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MEDIAN AND AVERAGE AS TOOLS FOR MEASURING, ELECTING AND RANKING: NEW PROSPECTS

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Abstract

Impossibility theorems expose inconsistencies and paradoxes related to voting systems. Recently, Michel Balinski and Rida Laraki proposed a new voting theory called Majority Judgment which tries to circumvent this limitation. In Majority Judgment, voters are invited to evaluate candidates in terms taken in a well-known common language. The winner is then the

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one that obtains the highest median. Since the Majority Judgment proposal was made, authors have detected insufficiencies with this new voting system. This article aims at reducing these insufficiencies by proposing a voting system to decide between the median-based voting and the mean-based one. It proposes, moreover, a new tie-breaking method computing intermedian ranks mean.

1. Introduction

In our current democracies, there is, according to Nobel Laureate Kenneth J. Arrow [4], essentially two ways by which social choices can be made: voting, typically used to make a political decision, and the market mechanism typically used to make economic decisions [29]. Our concern in this paper is exclusively the former, i.e., voting.

The Theory of Social Choice concerns methods for amalgamating the appreciations or evaluations of many individuals into one collective appreciation or evaluation [7]. Impossibility theorems stated by Arrow [5, 6], Sen [27, 28], Gibbard [21] and Satterthwaite [26] - only to quote these - highlight the fact that none of these aggregation methods fulfills the requirements of an ideal democracy.

Many results now known as paradoxes abound in the direction to show imperfections of the electoral systems. These results tackle usual modes of poll and those which are not practically used in any country in the world as well. Instead of discouraging researchers in Economy and Theory of Social Choice, paradoxes as well as impossibility theorems have on the contrary instigated them to conceive and even try out several new methods or aggregation functions. These methods are being proposed as alternative to those which are publicly valued.

As an example, approval voting was tested in France at the 2002 presidential elections first round by a team directed by Jean-François Laslier and Karine Van der Straeten [25, 23]. A second experiment was carried out under the same conditions in 2007 for approval voting and evaluation voting [3] by Antoinette Baujard and Herrade Igersheim [13, 14, 23]. The same day of the 2007 elections, another team led by Etienne Farvaque, Hugo Jayet and Lionel Ragot [20] tried out the single transferable vote. Various teams still undertook experimental votes parallel to the official one on April 22, 2012 [12, 9].

Among all modes of poll that are tested in France, one seems to dissociate others on philosophical level: it is about the one proposed by Michel Balinski and Rida Laraki. This new voting theory is median-based and uses several ways, more or less complicated, for ranking candidates. Its authors named it *Majority Judgment* (MJ) [7, 8, 11]. MJ was tested twice in France at the presidential elections of 2007 [11] and 2012 [9]. Philosophically, it is opposed to traditional vote which consists in operating summations of individual preferences votes in order to obtain a candidate's value in an election or - more formally - a given situation. It asks each voter to evaluate each candidate instead of simply voting for one or more candidates according to one's preferences.

The 6 February 2009 "Numbers Guy" column in the Wall Street Journal ("And the Oscar Goes to...Not Its Voting System") contained :≪ Prof. Balinski... calls (mean-based) range voting a "ridiculous method", because it can be manipulated by strategic voters ≫ [15]. However, though in its beginning, MJ already sustains praises and attacks as well from scientific community. Greeted in particular by Nobel Laureates Kenneth Arrow, Robert Aumann and Eric Maskin for giving the possibility of enriching and moderating the expression of one's political choice [10], it is also disparaged by some scientists, in particular:

● Jean-François Laslier from CNRS (Centre National de Recherche Scientifique /France) who details a whole theory exposing all the weaknesses inherent to the choice based on the evaluation of the best median [22, 24].

● Warren D. Smith, emblematic range voting (mean-based) defender who devoted a whole Web page to MJ weaknesses.

● Manzoor Ahmed Zahid proposes a thesis highlighting MJ pathologies and prefers to it a mean-based method which he calls *Borda Majority Count* (BMC). This method is similar to evaluation voting except that it has several levels thus giving to voters more latitude of nuances between their preferences [2]. Manzoor Ahmed Zahid has in his thesis seven paradoxical results relating to the MJ that the BMC could circumvent. However, as we will see further in this article, mean-based aggregation methods (including the BMC) are not either free from any reproach. They can, on the contrary, prove more dictatorial than usual modes of poll which are already much disparaged.

In this article, we propose a voting method reconciling both concepts: the mean-based concept and the median-based one. But before achieving this, let us recall

Borda Majority Count and Majority Judgment foundations.

2. Borda Majority Count (BMC)

Let a be an alternative and $\{g_1, g_2, \ldots, g_k\}$ be a set of grades in ascending order $g_1 > g_2 > \cdots > g_k$. Let p_j be the number of voters who assign grade g_j to a, where $j = 1, 2, \ldots, k$. One then defines the BMC of a by the formula $BMC(a) = \sum_{j=1}^{k} p_j.[k - j]$.

