Fundamental Journal of Modern Physics
ISSN: 2249-9768
Volume 13, Issue 2, 2020, Pages 97-120
This paper is available online at http://www.frdint.com/
Published online June 22, 2020

GRAVITATIONAL MASS IN ISOTROPIC UNIVERSES

J. P. PANSART*

IRFU, CEA
Université Paris-Saclay
F-91191 Gif-sur-Yvette
France
e-mail: jean-pierre.pansart@orange.fr

Abstract

This note looks at the gravitational field of a point mass embedded in an evolving isotropic universe.

1. Introduction

This note looks at the gravitational field of one point mass embedded in an evolving isotropic universe, and how the solutions to Einstein equations differ from the Schwarzschild solution. This problem has already been studied, see for instance [1], [2], and references therein. Here we do not consider flat space expanding universes only but also positive or negative curvature space universes. The tensors are computed in orthonormal local frames (tetrad). Using the so-called "isotropic

Keywords and phrases: gravitational field, isotropic universe, 3-d compact manifold.

*Retired from IRFU, CEA, Université Paris-Saclay, F-91191 Gif-sur-Yvette, France

Received May 25, 2020; Accepted June 18, 2020

coordinates” the Einstein tensor component G_0^0 involves the underlying 3-dimensional constant curvature space Laplacian. As expected, at short distances the classical potential is inversely proportional to the physical distance. This study includes the case of constant curvature compact manifolds without border. The Einstein equations are highly non-linear, one of the main motivation was, knowing two exact solutions of these equations, find an example of how to merge them non perturbatively.

Sections 1 and 2 show why the so-called “isotropic coordinates” are very convenient. Section 3 looks at the effect of adding a point mass to the background matter and approximate solutions are given in Section 4. These solutions are obtained at lowest order in the development of the parameter $\alpha = r_0/a$ where r_0 is the Schwarzschild radius and a the radius of the universe. Section 5 discusses the case of constant curvature compact manifolds without border.

For a quick reading, after Sections 1 and 2 skip directly to equations (2.4), (3.5), (3.6) and (4.3) which represents the main equation and (4.4) which gives solutions. Section 5 can not be shortened.

1. Notations, Einstein Tensor and Spherical Symmetry

The space-time coordinates $\{x^\alpha\}$ of a point x are labelled with Greek letters: $\alpha, \beta, \gamma, \ldots,$ $0 \leq \alpha, \beta, \gamma, \ldots < n.$ The time coordinate is: x^0. The vectors of the local natural frame are written: $\overrightarrow{e_\alpha}, \overrightarrow{e_\beta}, \ldots$. When tensors are expressed with respect to local orthonormal frames they are labelled with Latin letters: $a, b, c, \ldots$. The orthonormal local frame basis vectors are called: $\overrightarrow{h_a}$, and we set: $\overrightarrow{h_a} = h_a^\alpha \overrightarrow{e_\alpha}$. The metric tensor is $g_{\alpha\beta}$, and $g^{\alpha\beta}$ is its inverse. The determinant of the metric tensor is called $|g|$. The signature of the metric is: $(+ - - -)$. In the case of local orthonormal frames, the metric tensor is written: η_{ab} and its diagonal terms are: $\eta_{aa} = (+1, -1, -1, -1)$. Latin indices are lowered with η_{ab} and raised with the inverse tensor η^{ab}. The geometry of constant curvature spaces is studied in [3]. For more details about the basic geometric equations and notations and information about the symmetry properties of constant curvature spaces see [4], [5].

In the neighborhood of a given point, the local coordinates, with respect to the local orthonormal frame attached to this point, are given by the 1-forms: $\omega^a = h^a_\alpha dx^\alpha$, which satisfy the structure equations:

$$d\omega^a + \omega^a_{.\,b} \wedge \omega^b = 0, \tag{1.1}$$

where: $\omega^a_{.\,b} = \omega^a_{.\,b\gamma} dx^\gamma$ are the connexion 1-forms and the torsion is supposed to be zero.

The curvature 2-form is defined by:

$$\Omega^a_{.\,b} = d\omega^a_{.\,b} + \omega^a_{.\,c} \wedge \omega^c_{.\,b} = R^a_{.\,bcd}\omega^c \wedge \omega^d. \tag{1.2}$$

The space-time coordinates are: the conformal time η and the Riemann normal coordinates for the spatial part named (χ, θ, φ) corresponding to spherical coordinates with respect to some Oz axis, where O is the origin of the coordinates. In the following, partial derivation of a function with respect to η is represented by a dot over the function, and derivation with respect to χ by a prime sign. The vectors of the local orthonormal frames are chosen to be colinear with those of the local natural frames. The mass M which is supposed to have negligible size, is placed at the origin O of the coordinates.

Imposing the spherical symmetry, the 1-forms ω^a are:

$$\omega^0 = f(\eta, \chi)d\eta, \quad \omega^1 = g(\eta, \chi)d\chi, \quad \omega^2 = h(\eta, \chi)d\theta,$$

$$\omega^3 = h(\eta, \chi) s d\eta, \quad s = \sin\theta \tag{1.3}$$

and the metric is:

$$ds^2 = f^2 d\eta^2 - g^2 d\chi^2 - h^2(d\theta^2 + s^2 d\varphi^2). \tag{1.4}$$

The connection 1-forms are:

$$\omega^0{}_{.1} = \frac{f'}{g} d\eta + \frac{\dot{g}}{f} d\chi, \quad \omega^2{}_{.0} = \frac{\dot{h}}{f} d\theta, \quad \omega^3{}_{.0} = \frac{\dot{h}s}{f} d\varphi, \tag{1.5}$$

$$\omega^2{}_{.1} = \frac{h'}{g} d\theta, \quad \omega^3{}_{.1} = \frac{h's}{g} d\varphi, \quad \omega^3{}_{.2} = c d\varphi,$$

where: $c = \cos(\theta)$.

