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HIGHER-ORDER BANACH CONTRACTION MAPPING THEOREM IN η-CONE METRIC SPACE

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Abstract

The concept of η-cone metric space appeared in [1]. In the present paper, we prove the higher-order Banach contraction mapping theorem [2] in this setting.

1. Introduction and Preliminaries

Definition 1.1 (Huang and Zhang [3]). Let E be a real Banach space with norm $\|\cdot\|$ and P be a subset of E. P is called a cone if and only if

(a) P is closed, nonempty, and $P \neq \{\theta\}$, where θ is the zero vector in E;

(b) For any nonnegative real numbers a and b, and $x, y \in P$, we have $ax + by \in P$;

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(c) for $x \in P$, if $-x \in P$, then $x = \theta$.

Definition 1.2 (Huang and Zhang [3]). Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \quad \Leftrightarrow \quad y - x \in \operatorname{int}(P).$$

We shall write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{Int}(P)$, where $\operatorname{Int}(P)$ designates the interior of P.

Definition 1.3 (Huang and Zhang [3]). The cone P is said to be normal if there is a real number $C > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq C\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. In particular, we will say that P is a K-normal cone to indicate the fact that the normal constant is K.

Definition 1.4 (Huang and Zhang [3]). The cone P is said to be regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots y$$

for some $y \in E$, then there exists $x^* \in E$ such that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Definition 1.5 (Deimling [4]). The cone P is said to be minihedral if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum and hence any subset which is bounded from below has an infimum.

Remark 1.6. In this paper, we assume that the cone P is normal with constant K and P is such that $\operatorname{int}(P) \neq \emptyset$, and $\preceq$ is a partial ordering with respect to P. Hence the Banach space E and the cone P will be omitted, and the Banach space E will be assumed to be ordered with the order induced by the cone P.

Definition 1.7 (Gaba [1]). Let X be a nonempty set and $\eta : X \times X \mapsto [1, \infty)$ be a map. A function $d_\eta : X \times X \mapsto E$ will be called a η-cone metric space on X if

(a) $\theta \preceq d_\eta(x, y)$ for all $x \in X$ and $d_\eta(x, y) = \theta$ iff $x = y$;

(b) $d_\eta(x, y) = d_\eta(y, x)$ for all $x, y \in X$;

(c) $d_\eta(x, z) \preceq \eta(x, z)[d_\eta(x, y) + d_\eta(y, z)]$ for all $x, y, z \in X$.

Moreover, the pair (X, d_η) is called an η-cone metric space.

Remark 1.8 (Gaba [1]). If for all $x, y \in X$

(a) $\eta(x, y) = 1$, then we obtain the definition of cone metric space (Huang and Zhang [3]).

(b) $\eta(x, y) = L$, where $L \geq 1$, then we obtain the definition of cone metric type space (Cvetkovic et al. [6]).

(c) $\eta(x, y) = C$, where $C \geq 1$, $E = \mathbb{R}$ and $P = [0, \infty)$, then we obtain the definition of metric type space (Khamsi [7]).

Example 1.9 (Gaba [1]). Let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions defined on the interval $[a, b]$, $E = \mathbb{R}$ and $P = \mathbb{R}^+$, then (X, d_η) is an η-cone metric space with

$$d_\eta(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2$$

and

$$\eta(x, y) = |x(t)| + |y(t)| + 2.$$

Definition 1.10 (Gaba [1]). We say the following in an η-cone metric space (X, d_η):

(a) $\{x_n\}$ is convergent to $x \in X$ if for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d_\eta(x_n, x) \ll c$;

(b) $\{x_n\}$ is Cauchy if for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d_\eta(x_n, x_m) \ll c$;

(c) X is complete if every Cauchy sequence in X converges to an element of X.

Remark 1.11. Note that an η-cone metric is not always continuous (Example 4.1 of [Gaba [1]).

2. Main Result

Definition 2.1. Let X be a nonempty set. By the r-orbit of $T : X \mapsto X$ at x_0, we mean for any $r \in \mathbb{N}$, the set

$$I(x_0, T^r) = \{x_0, T^r x_0, T^{2r} x_0, \cdots\}.$$

Definition 2.2. Let (X, d_η) be an η-cone metric space. A map $T : X \mapsto X$ will be called an rth-order Banach contraction mapping if it satisfies

$$d_\eta(T^r x, T^r y) \leq \sum_{q=0}^{r-1} c_q d_\eta(T^q x, T^q y)$$

for all $x, y \in X$, where $0 \leq c_q < 1$ for all $0 \leq q \leq r-1$, and all $r \in \mathbb{N}$.

Proposition 2.3. *Let (X, d_η) be an η-cone metric space, and $T : X \mapsto X$ be an rth-order Banach contraction mapping. For every pair $x \neq y$, define*

$$Z := Z(x, y) = \max_{0 \leq v \leq r-1} \beta^{-v} \frac{d_\eta(T^v x, T^v y)}{d_\eta(x, y)}.$$

Then

$$Z = \max_{n \in \mathbb{N} \cup \{0\}} \beta^{-n} \frac{d_\eta(T^n x, T^n y)}{d_\eta(x, y)},$$

where $\beta \in [0, 1)$.

Remark 2.4. If in addition to Remark 1.8(a), $E = \mathbb{R}$ and $P = [0, \infty)$, then it follows that every metric space is an η-cone metric space. Thus Proposition 4.1 of [5] implies the Proposition immediately above.

Now by [2], we have the following alternate characterization of Definition 2.2.

