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INCLUSION RESULTS OF *p*-VALENT ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract

In this paper, we study some properties of the inclusion relationships for two subclasses and some other interesting properties of p-valent functions which are defined by linear operator.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a, p]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots \; (a \in \mathbb{C};\; p \in \mathbb{N} = \{1, 2, \ldots\}).$$

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Also, let $A(p)$ be the subclass of the functions $f(z) \in \mathcal{H}(\mathbb{U})$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}). \tag{1.1}$$

Let $f(z) \in A(p)$ be given by (1.1) and $g \in A(p)$ is given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z) \quad (z \in \mathbb{U}).$$

Also, let $\mathcal{P}_k(p, \rho)$ be the class of functions $h(z) = p + \sum_{k=p+1}^{\infty} c_k z^k$ which are analytic in $\mathbb{U}$ and satisfying the properties $h(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re}\{h(z)\} - \rho}{p - \rho} \right| d\theta \le k\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \rho < p$. The class $\mathcal{P}_k(p, \rho)$ was introduced by Aouf [2].

We note that

(i) $\mathcal{P}_k(1, \rho) = \mathcal{P}_k(\rho)$ $(k \ge 2, 0 \le \rho < 1)$ (see Padmanabhan and Parvatham [21]).

(ii) $\mathcal{P}_k(1, 0) = \mathcal{P}_k (k \ge 2)$ (see Pinchuk [24]).

(iii) $\mathcal{P}_2(p, \rho) = \mathcal{P}(p, \rho)$ $(0 \le \rho < p, p \in \mathbb{N})$, where $\mathcal{P}(p, \rho)$ is the class of functions with positive real part greater than ρ (see Aouf [2]).

(iv) $\mathcal{P}_2(p, 0) = \mathcal{P}(p)$ $(p \in \mathbb{N})$, where $\mathcal{P}(p)$ is the class of p-valent functions with positive real part (see [2]).

(v) $\mathcal{P}_2(1, \rho) = \mathcal{P}(\rho)$ $(0 \le \rho < 1)$, where $\mathcal{P}(\rho)$ is the class of functions with positive real part greater than ρ.

(vi) $\mathcal{P}_2(1, 0) = \mathcal{P}$ is the class of functions with positive real part.

Let $h(z) \in \mathcal{P}_k(p, \rho)$, then we can write $h(z)$ of the form

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z) \; (z \in \mathbb{U};\, h_1,\, h_2 \in \mathcal{P}(p, \rho)).$$

The classes $\mathcal{S}_k(p, \rho)$ and $\mathcal{C}_k(p, \rho)$ are related to the class $\mathcal{P}_k(p, \rho)$ and can be defined as

$$f(z) \in \mathcal{S}_k(p, \rho) \Leftrightarrow \frac{zf'(z)}{f(z)} \in \mathcal{P}_k(p, \rho) \quad (z \in \mathbb{U}) \tag{1.3}$$

and

$$f(z) \in \mathcal{C}_k(p, \rho) \Leftrightarrow \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_k(p, \rho) \quad (z \in \mathbb{U}). \tag{1.4}$$

El-Ashwah and Drbuk [12] defined the linear operator $\mathcal{J}_{\lambda, \ell}^{m, p}(a, c, A):$ $A(p) \to A(p)$ as follows:

$$\mathcal{J}_{\lambda, \ell}^{m, p}(a, c, A) f(z)$$

$$= z^p + \frac{\Gamma(c + Ap)}{\Gamma(a + Ap)} \sum_{k=p+1}^{\infty} \left(\frac{p + \ell + \lambda(k - p)}{p + \ell}\right)^m \frac{\Gamma(a + kA)}{\Gamma(c + kA)} a_k z^k, \tag{1.5}$$

where $m \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $A > 0$, $\lambda \ge 0$, $\ell > -p$, $a, c \in \mathbb{C}$ be such that $\mathrm{Re}(c - a) > 0$ and $\mathrm{Re}(a) > -Ap$.