For example, let us suppose that we have 5 grades: Excellent (E), Very Good (V G), Good (G), Acceptable (A), Reject (R). One assigns 4 points to E, 3 points to V G, 2 points to G, 1 point to A and 0 point to R. With this coding, let us suppose that 10 voters evaluated candidate a in the following way:

E	VG	G	A	R
1	2	3	3	1

Then $BMC(a) = 1 \times 4 + 2 \times 3 + 3 \times 2 + 3 \times 1 + 1 \times 0 = 19.$

3. Majority Judgment (MJ)

The idea of Balinski and Laraki to conceive a voting theory completely based on the median is justified by the fact that - according to authors - median is less easy to manipulate than average.

MJ goes beyond Arrovian framework: voters do not decide about candidates in terms of "… is preferred to …" any more but they evaluate them individually. In practice, in skating contest and wine competitions, voters are asked to evaluate each candidate in a language understood by all in order to rank all candidates. The same applies to Range Voting, though the social ranking is established in different ways. While in Range Voting simply the average of all evaluations of a candidate is taken as the social evaluation, Balinski and Laraki take the median value of all evaluations of a candidate as their starting point, in order to reduce the possibilities for successful strategic behavior. In case two or more candidates get the same median value, a tie-breaking rule then decides the final social ranking. The MJ is exactly what has been detailed.

MJ is recent and successful proposal that attracts the attention of many researchers from theoretical and experimental points of view. Some of its extensions - inclusive of cases where extension comes from the side of MJ tie-breaking rule - are given in [17], [18] and [19].

First of all, a common language for grading is required such that all grades are well understood by all persons involved. In their experiment at the 2007 French presidential elections, Balinski and Laraki took the following common language:

$$\{\text{Excellent, Very Good, Good, Acceptable, Poor, Reject}\}$$

Definition 1 (Common Language)**.** One calls "*common language*" a set $\mathcal{L} = \{g_1, g_2, \ldots, g_k\}$ strictly ordered by ">" such that $g_1 > g_2 > \cdots > g_k$ $(g_i \geq g_j \coloneqq g_i > g_j$ or $g_i = g_j)$.

Note that one can have as a common language an infinite set such as the interval $[0, 1]$ of the real numbers with its natural ordering.

We can notice the possibility for a voter to allot the same evaluation to more than two candidates. For this reason, a voter can say that he values a candidate *a* as Very Good, a candidate *b* as Very Good and candidate *c* as Good. In Arrow's framework, these evaluations will be rendered by:

"a is at least as good as b", "b is at least as good as a", "a is at least as good as c", "b is at least as good as c", "a is preferred to c", "b is preferred to c", "a is indifferent to b".

Definition 2 (Method of grading)**.** A function F is a "method of grading" if it assigns to any profile a single grade (in the same language) for every candidate. So,

$$F : \mathcal{L}^{m \times n} \to \mathcal{L}^m,$$

$$(g_{11}, g_{12}, \ldots, g_{1n}, g_{21}, \ldots, g_{mn}) \mapsto (g_1^*, g_2^*, \ldots, g_m^*),$$

where m is the number of candidates, n is the number of judges or voters and g_i^* $(1 \leqslant i \leqslant m)$ is the final rank obtained by the candidate i.

Properties of *F*

It was shown [8] that F enjoys the following properties:

- Neutrality: If each voter reverses his preference, the selected candidate changes too.

● Anonymity: If one permutes voters (or even with other voters), the elected candidate does not change.

● Pareto-consistency: If all the voters prefer candidate x to candidate y, then x must be the election winner.

● Monotoncity: If one or more voters reclassify a candidate x better, then x does not have ultimately to be less better placed in this election.

● Independence of irrelevant alternatives: To classify two candidates among several others, it must be enough to know preferences of each voter for these two candidates - their choices for others do not influence the classification between these candidates as well as the addition or the withdrawal of a candidate.

Definition 3 (Majority grade)**.** Let a_i be a candidate or competitor with grades $g_{i1}, g_{i2}, \ldots, g_{in}$, where $g_{i1} \geq g_{i2} \geq \ldots \geq g_{in}$. Then the "majority grade" or "majority rank" $f^{maj}(a_i)$ is by definition:

$$f^{maj}(a_i) = \begin{cases} f^{\frac{n+1}{2}}(g_{i1}, g_{i2}, \ldots, g_{in}), & \text{if } n \text{ is odd,} \\ f^{\frac{n+2}{2}}(g_{i1}, g_{i2}, \ldots, g_{in}), & \text{if } n \text{ is even.} \end{cases}$$

For example, if 5 judges allot grades 4, 8, 7, 9, 5 to a_i, then $f^{maj}(a_i) = f^3(9, 8, 7, 5, 4) = 7$ and if 8 judges allot grades 9, 7, 3, 6, 5, 4, 5, 8 to a_i, then $f^{maj}(a_i) = f^5(9, 8, 7, 6, 5, 5, 4, 3) = 5$.