The non-zero components of the Einstein tensor G^a_b, in the orthonormal frame $\overrightarrow{h_a}$, are:

$$G^0_0 = \frac{1}{f^2}\frac{\dot{h}}{h}\left[\frac{\dot{h}}{h} + \frac{2\dot{g}}{g}\right] + \frac{1}{g^2}\left[-\frac{2h''}{h} + \frac{2h'g'}{hg} - \frac{h'^2}{h^2}\right] + \frac{1}{h^2},$$

$$G^1_1 = \frac{1}{f^2}\left[2\frac{\ddot{h}}{h} - 2\frac{\dot{h}\dot{f}}{hf} + \frac{\dot{h}^2}{h^2}\right] - \frac{1}{g^2}\left[2\frac{f'h'}{fh} + \frac{h'^2}{h^2}\right] + \frac{1}{h^2},$$

$$G^2_2 = G^3_3 = \frac{1}{f^2}\left[\frac{\ddot{g}}{g} - \frac{\dot{g}\dot{f}}{gf} + \frac{\ddot{h}}{h} - \frac{\dot{h}\dot{f}}{hf} + \frac{\dot{h}\dot{g}}{hg}\right]$$

$$- \frac{1}{g^2}\left[\left(\frac{f'}{f}\right)' + \frac{f'^2}{f^2} - \frac{f'g'}{fg} + \frac{f'h'}{fh} + \left(\frac{h'}{h}\right)' + \frac{h'^2}{h^2} - \frac{g'h'}{gh}\right],$$

$$G^0_1 = -\frac{2}{fg}\left[\left(\frac{\dot{h}}{h}\right)' - \frac{\dot{h}}{h}\frac{f'}{f} + \frac{h'}{h}\left(\frac{\dot{h}}{h} - \frac{\dot{g}}{g}\right)\right]. \tag{1.6}$$

The last component can be simplified if: $\frac{\dot{h}}{h} = \frac{\dot{g}}{g}$, a condition equivalent to:

$$h(\eta,\ \chi) = g(\eta,\ \chi)k(\chi). \tag{1.7}$$

In the following, we choose: $k = sh(\chi)$ in the case of hyperbolic spaces $(\kappa = -1)$, $k = \chi$ for euclidean spaces $(\kappa = 0)$, and $k = \sin(\chi)$ for spherical spaces $(\kappa = 1)$. This is always possible by redefining the radial coordinate. Despites this variable change, the name χ is kept for the radial coordinate.

Using condition (1.7), equations (1.6) become:

$$G_0^0 = \frac{3}{f^2}\left(\frac{\dot{g}}{g}\right)^2 + \frac{3\kappa}{g^2} - \frac{1}{g^2}\left[2\left(\left(\frac{g'}{g}\right)' + \frac{2k'g'}{kg}\right) + \frac{g'^2}{g^2}\right], \tag{1.8a}$$

$$G_1^1 = \frac{1}{f^2}\left[2\frac{\ddot{g}}{g} - 2\frac{\dot{g}\dot{f}}{gf} + \frac{\dot{g}^2}{g^2}\right]$$

$$- \frac{1}{g^2}\left[2\frac{f'}{f}\left(\frac{g'}{g} + \frac{k'}{k}\right) + \left(\frac{g'}{g} + \frac{k'}{k}\right)^2 - \frac{1}{k^2}\right], \tag{1.8b}$$

$$G_2^2 = G_3^3 = \frac{1}{f^2}\left[2\frac{\ddot{g}}{g} - 2\frac{\dot{g}\dot{f}}{gf} + \frac{\dot{g}^2}{g^2}\right]$$

$$- \frac{1}{g^2}\left[\left(\frac{f'}{f} + \frac{g'}{g} + \frac{k'}{k}\right)' + \frac{k'}{k}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{k'}{k}\right) + \frac{f'^2}{f^2}\right], \tag{1.8c}$$

Equation (1.8a) is simplified if:

$$g = d^2 \tag{1.9}$$

and becomes:

$$G_0^0 = \frac{3}{f^2}\left(\frac{\dot{g}}{g}\right)^2 + \frac{3\kappa}{g^2} - \frac{1}{g^2}\frac{4}{d}({}^3\Delta d), \tag{1.10}$$

where:

$$^3\Delta = d'' + 2\frac{k'}{k}d' \tag{1.11}$$

is the three dimensional Laplacian of the isotropic space for zero angular momentum.

One consequence of (1.10) is that there is no empty space static physical solution which is asymptotically hyperbolic. Because if the space is empty and the solution is static, and if there is no cosmological constant, equation (1.10) gives: $({}^3\Delta d) - \frac{3\kappa}{4}d \sim \delta(x)$. For $\kappa = 0$ the solution is: $d \sim 1 + \alpha/\chi$, where α is a constant, which is the solution (2.2). But for $\kappa = -1$, the solution is: $d = (\gamma e^{\chi/2} + \delta e^{-\chi/2})/sh(\chi)$, where γ and δ are constants. Then, $g \to 0$ if $\chi \to \infty$ which is not a physical solution. Therefore, if $\kappa = -1$, the solution can not be static.

2. The Metric Coefficients

Until now we have used only the spherical symmetry hypothesis, (1.7) and (1.9) being only technical choices.

If $M = 0$, the metric (1.4) is the one of an evolving isotropic universe. For instance for hyperbolic spaces [6]:

$$ds^2 = a(\eta)^2(d\eta^2 - d\chi^2 - sh(\chi)^2(d\theta^2 + s^2 d\varphi^2)). \tag{2.1}$$

If $M \neq 0$ in a static empty universe, the choice (1.7) corresponds to the Schwarzchild metric in the so-called isotropic coordinates [6]:

$$ds^2 = \left(\frac{1 - \frac{r_0}{4r}}{1 + \frac{r_0}{4r}}\right)^2 c^2 dt^2 - \left(1 + \frac{r_0}{4r}\right)^4 (dr^2 + r^2(d\theta^2 + s^2 d\varphi^2)), \tag{2.2}$$

where: r is the radial coordinate, $r_0 = 2MG/c^2$ is the Schwarzschild radius, G is the gravitational constant, and c the speed of light.

