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GRAVITATIONAL FIELD OF SOURCES WITH SPIN (PART TWO)

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Abstract

In a former note, it was shown that a Lagrangian based on a Clifford algebra introduces naturally the Einstein-Hilbert Lagrangian and allows to take into account the intrinsic spin of gravitational sources. Here it is shown that, in weak field limit, the angular momentum parameter in the Lense-Thirring and Kerr metrics is the sum of the source rotation angular momentum and intrinsic spin as expected. In this second part the algebra used to build the equations of motion is extended, introducing new fields.

Introduction

In [1], it has been shown that the Einstein-Hilbert Lagrangian for gravity plus a term quadratic in the curvature tensor can result from a

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gauge theory based on a Clifford algebra represented by the Dirac matrices. This naturally introduces torsion and allows to take into account source intrinsic spin density. For zero torsion and in weak field situations it was shown that there is no contradiction with the Einstein-Hilbert Lagrangian only.

[2] was an attempt to take into account sources with spin but it had some weaknesses due to unjustified approximations. Here we reconsider the problem.

As for any non abelian gauge theory the field equations are non linear and this note does not pretend to solve them exactly. This study is restricted to the case of small enough spin density allowing perturbation development. Massive relativistic objects with high spin density are outside the scope of this note. In the limit of weak fields, it will be shown that the angular momentum parameter in the Lense-Thirring and Kerr metrics is the sum of the source rotation angular momentum plus its total intrinsic spin. This supports the idea that a Lagrangian based on Clifford algebra is a possible Lagrangian to consider. The algebra used in [1] and [2] is minimal. Section 5 considers an extension of this algebra. This introduces additional fields, which can provide some ground and justification for the so-called Scalar-tensor-vector gravity theories [3].

Section 1 sets the notations and Section 2 recalls the equations of motion used in [2]. The contorsion equations are discussed in Section 3 and Section 4 looks at the Einstein equations and how they are modified by the source spin.

1. Notations and basic Geometrical Equations

The notations used in this note are the same as those used in [1], [2], [4]. The space-time coordinates $\{x^\alpha\}$ of a point x are labelled with Greek letters $\alpha, \beta, \gamma, \ldots$, $0 \le \alpha, \beta, \gamma, \ldots < n$. The time coordinate is x^0 and, when it is necessary to distinguish spatial coordinates from the time coordinate, the letters μ, ν, ρ, η … are used. The vectors of the local natural frame are written $\overrightarrow{e_\alpha}$, $\overrightarrow{e_\beta}$, ... When tensors are expressed with respect to local orthonormal frames they are labelled with Latin letters a, b, c, ... The orthonormal local frame basis vectors are called $\overrightarrow{h_a}$, and we set $\overrightarrow{h_a} = h_a^\alpha \overrightarrow{e_\alpha}$. The metric tensor is $g_{\alpha\beta}$, and $g^{\alpha\beta}$ is its inverse. The signature of the metric is $(+ - - -)$. In the case of local orthonormal frames, the metric tensor is written η_{ab} and its diagonal terms are $\eta_{aa} = (+1, -1, -1, -1)$.

In the neighborhood of a given point, the local coordinates, with respect to the local orthonormal frame attached to this point, are given by the 1-forms $\omega^a = h^a_\alpha dx^\alpha$, which satisfy the structure equations:

$$d\omega^a + \omega^a_{.b} \wedge \omega^b = \Sigma^a \tag{1.1}$$

where $\omega^a_{.b} = \omega^a_{.b\gamma} dx^\gamma$ are the connexion 1-forms and Σ^a is the torsion 2-form. We shall also write $\omega^a_{.b} = \omega^a_{.bc}\omega^c \leftrightarrow \omega^a_{.bc} = \omega^a_{.b\gamma} h^\gamma_c$. The connexion 1-forms are related to the connexion coefficients by:

$$\omega^a_{.b\gamma} = \Gamma^\alpha_{.\beta\gamma} h^a_\alpha h^\beta_b + h^a_\delta \partial_\gamma h^\delta_b \tag{1.2}$$

The connexion coefficients are the sum of two terms:

$$\Gamma^\alpha_{.\beta\gamma} = \tilde{\Gamma}^\alpha_{.\beta\gamma} + \overline{S}^\alpha_{.\beta\gamma} \tag{1.3}$$

where the first term is the Christoffel symbol and the second is the contorsion tensor. The contorsion is anti symmetric with respect to the two first indices $\overline{S}_{\alpha\beta\gamma} + \overline{S}_{\beta\alpha\gamma} = 0$. The torsion tensor is:

$$S^{\alpha}_{.\beta\gamma} = \frac{1}{2}(\Gamma^{\alpha}_{.\beta\gamma} - \Gamma^{\alpha}_{.\gamma\beta}) = \frac{1}{2}(\overline{S}^{\alpha}_{.\beta\gamma} - \overline{S}^{\alpha}_{.\gamma\beta}) \qquad (1.4a)$$

and inversely:

$$\overline{S}^{\alpha}_{.\beta\gamma} = S^{\alpha}_{.\beta\gamma} - S_{\beta.\gamma}^{\ \alpha} - S_{\gamma.\beta}^{\ \alpha} \qquad S_{\beta.\gamma}^{\ \alpha} = g^{\alpha\delta}S_{\beta\delta\gamma} \qquad (1.4b)$$