Tie-Breaking [7]

When the majority grades of two candidates are different, the one with the higher majority grade naturally ranks ahead of the other. The majority ranking $\succ_{maj}$ between two candidates evaluated by the same jury is determined by a repeated application of the majority grade:

● If $f^{maj}(a) > f^{maj}(b)$, then $a \succ_{maj} b$.

● If $f^{maj}(a) = f^{maj}(b)$, then one grade (majority grade) is dropped from the grades of each of the competitors, and the procedure is repeated.

Balinski and Laraki [8] give the following example to illustrate this definition: Suppose *a* and *b* are evaluated by a jury of 7 voters as follows:

a	85	73	78	90	69	70	73
b	77	70	95	81	73	73	66

Then the ordered profile is:

a	90	85	78	**73**	73	70	69
b	95	81	77	**73**	73	70	69

After three iterations, one obtains $f_3^{maj}(a) = 78 > f_3^{maj}(b) = 77$ and then $a \succ_{maj} b$.

It is clear that the majority ranking always ranks a candidate ahead of another unless both are assigned an identical set of grades by judges.

In the case of many judges or voters (presidential elections for example), Balinski and Laraki present another simplified tie-breaking rule.

Definition 4 (Majority gauge)**.** The "majority gauge" of a candidate with $f^{maj}(a) = \alpha$ is a triplet (p_a, α^*, q_a), where p is the number or percentage of the candidate's grades above the majority grade, q is the number or percentage of the candidate's grades below the majority grade, and $\alpha^* = \begin{cases} \alpha^+ & \text{if } p > q, \\ \alpha^- & \text{if } p \le q. \end{cases}$

α^* is called the candidate's modified majority grade.

Let β be another majority grade, by definition, $\alpha^* > \beta^*$ iff $\alpha > \beta$ or $(\alpha = \beta$ and $\alpha^* = \alpha^+$ and $\beta^* = \alpha^-)$.

Balinski and Laraki use majority gauge to define the majority-gauge-ranking $\succ_{mg}$.

Definition 5 (Majority-gauge-ranking)**.** Let *a* and *b* be two candidates with majority-gauges (p_a, α_a^*, q_a) and (p_b, α_b^*, q_b), respectively. Then $a \succ_{mg} b$ or

$(p_a, \alpha_a^*, q_a) >_{mg} (p_b, \alpha_b^*, q_b)$ iff $\alpha_a^* > \alpha_b^*$ or $(\alpha_a^* = \alpha_b^* = \alpha^+$ and $p_a > p_b)$ or $(\alpha_a^* = \alpha_b^* = \alpha^-$ and $p_a < p_b)$.

Manzoor Ahmed Zahid [2] shows that majority-gauge-ranking may not slice between candidates in exceptional cases. A theorem stated by Balinski and Laraki (Theorem 14.1 in [8]) shows that: $a \succ_{mg} b \Rightarrow a \succ_{maj} b$.

Ahmed Zahid then takes an example which illustrates a case where $a \succ_{maj} b$, but neither $a \succ_{mg} b$ nor $b \succ_{mg} a$.

Candidate	p	Excellent	Very Good	Good	Acceptable	Poor	Reject	q	Total
a	5	2	3	**<u>3</u>**	1	3	3	7	15
b	6	3	3	**<u>2</u>**	0	2	5	7	15

The majority gauge for a is $(5, Good^-, 7)$ and that of b is $(6, Good^-, 7)$. As $q_a = q_b = 7$, majority-gauge-ranking does not give any decision while one easily checks that $a \succ_{maj} b$.

4. Mean-Median Compromise Method (MMCM)

Taking into account MJ and BMC, we work out an original method able to reach as well distinctly as jointly the two theories and which would circumvent a maximum of paradoxes presented farther in this paper. This method makes it possible to decide between the median and the average.

4.1. Concepts and description of the MMCM

The method that we present in this article is based on the same time on the median and the average. One proceeds by a division of the grades distribution for a candidate in 2^k intervals with the same amplitude. This division determines $2^k + 1$ grades which are the points that limit the intervals. The function returns as candidate evaluation a real number corresponding to the arithmetic mean of $2^k + 1$ selected grades while extreme values are excluded (Olympic average).

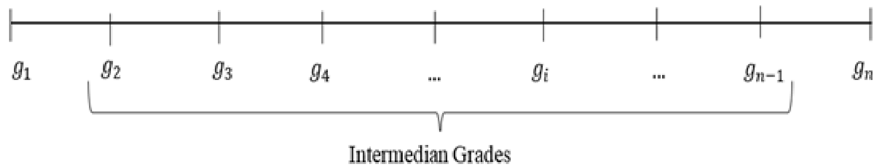

Figure 1. Division of the grades distribution in 2^k intervals.

Definition 6 (Olympic average)**.** Let $g_1, g_2, \ldots, g_n$ be a succession of *n* data such as $g_1 \geq g_2 \geq \cdots \geq g_n$. One calls "Olympic average" of these data the number:

$$\overline{g}_0 = \frac{1}{n-2}\sum_{i=2}^{n-1} g_i.$$

The Olympic average of *n* data is then the arithmetic average of these data, when the two extreme values (largest and smallest) are dropped.