In the first order perturbation theory, the metric of an evolving isotropic universe is, for the scalar modes and hyperbolic case [7]:

$$ds^2 = a(\eta)^2[(1 + 2\phi)d\eta^2 - (1 + 2\psi)(d\chi^2 + sh(\chi)^2(d\theta^2 + s^2 d\varphi^2))]. \quad (2.3)$$

In (2.2) let us make the variable change: $r = a\rho$ and: $t = a\eta$, the metric becomes:

$$ds^2 = a^2\left[\left(\frac{1 - \frac{\alpha}{\rho}}{1 + \frac{\alpha}{\rho}}\right)^2 d\eta^2 - \left(1 + \frac{\alpha}{\rho}\right)^4 (d\rho^2 + \rho^2(d\theta^2 + s^2 d\varphi^2))\right],$$

where: $\alpha = r_0/(4a)$ is a dimensionless parameter and the radial coordinate ρ and conformal time η are also dimensionless.

With these three examples in mind and (1.9), we make the following working hypothesis:

$$g = a(\eta)(1 + \alpha q(\eta, \chi))^2, \quad (2.4)$$

where α is chosen dimensionless, and since the problem involves only two length parameters (r_0 and a), we suppose that: $\alpha = r_0/a \sim M/a$.

Assumption (2.4) is general enough. We could have set: $g = g_0(\eta) + g_1(\eta, \chi, M)$ but this can be put in the form (2.4) once we take (1.9) into account.

Since α is small and dimensionless, we shall study the equations at the lowest order in α. (2.4) is not the most general possible form for g. We could have tried something like: $g = a(\eta)(1 + \alpha q(\eta, \chi) + u(\eta, \chi))^2$, where u is a function representing, for instance, wave propagation independently of M (at least in first approximation).

In (2.4), $a(\eta)$ as a function of η only, can still be understood as the universe radius only in the limit $\alpha \to 0$ or asymptotically when $\chi \to \infty$. $\alpha \ll 1$ does not mean that M is small but that a is large enough.

We define: $H = \dot{a}/a$ (although, H is usually defined as: $H = \frac{1}{a}\frac{da}{dt}$ where: $dt = a d\eta$). Then with: $\frac{\dot{\alpha}}{\alpha} = -\frac{\dot{a}}{a} = -H$, we have:

$$\frac{\dot{g}}{g} = H\left(\frac{1 - \alpha q}{1 + \alpha q}\right) + 2\frac{\alpha \dot{q}}{1 + \alpha q}. \tag{2.5}$$

3. The Matter Energy Momentum Tensor Equations

The Einstein equations are: $G_b^a - \Lambda\delta_b^a = \mu T_b^a$ where $\mu = 8\pi G/c^4$ and Λ is the cosmological constant.

The matter evolution is given by the conservation equations of the energy-momentum tensor: $D_\alpha T_\beta^\alpha = 0$. To write the Einstein equations with (1.8), we have to be careful and use the components of the energy momentum tensor in the orthonormal frames. Since the metric (1.4) is diagonal, the diagonal terms of the energy momentum tensor are the same in the natural frames as in the orthonormal frames. But this is not the case for the component T_0^1 and from now on, the components in the orthonormal frames are written with a bar over in order to avoid any confusion. The conservation equations are:

$$\partial_0 T_0^0 + 3\frac{\dot{g}}{g}(T_0^0 - T_1^1) + 2\frac{\dot{g}}{g}(T_1^1 - T_2^2)$$

$$+ \frac{f}{g}\left[\partial_1 \overline{T}_0^1 + 2\left(\frac{f'}{f} + \frac{g'}{g} + \frac{k'}{k}\right)\overline{T}_0^1\right] = 0, \tag{3.1a}$$

$$\partial_1 T_1^1 - \frac{f'}{f}(T_0^0 - T_1^1) + 2\left(\frac{g'}{g} + \frac{k'}{k}\right)(T_1^1 - T_2^2)$$

$$+ \frac{g}{f}\left[\partial_0 \overline{T}_1^0 + 4\left(\frac{\dot{g}}{g}\right)\overline{T}_1^0\right] = 0. \tag{3.1b}$$

The two other equations, for $\beta = 2$ and $\beta = 3$, are identically zero due to the spherical symmetry. The equations (1.6) are four and the conservation conditions $D_\alpha T_\beta^\alpha = D_\alpha G_\beta^\alpha = 0$ are two. Therefore there are two degrees of freedom which are f and g. Knowing g one should be able to deduce f from (1.8c) which is directly related to the pressure unlike (1.8b) (see hereafter).

In the classical non relativistic limit the first equation is the matter conservation equation, and the second is the Euler equation of fluid mechanics.

In the isotropic $(M = 0)$ and static cases: $G_1^0 = 0$. Then, in the latter equations, is it possible to neglect the contribution of $\overline{T}_1^0$ and $(T_1^1 - T_2^2)$ (pressure isotropy)?

As an example, let us consider the energy-momentum tensor of a continuous medium like a gas:

$$T^{ab} = (\varepsilon + p)u^a u^b - p\eta^{ab}. \tag{3.3}$$

Using the spherical symmetry we have:

$$T^{00} = \varepsilon + (\varepsilon + p)(u^1)^2, \qquad T^{11} = (\varepsilon + p)(u^1)^2 + p,$$

$$T^{22} = p, \qquad T^{01} = (\varepsilon + p)u^0 u^1.$$

If $u^1 = O(\alpha)$ then $(T_1^1 - T_2^2) = O(\alpha^2)$ and $\overline{T}_1^0 = O(\alpha)$.