Putting $a = c$ in (1.5) and by specializing the parameters λ, ℓ and p, we obtain the following operators studied by various authors:

(i) $\mathcal{J}_{\lambda, \ell}^{m, p}(a, a, A) f(z) = J_p^m(\lambda, \ell) f(z) (\lambda \ge 0,\, \ell > -p,\, p \in \mathbb{N}$ and $m = \{0, \pm 1, \pm 2, \ldots\})$ (see [25]).

(ii) $\mathcal{J}_{\lambda, \ell}^{m, p}(a, a, A) f(z) = I_p^m(\lambda, \ell) f(z)$ $(\ell \ge 0,\, \lambda \ge 0,\, p \in \mathbb{N}$ and $m \in \mathbb{N}_0$

$= \mathbb{N} \cup \{0\})$ (see [8]).

(iii) $\mathcal{J}_{1,\ell}^{m,p}(a, a, A)f(z) = I_p(m, \ell)f(z)(\ell \geq 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [16, 31]).

(iv) $\mathcal{J}_{\lambda,0}^{m,p}(a, a, A)f(z) = D_{\lambda,p}^{m}f(z)\ (\lambda \geq 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [3]).

(v) $\mathcal{J}_{1,0}^{m,p}(a, a, A)f(z) = D_p^m f(z)(p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [4, 15]).

(vi) $\mathcal{J}_{\lambda,\ell}^{-m,p}(a, a, A)f(z) = J_p^m(\lambda, \ell)f(z)\ (\ell \geq 0, \lambda \geq 0,\ p \in \mathbb{N}$ and $m \in \mathbb{N}_0)$ (see [5, 11, 30]).

(vii) $\mathcal{J}_{1,1}^{-m,p}(a, a, A)f(z) = D^m f(z)\ (m \in \mathbb{Z})$ (see [23]).

(viii) $\mathcal{J}_{1,\ell}^{m,1}(a, a, A)f(z) = I_\ell^m f(z)(\ell \geq 0$ and $m \in \mathbb{N}_0)$ (see [9, 10]).

(ix) $\mathcal{J}_{\lambda,0}^{m,1}(a, a, A)f(z) = D_\lambda^m f(z)(\lambda \geq 0$ and $m \in \mathbb{N}_0)$ (see [1]).

(x) $\mathcal{J}_{1,0}^{m,1}(a, a, A)f(z) = D^m f(z)(m \in \mathbb{N}_0)$ (see [28]).

(xi) $\mathcal{J}_{\lambda,1}^{-m,1}(a, a, A)f(z) = I_\lambda^{-m} f(z)\ (\lambda \geq 0$ and $m \in \mathbb{N}_0)$ (see [22, 6]).

(xii) $\mathcal{J}_{1,1}^{-m,1}(a, a, A)f(z) = I^m f(z)\ (m \in \mathbb{N}_0)$ (see [13]).

(xiii) $\mathcal{J}_{\lambda,\ell}^{0,1}(a-1, c-1, 1)f(z) = \mathcal{L}(a, c)f(z)$ (see [7]).

(xiv) $\mathcal{J}_{\lambda,\ell}^{0,p}(a-p, c-p, 1)f(z) = \mathcal{L}_p(a, c)f(z)$ (see [26]).

(xv) $\mathcal{J}_{1,1}^{-m,1}(a, a, A)f(z) = \mathcal{P}^m f(z)(m \in \mathbb{N}_0)$ (see [14]).

(xvi) $\mathcal{J}_{\lambda,\ell}^{0,1}(\beta, \alpha+\beta, 1)f(z) = \mathcal{Q}_\beta^\alpha f(z)\ (\alpha > 0; \beta > -1)$ (see [14]).

(xvii) $\mathcal{J}_{1,1}^{-1,1}(a, a, A)f(z) = \mathcal{J}_\ell f(z)\ (\ell > -1)$ (see [14]).

(xviii) $\mathcal{J}_{\lambda,\ell}^{0,p}(\beta, \alpha+\beta, 1)f(z) = \mathcal{Q}_{\beta,p}^\alpha f(z)\ (\alpha > 0; \beta > -p)$ (see [17, 29]).