Definition 7 (Amplitude of a division)**.** Let *N* be the set of *n* judges, one calls amplitude of a division the real number:

$$\rho = \frac{n+1}{2^k} \text{ with } k \geq 1.$$

k is a whole number fixed in advance which is called *degree of division*.

Definition 8 (Intermedian grade)**.** Let a_i be a candidate or competitor with grades $g_{i1}, g_{i2}, \ldots, g_{in}$, where $g_{i1} \geq g_{i2} \geq \cdots \geq g_{in}$. A grade g_{ij} is called "*intermedian*" if and only if $\exists m \in \mathbb{N}$ (with $1 \leq m \leq 2^k - 1$) such that $[\rho.m] = j$, where $[\rho.m]$ is the whole number that is nearest to $\rho.m$ and ρ the amplitude of division for a fixed degree of division k.

One notes $\mathcal{M}_k$ the set of non-redundant intermedian grades obtained from a degree of division k.

The so-defined $\mathcal{M}_k$ is the set of data involved in the Olympic average calculation of the points which are bounds (higher or lower) of 2^k intervals obtained after division.

Definition 9 (Average majority compromise)**.** Let a_i be a candidate or competitor with grades $g_{i1}, g_{i2}, \ldots, g_{in}$, where $g_{i1} \geq g_{i2} \geq \cdots \geq g_{in}$ and $\mathcal{M}_k = \{g_{i1}^*, g_{i2}^*, \ldots, g_{ij}^*\}$ be the set of his or her intermedian grades obtained from a degree of division k. Then the "average majority compromise", "the average majority evaluation" or "average majority rank" $f^{mm}(a_i)$ is by definition:

$$f^{mm}(a_i) = \frac{1}{j}\sum_{m=1}^{j} g_{im}^*.$$

For example, if 5 judges assign grades 4, 8, 7, 9, 5 to a_i, one fixes $k = 2$, $\rho = \frac{5+1}{2^2} = 1.5.$

The classified grades in a descending order are: 9, 8, 7, 5, 4.

$$\mathcal{M}_2 = \{f^{[1\times 1.5]}, f^{[2\times 1.5]}, f^{[3\times 1.5]}\} = \{f^2, f^3, f^5\} = \{8, 7, 4\}.$$

Therefore $f^{mm}(a_i) = \frac{8+7+4}{3} = \frac{19}{3} = 6.33.$

And if 8 judges assign grades 9, 7, 3, 6, 5, 4, 5, 8 to a_i,

for $k = 3$, $\rho = \frac{8+1}{2^3} = 1.125.$

The classified grades in the descending order are: 9, 8, 7, 6, 5, 5, 4, 3.

$$\mathcal{M}_3 = \{f^{[1\times 1.125]}, f^{[2\times 1.125]}, f^{[3\times 1.125]}, f^{[4\times 1.125]}, f^{[5\times 1.125]}, f^{[6\times 1.125]}, f^{[7\times 1.125]}\}$$

$$= \{f^{[1.125]}, f^{[2.25]}, f^{[3.375]}, f^{[4.5]}, f^{[5.625]}, f^{[6.75]}, f^{[7.875]}\}$$

$$= \{f^1, f^2, f^3, f^5, f^6, f^7, f^8\} = \{9, 8, 7, 5, 5, 4, 3\}.$$

Then $f^{mm}(a_i) = \frac{9+8+7+5+5+4+3}{7} = \frac{41}{7} = 5.8.$

4.2. Tie-breaking

When the average majority grades of two candidates are different, the one with the higher average majority grade is ranked before the other. The majority ranking

$\succ_{mm}$ between two candidates evaluated by the same jury is determined by a repeated application of the average majority ranking:

- One starts with $k = 2$
- If $f_k^{mm}(a) > f_k^{mm}(b)$, then $a \succ_{mm} b$
- If $f_k^{mm}(a) = f_k^{mm}(b)$, then the procedure is repeated for $k + 1$.

Let us take the following example to illustrate this definition:

Suppose that *a* et *b* are evaluated by a jury of 7 voters:

a	85	73	78	90	69	70	73
b	77	72	95	83	73	73	66

The ordered profile is :

a	90	85	78	73	71	70	69
b	95	83	77	73	73	72	66

$$\rho = \frac{7+1}{2^2} = \frac{8}{4} = 2,$$

$$\mathcal{M}_2 = \{f^2, f^4, f^6\},$$

$$\mathcal{M}_2(a) = \{85, 73, 70\} \text{ and } f_2^{mm}(a) = \frac{85+73+70}{3} = \frac{228}{3} = 76,$$

$$\mathcal{M}_2(b) = \{83, 73, 72\} \text{ and } f_2^{mm}(b) = \frac{83+73+72}{3} = \frac{228}{3} = 76.$$

$f_2^{mm}(a) = f_2^{mm}(b) = 76$. By definition, one repeats the procedure for $k = 3$ and obtains:

$$\rho = \frac{7+1}{2^3} = \frac{8}{8} = 1,$$

$$\mathcal{M}_3 = \{f^1, f^2, f^3, f^4, f^2, f^6, f^7\},$$

$$\mathcal{M}_3(a) = \{90, 85, 78, 73, 71, 70, 69\} \text{ and } f_3^{mm}(a) = \frac{90+85+78+73+71+70+69}{7}$$

$$= \frac{536}{7} = 76.57,$$

$\mathcal{M}_3(b) = \{95, 83, 77, 73, 73, 72, 66\}$ and $f_3^{mm}(b) = \dfrac{95 + 83 + 77 + 73 + 73 + 72 + 66}{7}$

$$= \frac{539}{7} = 77,$$

$f_3^{mm}(b) = 77 > f_3^{mm}(a) = 76.57$. Then $b >_{mm} a$.