Let us consider the last term of (3.1b), the square bracket is equal to

$\overline{T}_1^0 \partial_0 \ln(g^4 \overline{T}_1^0)$. For this term to be small one must have: $\overline{T}_1^0 \simeq O(\alpha)$ or/and: $g^4 \overline{T}_1^0 = \mathit{function}\,(\chi)$. Using (1.6) and (1.7) one has:

$$\mu \overline{T}_1^0 = G_1^0 = -\frac{2}{fg}\left(\frac{\dot{g}}{g}\right)\partial_\chi\left(\ln\left(\frac{\dot{g}}{gf}\right)\right). \tag{3.4}$$

Therefore $\mu \overline{T}_1^0$ will be small if: $\dfrac{\dot{g}}{gf} = \mathit{function}\,(\eta)(1 + O(\alpha))$. Using (2.5) this means: $f \sim \dfrac{1 - \alpha q}{1 + \alpha q}$. In order to keep some freedom we shall write:

$$f = a(1 - ar(\eta, \chi))/(1 + ar(\eta, \chi)). \tag{3.5}$$

The derivatives are:

$$\frac{g'}{g} = 2\alpha q'/(1 + \alpha q), \quad \frac{f'}{f} = -2\alpha r'/(1 - \alpha^2 r^2),$$

$$\frac{\dot{f}}{f} = H\left(1 + \frac{2\alpha r}{1 - \alpha^2 r^2}\right) - 2\frac{\alpha \dot{r}}{1 - \alpha^2 r^2},$$

$$\frac{1}{f}\frac{\dot{g}}{g} = \frac{H}{a}\frac{1 - \alpha q}{1 + \alpha q}\frac{1 + \alpha r}{1 - \alpha r} + 2\frac{\alpha \dot{q}}{a(1 + \alpha q)}\frac{1 + \alpha r}{1 - \alpha r}.$$

If:

$$r = q + O(\alpha), \tag{3.6}$$

then:

$$\frac{1}{f}\frac{\dot{g}}{g} = \frac{H}{a}(1 + O(\alpha^2)) + 2\frac{\alpha \dot{q}}{a} + O(\alpha^2)$$

which in (3.4) gives:

$$G_1^0 \simeq -4\alpha(\dot{q})'/a^2 + O(\alpha^2), \tag{3.7}$$

then: $G_1^0 \simeq O(\alpha)$.

With (3.5) and (3.6) we have: $\frac{f'}{f} + \frac{g'}{g} = 0 + O(\alpha^2)$. Equations (1.8b) and ((1.8c) become:

$$G_2^2 = G_3^3 = \frac{1}{f^2}\left[2\frac{\ddot{g}}{g} - 2\frac{\dot{g}\dot{f}}{gf} + \frac{\dot{g}^2}{g^2}\right] + \frac{\kappa}{g^2} + O(\alpha^2) \tag{3.8a}$$

and the same expression for G_1^1, therefore:

$$G_1^1 = G_2^2 + O(\alpha^2). \tag{3.8b}$$

Now we look at the last term of (3.1a).

$$-\mu\left[\partial_1 \overline{T}_0^1 + 2\left(\frac{f'}{f} + \frac{g'}{g} + \frac{k'}{k}\right)\overline{T}_0^1\right] = \left[\partial_1 G_1^0 + 2\frac{k'}{k}G_1^0\right] + O(\alpha^2) \tag{3.9}$$

(the minus sign is due to the exchange of the indices 0 and 1).

We compare the coefficient of (3.7) with μ:

$$\frac{1}{\mu}\frac{\alpha}{a^2} = \left(\frac{2MG}{c^2 a}\right)\frac{1}{a^2}\left(\frac{c^4}{8\pi G}\right) = \frac{Mc^2}{4a^3}. \tag{3.10}$$

This "density" is assumed to be negligible compared with ε. Therefore the terms containing $\overline{T}_0^1$ in the equations (3.1) will be neglected and these equations reduce to:

$$\mu\left(\partial_0 T_0^0 + 3\frac{\dot{g}}{g}(T_0^0 - T_1^1)\right) = 0 + O(\alpha^2), \tag{3.11a}$$

$$\mu\left(\partial_1 T_1^1 - \frac{f'}{f}(T_0^0 - T_1^1)\right) = 0 + O(\alpha^2). \tag{3.11b}$$

These equations say that u^1 plays a negligible role in (3.3) if a is large enough.

4. Approximate Solution to the Equations

T^{α}_{β} is the sum of several components which will be supposed to be uncoupled. The exact nature of these components is not important for our purpose. The discussion hereafter is given only as an example.

Let us consider a gas obeying Maxwell-Boltzmann statistics [8]. The particle density is: $n = 4\pi m^3 \nu K_2(m\beta)/(h^3 m\beta)$ where m is the molecule mass, h the Planck constant, $\beta = 1/(kT)$, T is the temperature, k the Boltzmann constant, and ν is the exponential of the chemical potential.

The pressure is: $p = n/\beta$ and the energy density is: $\varepsilon = \frac{4\pi m^3 \nu}{h^3}\left[\frac{3K_2(m\beta)}{(m\beta)^2} + \frac{K_1(m\beta)}{(m\beta)}\right]$, where: K_n are the modified Bessel functions.

If $T \to 0$, $p \to 0$ and $\varepsilon \simeq nm$. In this section, we consider only two cases: cold matter and radiation.

For the radiation: $p = \varepsilon/3$. Then equations (3.11) are satisfied if: $\varepsilon_r \sim a^{-4}(1 + \alpha q)^{-8}$.

The case of cold matter is identical to the case of a set of massive bodies without collision. (3.1b) is automatically satisfied and (3.1a) gives: $\varepsilon_c \sim a^{-3}(1 + 2\alpha q)^{-3}$.