It is readily verified from (1.5) that

$$\mathcal{J}_{\lambda,\ell}^{m,p}(a+1, c, A)f(z)$$

$$= \frac{a}{a+Ap}\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z) + \frac{A}{a+Ap}z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z))' \tag{1.6}$$

and

$$\mathcal{J}_{\lambda,\ell}^{m+1,p}(a, c, A)f(z)$$

$$= (1 - \frac{p\lambda}{p+\ell})\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z) + \frac{\lambda}{p+\ell}z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z))'. \tag{1.7}$$

Now, we can define the following classes of analytic functions by using the operator $\mathcal{J}_{\lambda,\ell}^{m,p}$ as follows:

$$\mathcal{S}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho)$$

$$= \{f(z) \in \mathcal{A}(p) : \mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z) \in \mathcal{S}_k(p, \rho), z \in \mathbb{U}\}, \tag{1.8}$$

and

$$\mathcal{C}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho)$$

$$= \{f(z) \in \mathcal{A}(p) : \mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z) \in \mathcal{C}_k(p, \rho), z \in \mathbb{U}\}. \tag{1.9}$$

Obviously, we know that

$$f(z) \in \mathcal{C}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho). \tag{1.10}$$

Lemma 1 [18]. *Let $\theta(u, v)$ be a complex-valued function such that*

$$\theta : D \to \mathbb{C}, D \subset \mathbb{C} \times \mathbb{C} \ (\mathbb{C} \text{ is the complex plane})$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u, v)$ satisfies the following conditions:

(i) $\theta(u, v)$ *is continuous in D.*

(ii) $(1, 0) \in D$ *and* $\mathrm{Re}\{\theta(1, 0)\} > 0$.

(iii) *for all* $\theta(iu_2, v_1) \in D$ *such that*

$$v_1 \leq -\frac{1}{2}(1 + u_2^2), \quad \mathrm{Re}\{\theta(iu_2, v_1)\} \leq 0.$$

Let

$$q(z) = 1 + q_1 z + q_2 z^2 + \dots \tag{1.11}$$

be analytic in U such that $(q(z), zq'(z)) \in D$ $(z \in U)$. If

$$\mathrm{Re}\left\{\theta(q(z), zq'(z))\right\} > 0 \ (z \in U),$$

then

$$\mathrm{Re}\left\{\theta(q(z), zq'(z))\right\} > 0 \text{ in } U.$$

Lemma 2 [19]**.** *Let* $p(z)$ *be analytic in U with* $p(0) = a$ *and* $\mathrm{Re}\{p(z)\} > 0$, $z \in U$. *Then, for* $s > 0$ *and* $\mu \in \mathbb{C} \setminus \{-1\}$,

$$\mathrm{Re}\left\{p(z) + \frac{szp'(z)}{p(z) + \mu}\right\} > 0, \quad (|z| < r_0), \tag{1.12}$$

where r_0 *is given by*

$$r_0 = \frac{|\mu + 1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{\frac{1}{2}}}}, \ A = 2(s+1)^2 + |\mu|^2 - 1. \tag{1.13}$$

Lemma 3 [20]**.** *Let* ψ *be convex function and let g be starlike in U. Then, for F analytic in U with* $F(0) = 1$, $((\psi * Fg)/(\psi * g))$ *is contained in the convex hull of* $F(U)$.

2. The Main Results

Unless otherwise mentioned, we assume throughout this paper that $m \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $k \geq 2$, $0 \leq \rho < p$, $p \in \mathbb{N}$, $A > 0$, $\lambda \geq 0$, $\ell > -p$, $a, c \in \mathbb{C}$ be such that $\mathrm{Re}(c - a) > 0$ and $\mathrm{Re}(a) > -Ap$.