In this example, the average majority evaluation gives exactly the same result as the arithmetic average. This is due to the fact that the set of intermedian grades of each candidate is equal to the set of grades.

Definition 10 (Maximum division index)**.** Let a_i be a candidate or competitor and $G_i = \{g_{i1}, g_{i2}, \dots, g_{in}\}$ be set of grades of a_i with $g_{i1} \geq g_{i2} \geq \cdots \geq g_{in}$ and $\mathcal{M}_k = \{g_{i1}^*, g_{i2}^*, \dots, g_{ij}^*\}$ be the set of his or her intermedian grades obtained with a degree of division k. Then, the smallest whole number k such as $G_i = \mathcal{M}_k$ is called "*maximum division index*" or "*total division index*". It is denoted ν.

In the latter example, the maximum division index is $\nu = 3$.

Theorem 1. *Let a_i be a candidate or competitor and $G_i = \{g_{i1}, g_{i2}, \dots, g_{in}\}$ with $g_{i1} \geq g_{i2} \geq \cdots \geq g_{in}$. If $k = 1$, then $f^{mm}(a_i) = f^{maj}(a_i)$.*

Proof. For $k = 1$, $\rho = \frac{n+1}{2}$. The only intermedian grade is $f^{[\frac{n+1}{2}]}$. It is exactly the value $f^{mm}(a_i)$ will return.

If n is odd $\left[\frac{n+1}{2}\right] = \frac{n+1}{2}$ and if n is even $\left[\frac{n+1}{2}\right] = \frac{n}{2} + 1 = \frac{n+2}{2}$.

It is clear that in this case $f^{mm}(a_i) = f^{maj}(a_i)$.

This theorem shows that if $k = 1$, the result given by the MMCM is the same as that given by the Balinski and Laraki method. The MMCM does not move away from the MJ philosophy in the sense that it is measure-based and not preference based (Arrow's framework). Moreover on this subject, Balinski and Laraki [7] argue that a judge may dislike a wine and yet give it a high grade because of its merit; he or she may also like a wine and yet, with great satisfaction, give it a low grade because of its demerits. A measure provides a common language, be it numerical, ordinal or verbal, to grade and classify. In this respect, Arrow's theorem means that, without a common language, there can be no consistent collective decision.

The MMCM can also give the same result as the MJ for other values of k different from 1.

Theorem 2. *Let* a_i *be a candidate or competitor and* $G_i = \{g_{i1}, g_{i2}, \ldots, g_{in}\}$ *with* $g_{i1} \geq g_{i2} \geq \cdots \geq g_{in}$. *If* $k = \nu$, then $f^{mm}(a_i) = \frac{BMC(a_i)}{n}$.

Proof. Let us consider $G_i = \{g_{i1}, g_{i2}, \ldots, g_{in}\}$. For $k = \nu$, $\mathcal{M}_k = G_i$. So

$$f^{mm}(a_i) = \frac{1}{n}\sum_{j=1}^{n} g_{ij}.$$

Yet

$$\sum_{j=1}^{n} g_{ij} = BMC(a_i).$$

Therefore

$$f^{mm}(a_i) = \frac{BMC(a_i)}{n}.$$

This result shows that, when division is total, the result of the MMCM is equal to the one returned by the BMC. MMCM is in this case an intermediate method between MJ (highest median-based) and BMC (highest average-based). For values of k varying between 1 and ν, it allows to weigh and balance between the advantages and disadvantages of one or the other method.

5. Mean-Median Compromise: Six Problems

In this section, we will apply MMCM to some examples which are considered as paradoxical results for MJ or for BMC and will see whether the MMCM is in accordance with what we hope.

By means of scientifically founded analysis and studies, many scientists such as

Jean-François Laslier, Warren D. Smith, Steven Brams or Manzoor Ahmed Zahid opposed as well MJ as any other form of aggregation based on “MaxMed”.

Steven Brams [14] is much more moderate than others because he accepts the MJ for ranking candidates when there is a small jury.

However, mean-based methods are failing as well as median-based one. The following examples stated in a form of problems illustrate weaknesses of mean-based methods and median-based one as well.