Finally, equation (1.10) for G^0_0 is (up to $O(\alpha^2)$ terms):

$$\frac{3}{a^2}(H^2 + \kappa) - \frac{4\alpha}{a^2}({}^3\Delta q + 3\kappa q - 3H\dot{q}) - \Lambda = \mu T^0_0 \tag{4.1a}$$

$$= \mu\left[\frac{\tau_c}{a^3}(1 + 2\alpha q)^{-3} + \frac{\tau_r}{a^4}(1 + 2\alpha q)^{-4}\right].$$

The order 0 gives the usual equation of an evolving isotropic universe:

$$\frac{3}{a^2}(H^2+\kappa)-\Lambda=\mu\left[\frac{\tau_c}{a^3}+\frac{\tau_r}{a^4}\right] \tag{4.1b}$$

and the order α equation is:

$$^3\Delta q+3\kappa q-3H\dot{q}=\frac{\mu}{2}\left(\frac{3\tau_c}{a}+\frac{4\tau_r}{a^2}\right)q. \tag{4.1c}$$

The other Einstein equation is (3.8) (up to $O(\alpha^2)$ terms):

$$G_2^2=\frac{2}{f^2}\left[\partial_\eta\left(\frac{\dot{g}}{g}\right)+\frac{\dot{g}}{g}\left(\frac{\dot{g}}{g}-\frac{\dot{f}}{f}\right)\right]+\frac{1}{f^2}\frac{\dot{g}^2}{g^2}+\frac{\kappa}{g^2}+O(\alpha^2)=-\mu p+\Lambda$$

$$=\frac{1}{a^2}(2\dot{H}+H^2+\kappa)+\frac{4\alpha}{a^2}(\ddot{q}+H\dot{q}+(\dot{H}-H^2-\kappa)q). \tag{4.2a}$$

Then, with the above hypotheses about the matter content:

$$\frac{1}{a^2}(2\dot{H}+H^2+\kappa)=-\frac{\mu}{3}\frac{\tau_r}{a^4}+\Lambda \tag{4.2b}$$

and:

$$\ddot{q}+H\dot{q}+(\dot{H}-H^2-\kappa)q=2\mu\tau_r q/(3a^2). \tag{4.2c}$$

In this equation:

$$\dot{H}-H^2-\kappa=\dot{H}+\frac{(H^2+\kappa)}{2}-\frac{3(H^2+\kappa)}{2}\simeq\frac{a^2}{2}(G_2^2-G_0^0) \tag{4.2d}$$

$$=-\mu\frac{a^2}{2}(p+\varepsilon)\sim-\mu/a$$

(where G_2^2 and G_0^0 are taken at lowest order).

Then in (4.2c), when a is large: $\ddot{q}+H\dot{q}\sim 0$ which has two solutions: $\dot{q}\simeq 0$ and $\dot{q}\sim 1/a$. Finally, when $a>>1$, (4.1c) is re-written:

$$^3\Delta q + 3\kappa q - Tq \simeq \delta(x), \tag{4.3}$$

where T represents the contribution of T_a^b. In the above example: $T = \frac{\mu}{2}\left(\frac{3\tau_c}{a} + \frac{4\tau_r}{a^2}\right)$. (4.3) is an eigenvalue equation. If T can not be neglected the eigenvalue depends on η.

In order to obtain (4.3), we have used a matter background composed of radiation and cold matter. Appendix A shows that (4.3) is still true if other types of matter are considered.

The Einstein equations are not solved exactly, but up to $O(\alpha^2)$ and $O(1/a)$ contributions. This is mainly due to the hypothesis (3.6).

Equation (4.3) can be solved by setting: $q \sim \psi(\eta, \chi)/sh(\chi)$ if $\kappa = -1$, and for $\kappa = 0$ and $\kappa = 1$, respectively: $q \sim \psi(\eta, \chi)/\chi$ and $q \sim \psi(\eta, \chi)/\sin(\chi)$. More generally, and for example, in the hyperbolic case, one can try: $q \sim u(\chi)\big/sh^{\gamma}(\chi)$ which leads to a Legendre equation for u if: $\gamma = -1/2$ [9]. This gives the same result.

We now assume that: $\Lambda = 0$, the solutions of (4.3) are:

if $\kappa = -1$: $q \sim e^{-|\nu|\chi}\big/sh(\chi)$, where: $\nu^2 = 4 + T$

and $\nu \to 2$ if $a \to \infty$; (4.4a)

if $\kappa = 0$: $q \sim e^{-|\nu|\chi}\big/\chi$, where: $\nu^2 = T$; (4.4b)

if $\kappa = +1$ from (4.1b) we have:

$$T - 3\kappa = \frac{\mu}{2}\left(\frac{3\tau_c}{a} + \frac{4\tau_r}{a^2}\right) - 3\kappa > \frac{3\mu}{2}\left(\frac{\tau_c}{a} + \frac{\tau_r}{a^2}\right) - 3\kappa = \frac{9}{2}H^2 + \frac{3}{2} \geq \frac{3}{2}.$$

Then: $q \sim e^{-|\nu|\chi}\big/\sin(\chi)$, where: $\nu^2 = T - 4$. (4.4c)

The coordinates (χ, θ, φ) are merely labels used to identify space points. If $M = 0$, the radial distance is $a\chi$. When $\chi << 1$, $q \sim 1/\chi$, whatever κ. Equation (2.4), with $\alpha \sim 1/a$, shows that, at first order and at small χ the classical potential is inversely proportional to the distance even if the background space-time is not static and not Euclidean.

We have not checked the stability of these solutions with respect to perturbations.

5. Compact Manifolds

Spaces of constant curvature are studied in [3]. Let V be a compact 3-dimensional constant curvature manifold without border and $\widetilde{V}$ its universal covering. Let Γ be the universal covering group and γ its elements. Functions defined on V can be extended to $\widetilde{V}$ where they must be periodic under the action of Γ. Γ is a discrete group of isometries, therefore the spherical symmetry used until now is no longer true.

In [11], we used cylindrical coordinates which are more convenient than spherical coordinates. They are defined as follows: The symmetry axis, called the Oz axis, is taken to be the base geodesic of the generator γ_0 of Γ having the longest transvection (but this is not compulsory). The distance of a point P to its orthogonal projection on Oz is the radius and is called ρ. The abscissa of this projection is called z and the azimuthal angle is φ.

