
2	13	23	32	35	44	54	57
3	12	22	25	34	45	54	64
4	11	17	30	37	42	56	63
5	10	24	31	36	43	49	62
6	9	19	28	39	48	50	61
7	16	18	29	38	41	51	60
8	15	21	26	33	46	52	59

5. Conclusions

The method of Mutually Orthogonal Latin Squares of Resolvable Balanced Incomplete Block Designs was used to construct designs for p either being prime order or prime power and it equally restored lost of balance in the example given by Hinkelmann and Kempthorne [6, Chapter 3]. Finally, it mitigates the tediousness encountered in the construction when $v > 16$.

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