Definition 2.5. Let (X, d_η) be an η-cone metric space, a map $T : X \mapsto X$ will be called an *r*th-order Banach contraction mapping if for all $x, y \in X$ and all $r \in \mathbb{N}$, the following inequality holds:

$$d_\eta(T^r x, T^r y) \leq Z\beta^r d_\eta(x, y),$$

where $\beta \in [0, 1)$ and $Z \geq 1$ is given by Proposition 2.3.

Our main result is as follows.

Theorem 2.6. *Let* (X, d_η) *be a complete* η*-cone metric space such that* d_η *is continuous. Suppose* $T : X \mapsto X$ *satisfies Definition* 2.5. *Moreover, for any* $x_0 \in X$, *suppose that* $\lim_{n, m \to \infty} \eta(x_n, x_m) < \frac{1}{Z\beta^r}$, *where* $x_n, x_m \in I(x_0, T^r)$. *Then there is a unique* $w^* \in X$ *such that* $T^r w^* = w^*$ *for any* $r \in \mathbb{N}$. *Moreover, for each* $x \in X$ *and any* $r \in \mathbb{N}$, $\lim_{n \to \infty} T^{rn} x = w^*$.

Proof. Let x_0 be arbitrary and define the sequence $\{x_n\}$ by $x_n = T^{rn} x_{n-1}$ for any $r \in \mathbb{N}$. Observe

$$d_\eta(x_n, x_{n+1}) = d_\eta(T^{rn} x_{n-1}, T^{r(n+1)} x_n) \leq Z\beta^r d_\eta(x_{n-1}, x_n),$$

$$d_\eta(x_{n+1},\, x_{n+2}) = d_\eta(T^{r(n+1)}x_n,\, T^{r(n+2)}x_{n+1}) \le (Z\beta^r)^2 d_\eta(x_{n-1},\, x_n),$$

$$\vdots$$

$$d_\eta(x_n,\, x_{n+1}) \le (Z\beta^r)^n d_\eta(x_0,\, x_1)$$

for all $n = 1, 2, 3, \cdots$. Since $Z\beta^r < 1$, consequently, the sequence $\{T^{rn}x_{n-1}\}$ is Cauchy. By the completeness of *X*, there exists $w^* \in X$ such that $\lim_{n\to\infty} T^{rn}x_{n-1} = w^*$. Now we show existence of the *r*-fixed point, that is, $T^r w^* = w^*$. Observe that

$$\begin{aligned} d_\eta(T^r w^*,\, w^*) &\le \eta(T^r w^*,\, w^*)[d_\eta(T^r w^*,\, x_n) + d_\eta(x_n,\, w^*)] \\ &\le \eta(T^r w^*,\, w^*)[d_\eta(T^r w^*,\, T^{rn}x_{n-1}) + d_\eta(x_n,\, w^*)] \\ &\le \eta(T^r w^*,\, w^*)[Z\beta^r d_\eta(w^*,\, x_{n-1}) + d_\eta(x_n,\, w^*)]. \end{aligned}$$

Now taking norm to inequality in the above, and then taking limits as $n \to \infty$, we deduce that

$$\|d_\eta(T^r w^*,\, w^*)\| = 0$$

which implies $T^r w^* = w^*$. For uniqueness, suppose $T^r u^* = u^*$, but $u^* \ne w^*$, then we have

$$d_\eta(u^*,\, w^*) = d_\eta(T^r u^*,\, T^r w^*) \le Z\beta^r d_\eta(u^*,\, w^*)$$

which implies

$$(1 - Z\beta^r)d_\eta(u^*,\, w^*) \le 0$$

but $d_\eta \ge 0$ and $Z\beta^r \ne 1$, thus $d_\eta(u^*,\, w^*) = 0$, that is, $u^* = w^*$ and uniqueness follows, and the proof is complete.

Now we have the following in support of the main result.

Example 2.7. Let $X = [0, \infty)$, $E = \mathbb{R}$, and $P = [0, \infty)$. Let us define for all $x, y \in X$, $d_\eta : X \times X \mapsto \mathbb{R}$ and $\eta : X \times X \mapsto [1, \infty)$ as:

$$d_\eta(x, y) = (x - y)^2; \, \eta(x, y) = x + y + 2.$$

Then d_η is an η-cone metric on X. Moreover, (X, d_η) is complete. Define $T^r x = \frac{x}{2r}$ for any $r \in \mathbb{N}$, and set $\frac{1}{3} = \beta \in [0, 1)$. Observe from the conclusion of Proposition 2.3, we have

$$\begin{aligned} Z &= \max_{n \in \mathbb{N} \cup \{0\}} \left(\frac{1}{3}\right)^{-n} \frac{\left(\frac{x}{2^n} - \frac{y}{2^n}\right)^2}{(x - y)^2} \\ &= \max_{n \in \mathbb{N} \cup \{0\}} 3^n \frac{1}{2^{2n}} \frac{(x - y)^2}{(x - y)^2} \\ &= \max_{n \in \mathbb{N} \cup \{0\}} \frac{3^n}{2^{2n}} \\ &= \max\{1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \cdots\} \\ &= 1. \end{aligned}$$

Thus for any $r \in \mathbb{N}$, we have

$$d_\eta(T^r x, T^r y) = \frac{1}{2^{2r}}(x - y)^2 < \frac{1}{3^r}(x - y)^2 = Z\beta^r d_\eta(x, y).$$

Note that for each $x \in X$, and any $r \in \mathbb{N}$, $T^{rn} x = \frac{x}{2^{rn}}$. Thus we obtain

$$\lim_{n, m \to \infty} \eta(x_n, x_m) = \lim_{n, m \to \infty} \left(\frac{x}{2^{rn}} + \frac{x}{2^{rm}} + 2 \right) < 3^r.$$

It follows all the conditions of the previous theorem hold, and the unique *r*-fixed point is given by $0 \in X$.

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