We now consider the example of space of constant negative curvature but the conclusions are the same for the other cases. $\widetilde{V} = H^3$ is an hypersphere of the Minkowski space M^4. The correspondence between the cylindrical coordinates and the M^4 coordinates (with signature

$(+ - - -)$ is:

$$x^0 = ch(\rho)ch(z), \quad x^1 = sh(\rho)c_\varphi, \quad x^2 = sh(\rho)s_\varphi, \quad x^3 = ch(\rho)sh(z), \quad (5.1)$$

where: $c_\varphi = \cos(\varphi)$, $s_\varphi = \sin(\varphi)$. Let x be a point whose coordinates in M^4 are $x = (x^0, x^1, x^2, x^3)$ and γ an element of Γ. The image x' of x under the action of γ is $x' = [\gamma]x$ where $[\gamma]$ is the matrix representing γ in M^4.

We define:

$$\omega_0^1 = ad\rho, \quad \omega_0^2 = ash(\rho)d\varphi, \quad \omega_0^3 = ach(\rho)dz. \quad (5.2)$$

These ω_0^a represent the coordinates of points in a neighbourhood U of x with respect to an orthonormal frame whose basis vectors $\overrightarrow{h_a}(x)$ are collinear with the vectors of the natural basis. γ is an isometry, the image of U is given by: $\omega_0^a(x)\overrightarrow{h'_a}(x')$ where $\overrightarrow{h'_a}(x')$ are the basis vectors of the adapted frame obtained by transporting the vectors $\overrightarrow{h_a}(x)$ and where the ω_0^a are unchanged.

Since γ is an isometry, the adapted frame is orthonormal and the local coordinates are $\omega_0^a(x') = A_b^a\omega_0^b(x)$ where A is a rotation matrix. For instance ([4] Appendix A): $\overrightarrow{h'_a}(\gamma_0 x) = \gamma_0(\overrightarrow{h_a}(x)) = \overrightarrow{h_a}(x')$ and A is the unit matrix.

Now let us assume that:

$$\omega^1 = gd\rho, \quad \omega^2 = \bar{h}sh(\rho)d\varphi, \quad \omega^3 = \bar{l}ch(\rho)dz, \quad (5.3)$$

where g, $\bar{h}$, $\bar{l}$ are functions of ρ, φ, z. The rotation A mingles the components $\omega_0^a(x)$ to obtain $d\rho$, $d\varphi$, dz at x'. The components ω^a are transformed likewise. The ω^a keeps their forms if:

$$g = \bar{h} = \bar{l} \tag{5.4}$$

and if g is invariant under the action of γ.

In Section 1, the radial coordinate was redefined in order to simplify the Einstein tensor components and to be able to use the convenient "isotropic coordinates". Here the coordinates (ρ, φ, z) of the constant curvature space are not redefined in order to keep the transformation relation $x' = [\gamma]x$ unchanged.

We write: $\omega^0 = f d\eta$ as before and:

$$\omega^1 = g d\rho, \quad \omega^2 = h d\varphi, \quad \omega^3 = l dz. \tag{5.5}$$

In the hyperbolic example: $h = \bar{h} sh(\rho)$, $l = \bar{l} ch(\rho)$.

In the rest of this section the sign $'$ means the partial derivative with respect to ρ.

The connection 1-forms are:

$$\omega^0{}_{.1} = \frac{f'}{g} d\eta + \frac{\dot{g}}{f} d\rho, \quad \omega^0{}_{.2} = \frac{f_\varphi}{h} d\eta + \frac{\dot{h}}{f} d\varphi, \quad \omega^0{}_{.3} = \frac{f_z}{l} d\eta + \frac{\dot{l}}{f} dz, \tag{5.6}$$

$$\omega^1{}_{.2} = \frac{g_\varphi}{h} d\rho - \frac{h'}{g} d\varphi, \quad \omega^1{}_{.3} = \frac{g_z}{l} d\rho - \frac{l'}{g} dz, \quad \omega^2{}_{.3} = \frac{h_z}{l} d\varphi - \frac{l_\varphi}{h} dz.$$

Since all the calculations follow the same path as before, we do not give all the details but only the main points. With (5.4) and the hypothesis (1.9) for g we obtain, like (1.10):

$$G^0_0 = \frac{3}{f^2}\left(\frac{\dot{g}}{g}\right)^2 + \frac{3\kappa}{g^2} - \frac{4}{g^2}\frac{1}{d}\left({}^3\Delta d\right), \tag{5.7}$$

where:

$$^{3}\Delta = d'' + \left(\frac{sh\rho}{ch\rho} + \frac{ch\rho}{sh\rho}\right)d' + \frac{d_{\varphi\varphi}}{sh^2\rho} + \frac{d_{zz}}{ch^2\rho} \tag{5.8}$$

and: $d_{\varphi\varphi}$ is the second derivative with respect to φ and idem for d_{zz}.

In the previous sections we considered also the component G_1^0. Here it has the same form as in (1.6):

$$G_1^0 = -\frac{2}{fg}\left[\left(\frac{\dot{g}}{g}\right)' - \frac{f'}{f}\frac{\dot{g}}{g}\right].$$

As in (2.4) and (3.5), we write:

$$g = a(\eta)(1 + \alpha q)^2, \qquad f = a\,(1 - \alpha r)/(1 + \alpha r),$$

where q and r are functions of (ρ, φ, z).

With (3.6) we obtain (3.8):

$$G_1^1 \simeq G_2^2 \simeq G_3^3 = \frac{1}{f^2}\left[2\frac{\ddot{g}}{g} - 2\frac{\dot{g}\,\dot{f}}{g\,f} + \frac{\dot{g}^2}{g^2}\right] + \frac{\kappa}{g^2} + O(\alpha^2).$$

The nondiagonal components of G_a^b, except G_1^0, which are strictly zero in the spherical symmetry case are still 0 but up to $O(\alpha^2)$ terms only.