Problem 1. Three friends wish to buy a common dish to share but one of them is a muslim and thus cannot eat the pork meat. All of them like the beef meat but the non-muslims slightly prefer the pork meat. Their profile is as follows:

	A1	A2	A3
Pork	9	9	0
Beef	8	8	9

The median for the pork is 9 and that for the beef is 8. However, the beef meat is the only one which meets the approval of all. In this case, the average is quite preferable to the median. The MMCM will give the same result as the BMC. Indeed, $\nu = 2$.

Problem 2. Five judges evaluate the performance of two skaters. The table of the results is presented as follows:

	Judge 1	Judge 2	Judge 3	Judge 4	Judge 5
A	10	5	6	4	2
B	4	7	7	5	3

The average mark of skater A is 5.4 and that of the skater B is 5.2 and one can then deduce that $A > B.$ Yet 4 judges out of 5 (either 80%) estimate that B is better than A. The median gives $B > A.$

Judge 1, because of his too high evaluation for A, imposed his preference to all the jury. This is not right for a community wanting to be democratic.

The MMCM will give the desired result as well. Indeed, the ordered profile is:

A	10	6	5	4	2
B	7	7	5	4	3

$$\mathcal{M}_2 = \{f^2, f^3, f^5\},$$

$$f_2^{mm}(A) = \frac{6+5+2}{3} = 4.33 \text{ and } f_2^{mm}(B) = \frac{7+5+3}{3} = 5.$$

Here, MMCM give the same result as MJ: $B \succ_{mm} A$.

One could think that this example is only imaginary and that it cannot occur in the real world. The following example is somewhat realistic.

Problem 3. In the Balinski and Laraki's framework, we get the following results for two candidates:

	5	4	3	2	1	0
Candidate A	25%	10%	18%	5%	12%	30%
Candidate B	0%	25%	30%	10%	25%	10%

$$BMC(Candidate\ A) = \frac{25\times 5 + 10\times 4 + 18\times 3 + 5\times 2 + 12\times 1 + 30\times 0}{100} = 2.41,$$

$$BMC(Candidate\ B) = \frac{0\times 5 + 25\times 4 + 30\times 3 + 10\times 2 + 25\times 1 + 10\times 0}{100} = 2.35.$$

The average gives candidate A victorious whereas it is the candidate B who is better accepted by the community. 30% of the population assigned mark 0 to candidate A against 10% only for the candidate B. Except for the maximum note, for all other marks higher than 0, B has a proportion of the population higher than that of A. The 25% of the population that expressed a strong preference for A carried off on 30% that rejected him. In Africa, for example, the winning of A would be likely to disturb social peace and stability.

Finally, probably Balinski was not completely wrong to qualify this method (mean-based) "ridiculous"!

According to MJ, candidate B is victorious. MMCM corroborates this result:

$$\rho = \frac{100+1}{2^2} = 25.25.$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\},$$

$$\mathcal{M}_2(A) = \{5, 3, 0\} \text{ and } \mathcal{M}_2(B) = \{4, 3, 1\},$$

$$f_2^{mm}(A) = f_2^{mm}(B) = 2.66.$$

As there is *ex aequo*, the procedure is repeated with $k = 3$.

$$\rho = \frac{100+1}{2^3} = 12.625,$$

$$\mathcal{M}_3 = \{f^{13}, f^{25}, f^{38}, f^{51}, f^{63}, f^{76}, f^{88}\},$$

$$\mathcal{M}_3(A) = \{5, 5, 3, 3, 1, 0, 0\} \text{ and } f_3^{mm}(A) = \frac{17}{7} = 2.42,$$

$$\mathcal{M}_3(B) = \{4, 4, 3, 3, 2, 1, 1\} \text{ and } f_3^{mm}(B) = \frac{18}{7} = 2.57,$$

$$f_3^{mm}(B) > f_3^{mm}(A) \Rightarrow B >_{mm} A.$$

Problem 4. This illustration was given by Zahid [1].

	Excellent	Very Good	Good	Acceptable	Poor	Reject
A	9%	41%	50%	0%	0%	0%
B	4%	47%	3%	5%	12%	29%

According to the MJ, B wins. However A is judged at least "Good" by all the voters and B is more rejected than A by 29% against 0% for A. According to Borda Majority Count, A would be the winner. Thus, Mean-based methods are not as "ridiculous" as Balinski says it!

Here again, MMCM corroborate the BMC result:

$$\rho = \frac{100+1}{2^2} = 25.25,$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\},$$

$$\mathcal{M}_2(A) = \{\text{Very Good, Good, Good}\}$$

and

$$\mathcal{M}_2(B) = \{\text{Very good, Very Good, Reject}\}.$$

By allotting the appropriate mark to each grade, one obtains:

$$\mathcal{M}_2(A) = \{4,\ 3,\ 3\} \text{ and } f_2^{mm}(A) = \frac{4+3+3}{3} = 3.33,$$

$$\mathcal{M}_2(B) = \{4,\ 4,\ 0\} \text{ and } f_2^{mm}(B) = \frac{4+4+0}{3} = 2.66,$$

$$f_2^{mm}(A) > f_2^{mm}(B) \Rightarrow A >_{mm} B.$$

Problem 5. This example is drawn from Ahmed Zahid's PhD thesis [2].