Theorem 1. *Let* $a = a_1 + ia_2, \dfrac{\mathrm{Re}(a)}{A} > 0$ *and* $f \in \mathcal{A}(p)$, *then*

$$\mathcal{S}^m_{k,\lambda,\ell}(a+1, c, A, p, \rho) \subset \mathcal{S}^m_{k,\lambda,\ell}(a, c, A, p, \rho). \tag{2.1}$$

Proof. We begin by setting

$$\frac{z(\mathcal{J}^{m,p}_{\lambda,\ell}(a, c, A)f(z))'}{\mathcal{J}^{m,p}_{\lambda,\ell}(a, c, A)f(z)} = (p-\rho)h(z) + \rho.$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\{(p-\rho)h_1(z) + \rho\} - \left(\frac{k}{4} - \frac{1}{2}\right)\{(p-\rho)h_2(z) + \rho\},$$

where h_i is analytic in U and $h_i(0) = 1 (i = 1, 2)$. By using the identity (1.6) in (2.2) and differentiating the resulting, we obtain

$$\frac{z(\mathcal{J}^{m,p}_{\lambda,\ell}(a+1, c, A)f(z))'}{\mathcal{J}^{m,p}_{\lambda,\ell}(a+1, c, A)f(z)} = \left\{\rho + (p-\rho)h(z) + \frac{(p-\rho)zh'(z)}{(p-\rho)h(z) + \rho + \frac{a}{A}}\right\} \in \mathcal{P}_k. \tag{2.3}$$

This implies that

$$h_i(z) + \frac{zh_i'(z)}{(p-\rho)h_i(z) + \rho + \frac{a}{A}} \in \mathcal{P} \quad (i = 1, 2; z \in U). \tag{2.4}$$

Now, from the functoinal $\psi(u, v)$ by setting $u = h_i(z) = u_1 + iu_2$ and $v = zh_i'(z) = v_1 + iv_2$. Thus

$$\psi(u, v) = u + \frac{v}{(p-\rho)u + \rho + \frac{a}{A}}. \tag{2.5}$$

The first and second conditions of Lemma 1 are satisfied by using $\psi(u, v)$. To prove the third condition:

$$\mathrm{Re}\,\{\psi(iu_2, v_1)\} = \mathrm{Re}\left\{\frac{v_1}{(p-\rho)iu_2 + \rho + \frac{a}{A}}\right\}.$$

$$\leq -\frac{\left(\rho + \frac{a_1}{A}\right)(1 + u_2^2)}{2[((p-\rho)u_2 + \frac{a_2}{A})^2 + (\rho + \frac{a_1}{A})^2]} < 0 \tag{2.6}$$

Therefore, by applying Lemma 1, $h_i \in \mathcal{P}(i = 1, 2)$ and thus $h \in \mathcal{P}_k(z \in U)$. The proof of Theorem 1 is completed.

Theorem 2. *Let* $\frac{p + \ell}{\lambda} > p - \rho$ *and* $f \in \mathcal{A}(p)$, *then*

$$\mathcal{J}_{k,\lambda,\ell}^{m+1}(a, c, A, p, \rho) \subset \mathcal{J}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho). \tag{2.7}$$

Proof. Making use of (1.7), the proof of Theorem 2 is similar to that of Theorem 1, so it is omitted.

Theorem 3. *Let* $a = a_1 + ia_2$, $\frac{\mathrm{Re}(a)}{A} > 0$ *and* $f \in \mathcal{A}(p)$, *then*

$$\mathcal{C}_{k,\lambda,\ell}^{m}(a+1, c, A, p, \rho) \subset \mathcal{C}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho). \tag{2.8}$$

Proof. Applying (1.10) and Theorem 1, we assume that

$$f \in \mathcal{C}_{k,\lambda,\ell}^{m}(a+1, c, A, p, \rho) \Leftrightarrow \frac{zf'}{p} \in \mathcal{S}_{k,\lambda,\ell}^{m}(a+1, c, A, p, \rho)$$

$$\Rightarrow \frac{zf'}{p} \in \mathcal{S}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho)$$

$$\Leftrightarrow f \in \mathcal{C}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho).$$

This completes the proof of Theorem 3.