The conservation equations of T_a^b have the same structure as (3.1):

$$\partial_0 T_0^0 + 3\frac{\dot{g}}{g}(T_0^0 - T_1^1) + \frac{\dot{g}}{g}(2T_1^1 - T_2^2 - T_3^3)$$

$$+ \frac{f}{g}\left[\partial_1 \overline{T}_0^1 + \left(2\frac{f'}{f} + 2\frac{g'}{g} + \frac{ch\rho}{sh\rho} + \frac{sh\rho}{ch\rho}\right)\overline{T}_0^1\right] = 0,$$

$$\partial_1 T_1^1 - \frac{f'}{f}(T_0^0 - T_1^1) + \frac{g'}{g}(2T_1^1 - T_2^2 - T_3^3) + \frac{ch\rho}{sh\rho}(T_1^1 - T_2^2)$$

$$+ \frac{sh\rho}{ch\rho}(T_1^1 - T_3^3) + \frac{g}{f}\left[\partial_0 \overline{T}_1^0 + 4\left(\frac{\dot{g}}{g}\right)\overline{T}_1^0\right] = 0.$$

Finally we obtain:

$$^{3}\Delta q + 3\kappa q - Tq \sim \sum_{\gamma} \delta(x - \gamma y), \tag{5.9a}$$

where y is the point of V where the mass M is located, and γy are the image of y in $\widetilde{V}$.

In order to avoid any confusion with the spherical solutions (4.4), we rename $q : q \to q_c$ and re-write (5.9):

$$^{3}\Delta q_c + \sigma q_c \sim \sum_{\gamma} \delta(x - \gamma y), \tag{5.9b}$$

where $\sigma = 3\kappa - T$.

In the following the orthonormal eigenfunctions of the Laplacian on V are called ψ_r: $^{3}\Delta\psi_r = -\lambda_r\psi_r$, where: $\lambda_r \geq 0$. In (5.9b) q_c is a Green function. The general solution of (5.9b) is:

$$q_c \sim \sum_{s} \frac{\overline{\psi}^{s}(y)\psi_s(x)}{\sigma - \lambda_s}, \tag{5.10}$$

q_c is invariant by Γ as required.

In classical mechanics the solutions (4.4) represent potentials, and it is tempting to add the potentials produced by the images of the mass M in $\widetilde{V}$. In principle we should not do that because General Relativity is a non linear theory. However, we shall compute: $Q(x) = \sum_{\gamma} q(x, \gamma y)$ where q in (4.4) is a function of the distance in V between the point x where one computes the potential and the mass source. The Seldberg trace formula [10] links the invariant sum Q (with respect to Γ) to the spectrum of the Laplacian operator:

$$Q(x) = \sum_{\gamma} q(x, \gamma y) = \sum_{s} h(\lambda_s)\overline{\psi}^{s}(y)\psi_s(x). \tag{5.11}$$

If the background space-time is hyperbolic $(\kappa = -1)$:

$$h(\lambda_s) = 4\pi\int_0^{\infty} q(\chi)\Phi_\beta^0 sh^2(\chi)d\chi,$$

where: $\Phi_\beta^0 = \sin(\beta\chi)/(\beta sh(\chi))$, $\lambda_s = (1+\beta^2)$ is the 3-d Laplacian eigenfunction in spherical coordinates with null angular momentum associated to $\lambda_s \neq 0$. If $\lambda = 0$, this function is replaced by 1.

With (4.4a) the sum Q is defined, and the calculation of the integrals $h(\lambda_s)$ gives $Q \sim q_c$ (5.10). This means that adding the classical potentials is a good approximation.

In the Eclidean case $(\kappa = 0)$: $h(\lambda_s) = 4\pi\int_0^{\infty} q(\chi)\Phi_\beta^0\chi^2 d\chi$ with: $\Phi_\beta^0 = \sin(\beta\chi)/(\beta\chi)$, $\lambda_s = \beta^2$.

The calculation is the same as for $\kappa = -1$. However, this integral is not defined if $q \sim 1/\chi$ $(\nu = 0$ in 4.4b)) and if $\lambda = 0$. This corresponds to the fact that the sum (5.11) does not converge in that case. If we call V_0 the volume of the fundamental domain, associated to V the number of image domains between χ and $\chi + \delta\chi$ is, at large χ: $4\pi\chi^2\delta\chi/V_0$ which grows faster than $q \sim 1/\chi$.

If the background space-time is spherical $(\kappa = +1)$:

$$h(\lambda_s) = 4\pi\int_0^{\chi_{\mathrm{sup}}} q(\chi)\Phi_\beta^0 \sin^2(\chi)d\chi,$$

where if $\lambda_s \neq 0$: $\Phi_\beta^0 = \sin(\beta\chi)/(\beta\sin(\chi))$, $\lambda_s = \beta^2 - 1$.

In principle the upper bound of the integral should be $\chi_{\mathrm{sup}} = \pi$ but in fact we have to take into account all the geodesic of S^3 joining x to y.

Taking: $\chi_{\sup} = \infty$ gives (5.10).

6. Appendix A. The Scalar Field

The aim of this appendix is not to study the physics of a self-interacting scalar field in its own gravitational field, for that see [7] and references therein.

Considering a gravitational mass M during the inflation phase has no meaning, but here we add a scalar field to the ordinary matter (Section 4) to justify the form of equation (4.3) in another situation.

We consider a free scalar field ϕ whose Lagrangian is:

$$L = \partial_\alpha \phi^+ g^{\alpha\beta} \partial_\beta \phi - V(\phi^+ \phi), \tag{A.1}$$

the action being: $S = \int L\sqrt{|g|}d^n x.$ V is the potential which contains mass term and self-coupling. The notation V is used because it is usual for a potential but has nothing to do with Section 5.