	Excellent	Very Good	Good	Acceptable	Poor	Reject
A	1%	98%	1%	0%	0%	0%
B	50%	1%	0%	0%	0%	49%

According to the MJ, B wins whereas he is rejected by 49% of the population and A is judged at least "Good" by all the voters!

The BMC circumvent this paradox: A is the winner by the average. If we process this problem with the MMCM, we obtain exactly the same result:

$$\rho = \frac{100+1}{2^2} = 25.25,$$

$$\mathcal{M}_2 = \{f^{25},\ f^{51},\ f^{76}\},$$

$$\mathcal{M}_2(A) = \{\text{Very Good, Very Good, Good}\}$$

and

$$\mathcal{M}_2(B) = \{\text{Excellent, Very Good, Reject}\}.$$

By assigning the appropriate mark to each grade, one obtains:

$$\mathcal{M}_2(A) = \{4,\ 4,\ 4\} \text{ and } f_2^{mm}(A) = \frac{4+4+4}{3} = 4,$$

$$\mathcal{M}_2(B) = \{5,\ 4,\ 0\} \text{ and } f_2^{mm}(B) = \frac{5+4+0}{3} = 3,$$

$$f_2^{mm}(A) > f_2^{mm}(B) \Rightarrow A >_{mm} B.$$

MMCM circumvents also this paradox of the MJ.

Problem 6. Zahid [1, 2] presents another example similar to the precedent.

	Excellent	Very Good	Good	Acceptable	Poor	Reject
A	50%	0%	0%	0%	0%	50%
B	0%	0%	0%	0%	51%	49%

According to MJ, B wins whereas 100% of the electorate estimates that he is at most "Poor" (with 49% of rejection) and that A is considered to be "Excellent" by half of the electorate. The BMC is, here, preferable to MJ because the average of A is higher than that of B.

If we process this example by MMCM, we obtain:

$$\rho = \frac{100+1}{2^2} = 25.25,$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\},$$

$$\mathcal{M}_2(A) = \{\text{Excellent, Reject, Reject}\} \text{ et } \mathcal{M}_2(B) = \{\text{Poor, Poor, Reject}\}.$$

By assigning the appropriate mark to each rank, one obtains:

$$\mathcal{M}_2(A) = \{5, 0, 0\} \text{ and } f_2^{mm}(A) = \frac{5+0+0}{3} = 1.66,$$

$$\mathcal{M}_2(B) = \{1, 1, 0\} \text{ and } f_2^{mm}(B) = \frac{1+1+0}{3} = 0.66,$$

$$f_2^{mm}(A) > f_2^{mm}(B) \Rightarrow A >_{mm} B.$$

Examples 1 to 6 constitute paradoxical results declining with sufficiency the individual capacity of both mean and median approaches to provide logically plausible results in all the cases. These problems are all solved by a method of compromise: the MMCM.

6. Conclusion

In this article, we presented a new voting system obtained by a combination of Borda Majority Count [1, 2] and Majority Judgement [7, 8, 9]. The extension of each of the two above mentioned methods consists of dividing the ordered list of the grades or marks into *m* equal parts and to retain only the bounds of the internal parts. The value returned by the Mean-Median Compromise Method (MMCM) is the average of the selected grades or marks (intermedian grades). We propose also a tie-breaking mechanism by increasing the number of parts in the list of grades. The suggested tie-breaking rule differs as well from the BMC as the MJ.

MMCM coincides with BMC when the number of division is equal to the maximum division index and agrees with the MJ when the number of division is equal to 1. We thought fit that, as coincidence with the MJ can be also observed for values of number of division different from 1, it is necessary to begin the process with a number of division equal to 2.

The article refers to six didactic examples made from situations discussed for BMC and MJ. For each evoked paradox, if one of the two opposite methods circumvents it, MMCM circumvents it too. MMCM then appears like a better compromise between both methods. It keeps, indeed, the common properties of its parents, namely:

● it maximizes the voter-expressivity;

● it fulfills the criteria of anonymity, neutrality, unanimity, monotoniciy and Arrow's Independence of the Irrelevant Alternatives;

● it is reduced to the approval voting if the allowed scores are restricted to the two values 0 and 1, i.e., each voter "approves (1)" or "disapproves (0)" each candidate;

● it proceeds in a round;

● it can cause the enthusiasm and the participation of the voters to use a new method of really democratic election where they can fully express their opinion;

● it enjoys immunity to candidate cloning;

● it has the enjoyable property that "betraying one's favorite candidate does not pay the voter!" - voter does not take advantage of strategic vote any more.

● it easily allows having no opinion votes;

● it evades to some extent the Arrow and Gibbard-Satterthwaite impossibility theorems.

However, it also inherits the insufficiencies of its parents, namely:

● it is not Condorcet-consistent (Cf. Problem 1);

● it is not join-consistent and does not circumvent the no-show paradox (considering that MJ does not fulfill them);

● the offsetting effect of the arithmetic mean (which hurts BMC), although reduced, always exists.