Theorem 4. *Let* $\frac{p + \ell}{\lambda} > p - \rho$ *and* $f \in \mathcal{A}(p)$, *then*

$$\mathcal{C}_{k,\lambda,\ell}^{m+1}(a, c, A, p, \rho) \subset \mathcal{C}_{k,\lambda,\ell}^{m}(a, c, A, p, \rho). \tag{2.9}$$

Proof. The proof of Theorem 4 is similar that to the proof of Theorem 3. Now, we study the closure properties of generalized integral operator defined by Saitoh et al. [27] as follows:

$$\mathcal{L}_{c,p} f(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) dt \ (f \in \mathcal{A}(p), \ c > -p). \tag{2.10}$$

Theorem 5. *If* $f \in \mathcal{S}^{m}_{k, \lambda, \ell}(a, c, A, p, \rho)$, *then* $\mathcal{L}_{c, p} f(z) \in \mathcal{S}^{m}_{k, \lambda, \ell}(a, c, A, p, \rho)$ $(c > -p)$, *where the operator* $\mathcal{J}_{c, p}$ *is defined by* (2.10).

Proof. Suppose $f \in \mathcal{S}^{m}_{k}(a, c, A, p, \rho)$ and putting

$$\frac{z(\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A)\mathcal{L}_{c, p} f(z))'}{\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A)\mathcal{L}_{c, p} f(z)} = (p - \rho)h(z) + \rho \tag{2.11}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\{(p - \rho)h_1(z) + \rho\} - \left(\frac{k}{4} - \frac{1}{2}\right)\{(p - \rho)h_2(z) + \rho\},$$

where h is analytic in U and $h(0) = 1.$ By using (2.10), we have

$$z(\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A)\mathcal{L}_{c, p} f(z))' = (c + p)\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A) f(z) - c\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A)\mathcal{L}_{c, p} f(z) \tag{2.12}$$

Then by using (2.11) and (2.12), we get

$$(c + p)\frac{\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A) f(z)}{\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A)\mathcal{L}_{c, p} f(z)} = (p - \rho)h(z) + \rho + c \tag{2.13}$$

Differentiating (2.13) logarithmically with respect to z, we obtain

$$\frac{1}{p - \rho}\left(\frac{z(\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A) f(z))'}{\mathcal{J}^{m, p}_{\lambda, \ell}(a, c, A) f(z)} - \rho\right) = h(z) + \frac{zh'(z)}{(p - \rho)h(z) + \rho + c} \in \mathcal{P}_k. \tag{2.14}$$

This implies that

$$h_i(z) + \frac{zh_i'(z)}{(p - \rho)h_i(z) + \rho + c} \in \mathcal{P}\ (i = 1, 2;\ z \in U). \tag{2.15}$$

From the functoinal $\psi(u, v)$ by setting $u = h_i(z) = u_1 + iu_2$ and $v = zh_i'(z) = v_1 + iv_2$. Thus

$$\psi(u, v) = u + \frac{v}{(p - \rho)u + \rho + c}. \tag{2.16}$$

the conditions of Lemma 1 are satisfied by using $\psi(u, v)$ as the proof of Theorem 1.

Hence $h_i \in \mathcal{P}$ $(i = 1, 2)$ and thus $h \in \mathcal{P}_k(z \in U)$, which implies that $\mathcal{L}_{c,\,p} f(z) \in \mathcal{S}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$. The proof of Theorem 5 is completed.

Next, we introduce an inclusion property for the subclass $\mathcal{C}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$ involving the operator $\mathcal{L}_{c,\,p} f(z)$ which is given by the following theorem:

Theorem 6. *If* $f \in \mathcal{C}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$, *then* $\mathcal{L}_{c,\,p} f(z) \in \mathcal{C}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$ $(c > -p)$, *where the operator* $\mathcal{J}_{c,\,p}$ *is defined by* (2.10).

Proof. From Theorem 5 it follows that

$$f \in \mathcal{C}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$$

$$\Rightarrow \mathcal{L}_{c,\,p}\left(\frac{zf'(z)}{p}\right) \in \mathcal{S}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho) \text{ (by Theorem 5)}$$

$$\Leftrightarrow z\left(\frac{\mathcal{L}_{c,\,p} f(z)}{p}\right)' \in \mathcal{S}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$$

$$\Leftrightarrow \mathcal{L}_{c,\,p} f(z) \in \mathcal{C}^m_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho),$$

which evidently proves Theorem 6.