The Euler-Lagrange equations give:

$$\Delta\phi = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} g^{\alpha\beta} \partial_\beta \phi) = -V_\phi \phi,$$

where: $V_\phi = \partial_{(\phi^+\phi)} V.$

$$\Delta\phi = \frac{1}{f^2}\left[\ddot{\phi} + \left(\frac{3\dot{g}}{g} - \frac{\dot{f}}{f}\right)\dot{\phi}\right] - \frac{1}{g^2}\left[{}^3\Delta\phi + \left(\frac{f'}{f} + \frac{g'}{g}\right)\phi'\right], \tag{A.2}$$

where (Section 3): $\frac{f'}{f} + \frac{g'}{g} = 0 + O(\alpha^2)$ and: $\frac{3\dot{g}}{g} - \frac{\dot{f}}{f} = 2H - 8H\alpha q + 8\alpha\dot{q} + O(\alpha^2)$.

The energy-momentum tensor is (for real ϕ):

$$T_b^a = 2\eta^{ad} h_b^\beta h_d^\gamma \partial_\gamma\phi\partial_\beta\phi - L\delta_b^a.$$

As before the spherical symmetry implies $\partial_2\phi = \partial_3\phi = 0$, then, in the orthonormal frames defined in Section 1:

$$T_0^0 = \frac{\dot{\phi}^2}{f^2} + \frac{(\phi')^2}{g^2} + V, \quad T_1^1 = -\frac{\dot{\phi}^2}{f^2} - \frac{(\phi')^2}{g^2} + V,$$

$$T_2^2 = T_3^3 = -L = -\frac{\dot{\phi}^2}{f^2} + \frac{(\phi')^2}{g^2} + V, \quad \overline{T}_1^0 = \frac{2}{fg}\dot{\phi}\phi'.$$

If $M = 0$, (isotropic space) equation (A.2) gives:

$$\frac{1}{a^2}(\ddot{\phi}_0 + 2H\dot{\phi}_0) + V_\phi \phi_0 = 0. \tag{A.3}$$

If $M \neq 0$, we write: $\phi = \phi_0 + \alpha\phi_1$. The equation of ϕ_1 is:

$$\frac{1}{a^2}[\ddot{\phi}_1 - (\dot{H} + H^2)\phi_1 - {}^3\Delta\phi_1] = -V_\phi(\phi_1 - 4\phi_0 q) + \frac{8}{a^2}\left(H\dot{\phi}_0 q + \dot{q}\dot{\phi}_0\right).$$

In this equation the technique of variable separation can be used if $\phi_1 \sim q$ and if q is an eigenfunction of ${}^3\Delta$.

The Einstein equation for G_0^0 gives:

$$\frac{3}{a^2}[H^2 + \kappa] - \Lambda = \mu\left(\frac{\dot{\phi}_0^2}{a^2} + V(\phi_0)\right) + \mu(T_{m0})_0^0, \tag{A.4a}$$

where $(T_{m0})_0^0$ represents the energy-momentum tensor contribution of the other matter components at lowest order.

$$\frac{1}{a^2}({}^3\Delta q + 3\kappa q - 3H\dot{q}) = -\frac{\mu}{2}\left[\frac{\dot{\phi}_0}{a^2}\left(\dot{\phi}_1 - H\phi_1\right) + V_\phi\phi_0\phi_1\right]$$

$$- \frac{\mu}{a^2}\dot{\phi}_0^2 q + \mu(T_{m1})_0^0 + O(\alpha^2). \tag{A.4b}$$

$(T_{m1})^a_b$ is the contribution of the other matter components at order α.

The Einstein equation for G_2^2 gives:

$$\frac{1}{a^2}[2\dot{H} + H^2 + \kappa] - \Lambda = \mu\left(V(\phi_0) - \frac{\dot{\phi}_0^2}{a^2}\right) + \mu(T_{m0})_2^2, \tag{A.4c}$$

$$\frac{2}{a^2}(\ddot{q} + H\dot{q}) = \mu V_\phi \phi_0 \phi_1 - \frac{\mu}{a^2}\dot{\phi}_0\left(\dot{\phi}_1 - H\phi_1\right) + \mu(T_{m1})_2^2 + O(\alpha^2). \tag{A.4d}$$

Even the simple case where V is reduced to a mass term ($V = m^2\phi^2$) is complicated [7]. Here we suppose that ϕ is a free non self-interacting massless field: $V = 0$. Then, from (A.3): $\dot{\phi}_0 \sim a^{-2}$ and if we choose: $\phi_1 \sim q$, equations (A.4) show that (4.3) is still true.

References

[1] P. Castelo Ferreira, A locally anisotropic metric for matter in an expanding universe, arXiv:0907.0847v2 [astro-ph.CO] 21jul2009; Physics Letters B 684 (2010), 73-76.

[2] C. Gilbert, The gravitational field of a star in the expanding universe, Monthly Notices of the Royal Astronomical Society 116 (1956), 678.

[3] J. A. Wolf, Spaces of Constant Curvature, Fifth edition, Publish or Perish, Inc., 1984.

[4] J. P. Pansart, Electromagnetic field in 3-dimensional compact hyperbolic manifolds, Fundamental J. Modern Physics 12(1) (2019), 1-34.

[5] J. P. Pansart, A Clifford algebra gauge invariant Lagrangian for gravity. Part 2. Compatibility with General Relativity tests, HAL Id: hal-01261539, version 2; Arxiv: 1602.02131.

[6] L. Landau and E. Lifchitz, Classical Theory of Fields, Fourth revised English edition, Butterworth Heinemann, ed.

[7] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Theory of cosmological perturbations, Physics Reports 215 (1992), 203-333.

[8] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, Dover.

[9] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer Verlag.

[10] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, Inc., 1984.

[11] J. P. Pansart, Numerical calculation of the lowest eigenmodes of the Laplacian in compact orientable 3-dimensional hyperbolic spaces, arXiv: 0809.0591.