The above-mentioned weaknesses are not in the same magnitude as for BMC and the MJ. Besides, Condorcet-consistency, as Problem 1 shows it, is not always a desirable condition in an election. It would not be astonishing that one finds cases where the MMCM is vulnerable to join-inconsistency and the No-Show paradox which hurts MJ.

In a forthcoming publication, we will analyze MMCM and will study its vulnerability vis-a-vis paradoxes from which the BMC and the MJ suffer. We hope to show that the MMCM is less vulnerable to the tyranny of the majority which hurts MJ and expose the fact that BMC gives too much power to strategic voters who assign extreme grades. MMCM stands it.

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References

[1] Manzoor Ahmed Zahid, Majority judgment and pardoxical results, Int. J. Arts and Sciences 4(20) (2009), 121-131.

[2] Manzoor Ahmed Zahid, A new framework for elections, PhD thesis, Tilburg University, 2012.

[3] José Carlos R. Alcantud and Annick Laruelle, Dis&approval voting: a characterization, Social Choice and Welfare 43(1) (2014), 1-10.

[4] Kenneth J. Arrow, A difficulty in the concept of social welfare, J. Political Economy 58(4) (1950), 328-346.

[5] Kenneth J. Arrow, Social Choice and Individual Values, Yale University Press, 1951.

[6] Kenneth J. Arrow, Social Choice and Individual Values, Yale University Press, 1963.

[7] Michel Balinski and Rida Laraki, A theory of measuring, electing and ranking, Proceedings of the National Academy of Sciences of the United States of America, 2007, pp. 8720-8725.

[8] Michel Balinski and Rida Laraki, Majority Judgment: Measuring, Ranking and Electing, MIT Press, 2010.

[9] Michel Balinski and Rida Laraki, Jugement majoritaire vs vote majoritaire, Cahier 2012-37, CNRS, 2012.

[10] Michel Balinski and Rida Laraki, Rendre les élections aux électeurs: le jugement majoritaire, Technical report, Terra Nova, 2012.

[11] Michel Balinski and Rida Laraki, In Situ and Laboratory Experiments on Electoral Law Reform, chapter 2: Election by Majority Judgment: Experimental Evidence, Studies in Public Choice, Springer, 2011.

[12] Antoinette Baujard, Frédéric Gavrel, Herrade Igersheim, Jean-François Laslier and Isabelle Lebon, Who's favored by evaluative voting? an experiment conducted during the 2012 french presidentiel election, Cahier 2013-05, CNRS, 2013.

[13] Antoinette Baujard and Herrade Igersheim, Expérimentation du vote par note et du vote par approbation lors de l'élection présidentielle française du 22 avril 2007, Technical report, Centre d'analyse stratégique, 2007.

[14] Antoinette Baujard and Herrade Igersheim, Expérimentation du vote par note et du vote par approbation le 22 avril 2007, premiers resultants, Revue Économique 60(1) (2009), 189-201.

[15] C. Bialik, And the oscar goes to... not its voting system, The Wall Street Journal, February 6, 2009.

[16] Steven Brams, Grading candidates, American Scientist, 2011.

[17] Edurne Falco and José Luis Garcia-Lapresta, A distance-based extension of the majority judgement voting system, Acta Universitatis Matthiae Belii, series Mathematics 18 (2011), 17-27.

[18] Edurne Falco, José Luis Garcia-Lapresta and Llorenç Rosello, Allowing voters to be imprecise: a proposal using multiple linguistic terms, Information Sciences, 2013.

[19] Edurne Falco, José Luis Garcia-Lapresta and Llorenç Rosello, Aggregating imprecise linguistic expressions, Human-Centric Decision Making Models for Social Sciences, P. Guo and W. Pedrycz, eds., 2014.

[20] Etienne Farvaque, Hugo Jayet and Lionel Ragot, Quel mode de scrutin pour quel "vainqueur"? une expérience sur le vote préférentiel transferable, Revue d'économie Politique 119(2) (2009), 221-246.

[21] Allan Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973), 587-601.

[22] Jean-François Laslier, A note on choosing the alternative with the best median evaluation, Cahier 2010-17, CNRS, 2010.

[23] Jean-François Laslier, In Situ and Laboratory Experiments on Electoral Law Reform, chapter 5: Lessons from In Situ Experiments during French Elections, Studies in Public Choice, Springer, 2011.

[24] Jean-François Laslier, On choosing the alternative with the best median evaluation, Public Choice 153(3) (2012), 269-277.

[25] Jean-François Laslier and Karine Van der Straeten, Une expérience de vote par assentiment, Revue française de science politique 54(1) (2014), 99-130.

[26] M. A. Satterthwaite, Strategy-proofness and arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, J. Economic Theory 10 (1975), 187-217.

[27] Amartya K. Sen, Collective Choice and Social Welfare, Holden-Day, London, 1970.

[28] Amartya K. Sen, The impossibility of a paretian liberal, J. Political Economy 78 (1970), 152-157.

[29] Allan Taylor, Social Choice and the Mathematics of Manipulation, Cambridge University Press, New York, 2005.