Theorem 7. *Let* $a = a_1 + ia_2$, *and* $\frac{\operatorname{Re}(a)}{A} > 0$. *If* $f \in \mathcal{C}_{k,\lambda,\,\ell}(a+1,\, c,\, A,\, p,\, \rho)$ $(z \in U)$, *then* $f \in \mathcal{C}_{k,\lambda,\,\ell}(a,\, c,\, A,\, p,\, \rho)$ *for*

$$|z| < r_0 = \frac{|\mu + 1|}{\sqrt{D + (D^2 - |\mu^2 - 1|)^{\frac{1}{2}}}}, \tag{2.17}$$

where $D = 2(s+1)^2 + |\mu|^2 - 1$, *with* $\mu = ((\rho + \frac{a}{A})/(p - \rho)) \neq -1$ *and* $s = (1/(p - \rho))$.

Proof. Let $f \in \mathcal{C}^m_{k,\lambda,\,\ell}(a+1,\, c,\, A,\, p,\, \rho)$ $(z \in U)$ and let

$$\frac{z(\mathcal{J}^{m,\,p}_{\lambda,\,\ell}(a,\, c,\, A) f(z))'}{\mathcal{J}^{m,\,p}_{\lambda,\,\ell}(a,\, c,\, A) f(z)} = (p - \rho) h(z) + \rho \tag{2.18}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\{(p-\rho)h_1(z) + \rho\} - \left(\frac{k}{4} + \frac{1}{2}\right)\{(p-\rho)h_2(z) + \rho\},$$

where h_i is analytic in U and $h_i(0) = 1 (i = 1, 2)$ and $\mathrm{Re}\{h_i(z)\} > 0\, (i = 1, 2)$. By using the identity (1.6) in (2.18) and differentiating the resulting, we have

$$\frac{1}{p-\rho}\left(\frac{z(\mathcal{J}^{m,p}_{\lambda,\ell}(a+1, c, A)f(z))'}{\mathcal{J}^{m,p}_{\lambda,\ell}(a+1, c, A)f(z)} - \rho\right)$$

$$= h(z) + \frac{(1/(p-\rho))zh'(z)}{h(z) + ((\rho + \frac{a}{A})/(p-\rho))}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\{h_1(z) + \frac{(1/(p-\rho))zh_1'(z)}{h_1(z) + ((\rho + \frac{a}{A})/(p-\rho))}\}$$

$$- \left(\frac{k}{4} + \frac{1}{2}\right)\left\{h_2(z) + \frac{(1/(p-\rho))zh_2'(z)}{h_2(z) + ((\rho + \frac{a}{A})/(p-\rho))}\right\}, \tag{2.19}$$

where $\mathrm{Re}\,\{h_i(z)\} > 0\, (i = 1, 2)$. Applying Lemma 2 with $s = (1/(p-\rho))$ and $\mu = ((\rho + \frac{a}{A})/(p-\rho)) \neq -1$, we have

$$\mathrm{Re}\left\{h_i(z) + \frac{(1/(p-\rho))zh_i'(z)}{h_i(z) + ((\rho + \frac{a}{A})/(p-\rho))}\right\} > 0\ (|z| < r_0), \tag{2.20}$$

where r_0 is given by (2.17). This completes the proof of Theorem 7.

Theorem 8. *Let* $\frac{p+\ell}{\lambda} > p - \rho$. *If* $f \in \mathcal{C}^{m+1}_{k,\lambda,\ell}(a, c, A, p, \rho)\ (z \in U)$, *then* $f \in \mathcal{C}^{m}_{k,\lambda,\ell}(a, c, A, p, \rho)$ *for*

$$|z| < r_0 = \frac{|\mu + 1|}{\sqrt{D + (D^2 - |\mu^2 - 1|)^{\frac{1}{2}}}}, \tag{2.21}$$

where $D = 2(s+1)^2 + |\mu|^2 - 1$, *with* $\mu = ((\rho + (\frac{p+\ell}{\lambda} - \rho))/(p-\rho)) \neq -1$ *and* $s = (1/(p-\rho))$.

Proof. Let $f \in \mathcal{C}^{m+1}_{k,\lambda,\ell}(a, c, A, p, \rho)$ $(z \in U)$ and let

$$\frac{z(\mathcal{J}^{m,p}_{\lambda,\ell}(a, c, A)f(z))'}{\mathcal{J}^{m,p}_{\lambda,\ell}(a, c, A)f(z)} = (p-\rho)h(z) + \rho. \tag{2.22}$$

$$= \left(\frac{k}{4}+\frac{1}{4}\right)\{(p-\rho)h_1(z)+\rho\} - \left(\frac{k}{4}-\frac{1}{4}\right)\{(p-\rho)h_2(z)+\rho\},$$

where h_i is analytic in U and $h_i(0) = 1 (i = 1, 2)$ and $\mathrm{Re}\{h_i(z)\} > 0 (i = 1, 2)$. By using the identity (1.7) in (2.22) and differentiating the resulting, we have

$$\frac{1}{p-\rho}\left(\frac{z(\mathcal{J}^{m+1,p}_{\lambda,\ell}(a, c, A)f(z))'}{\mathcal{J}^{m+1,p}_{\lambda,\ell}(a, c, A)f(z)} - \rho\right)$$

$$= h(z) + \frac{(1/(p-\rho))zh'(z)}{h(z) + ((\rho + (\frac{p+\ell}{\lambda} - p))/(p-\rho))}$$

$$= \left(\frac{k}{4}+\frac{1}{4}\right)\{h_1(z) + \frac{(1/(p-\rho))zh_1'(z)}{h_1(z) + ((\rho + (\frac{p+\ell}{\lambda} - p))/(p-\rho))}\}$$

$$- \left(\frac{k}{4}-\frac{1}{4}\right)\left\{h_2(z) + \frac{(1/(p-\rho))zh_2'(z)}{h_2(z) + ((\rho + (\frac{p+\ell}{\lambda} - p))/(p-\rho))}\right\}, \tag{2.23}$$

where $\mathrm{Re}\{h_i(z)\} > 0$ $(i = 1, 2)$. Applying Lemma 2 with $s = (1/(p-\rho))$ and $\mu = ((\rho + (\frac{p+\ell}{\lambda} - p))/(p-\rho)) \neq -1$, we have

$$\mathrm{Re}\left\{h_i(z) + \frac{(1/(p-\rho))zh_i'(z)}{h_i(z) + ((\rho + (\frac{p+\ell}{\lambda} - p))/(p-\rho))}\right\} > 0 \ (|z| < r_0), \tag{2.24}$$

where r_0 is given by (2.21). This completes the proof of Theorem 8.

Theorem 9. *Let* ϕ *be a convex functions and* $f \in \mathcal{S}_2(p, \rho)$. *Then* $G \in \mathcal{S}_2(p, \rho)$, *where* $G = \phi * f$.

Proof. Let $G = \phi * f$. Then

$$\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z) = \mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)(\phi * f)(z) = \phi(z) * \mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z). \tag{2.25}$$

Since $f \in \mathcal{S}_2(p, \rho)$, Then $\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)\, f \in \mathcal{S}_2(p, \rho)$. By logarithmic differentiation of (2.25) and after some calculations, we have

$$\frac{z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z))'}{p\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z)} = \frac{\phi(z) * F(z)\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z)}{\phi(z) * \mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z)}, \tag{2.26}$$

where $F(z) = z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z))' / p\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)f(z)$ is analytic in U and $F(0) = 1$. From Lemma 3, we can see that $z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z))' / p\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z)$ is contained in the convex hull of $F(U)$. Since $z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z))' / p\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z)$ is analytic in U and

$$F(U) = \Omega = \left\{\omega := \frac{z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)\omega(z))'}{p\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)\omega(z)} \in \mathcal{P}(\gamma)\right\}, \tag{2.27}$$

then $z(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z))' / p\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)G(z)$ lies in Ω, this implies that $G = \phi * f. \in \mathcal{S}_2(p, \rho)$. This completes the proof of Theorem 9.

Concluding Remark

The results presented in this paper can lead to several applications by appropriated linear operators which are mentioned in the introduction.

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