The torsion 2-form is:

$$\Sigma^{a} = \Sigma^{a}_{.bc}\,\omega^{b} \wedge \omega^{c} = -h^{a}_{\alpha}S^{\alpha}_{.\beta\gamma}dx^{\beta} \wedge dx^{\gamma} \qquad (1.5)$$

Using (1.2), we set:

$$\widetilde{\Gamma}^{a}_{.b\gamma} = \widetilde{\Gamma}^{\alpha}_{.\beta\gamma}h^{a}_{\alpha}h^{\beta}_{b} + h^{a}_{\delta}\partial_{\gamma}h^{\delta}_{b} \qquad (1.6)$$

The curvature 2-form is defined by:

$$\Omega^{a}_{.b} = d\omega^{a}_{.b} + \omega^{a}_{.c} \wedge \omega^{c}_{.b} = \frac{1}{2}R^{a}_{.bcd}\omega^{c} \wedge \omega^{d} = \frac{1}{2}R^{a}_{.b\gamma\delta}dx^{\gamma} \wedge dx^{\delta}$$

$$\leftrightarrow R^{a}_{.b\gamma\delta} = \partial_{\gamma}\Gamma^{a}_{.b\delta} - \partial_{\delta}\Gamma^{a}_{.b\gamma} + \Gamma^{a}_{.e\gamma}\Gamma^{e}_{.b\delta} - \Gamma^{a}_{.e\delta}\Gamma^{e}_{.b\gamma} \qquad (1.7)$$

In this note, we shall assume axial symmetry and use cylindrical coordinates [4]. These coordinates are defined with respect to a geodesic called the symmetry axis and named the Oz axis. The distance of a point P to its orthogonal projection on Oz is the radius called ρ. The abscissa of this projection is called z and the azimuthal angle is φ. We assume that:

$$\omega^{0} = fd\eta + xd\varphi \qquad \omega^{1} = gd\rho \qquad \omega^{2} = hd\varphi \qquad \omega^{3} = ldz \qquad (1.8)$$

where η is the conformal time, and f, g, h, l are functions of η, ρ, φ, z. This choice corresponds to:

$$h^a_\alpha = \begin{bmatrix} f & 0 & x & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & l \end{bmatrix}, \qquad h^\alpha_a = \begin{bmatrix} \frac{1}{f} & 0 & -\frac{x}{fh} & 0 \\ 0 & \frac{1}{g} & 0 & 0 \\ 0 & 0 & \frac{1}{h} & 0 \\ 0 & 0 & 0 & \frac{1}{l} \end{bmatrix},$$

where the upper index is the line index and the lower index is the column one.

$$g_{\alpha\beta} = \begin{bmatrix} f^2 & 0 & xf & 0 \\ 0 & -g^2 & 0 & 0 \\ xf & 0 & -h^2 + x^2 & 0 \\ 0 & 0 & 0 & -l^2 \end{bmatrix},$$

$$g^{\alpha\beta} = \begin{bmatrix} \frac{1}{f^2}\left(1 - \frac{x^2}{h^2}\right) & 0 & \frac{x}{fh^2} & 0 \\ 0 & -\frac{1}{g^2} & 0 & 0 \\ \frac{x}{fh^2} & 0 & -\frac{1}{h^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{l^2} \end{bmatrix}.$$

The square root of the metric tensor determinant is $|g| = fghl$

A solution to the equations (1.1) without torsion is:

$$\omega^0_{.1} = \frac{f_\rho}{g} d\eta + \frac{g_\eta}{f} d\rho + \frac{1}{2g}\left(x_\rho + x\frac{f_\rho}{f}\right) d\varphi$$

$$\omega^0_{.2} = \frac{1}{h}(f_\varphi - x_\eta) d\eta + \frac{1}{2h}\left(x\frac{f_\rho}{f} - x_\rho\right) d\rho$$

$$+ \frac{1}{f}\left(h_\eta + \frac{x}{h}(f_\varphi - x_\eta)\right) d\varphi + \frac{1}{2h}\left(x\frac{f_z}{f} - x_z\right) dz$$

$$\omega^0_{.3} = \frac{f_z}{l}\,d\eta + \frac{1}{2l}\left(x_z + x\,\frac{f_z}{f}\right)d\varphi + \frac{l_\eta}{f}\,dz$$

$$\omega^1_{.2} = \frac{f}{2gh}\left(x_\rho - x\,\frac{f_\rho}{f}\right)d\eta + \frac{1}{h}\left(g_\varphi - x\,\frac{g_\eta}{f}\right)d\rho$$

$$- \frac{1}{g}\left(h_\rho + \frac{x}{2h}\left(x\,\frac{f_\rho}{f} - x_\rho\right)\right)d\varphi$$

$$\omega^1_{.3} = \frac{g_z}{l}\,d\rho - \frac{l_\rho}{g}\,dz$$

$$\omega^2_{.3} = \frac{f}{2hl}\left(x\,\frac{f_z}{f} - x_z\right)d\eta + \frac{1}{l}\left(h_z + \frac{x}{2h}\left(x\,\frac{f_z}{f} - x_z\right)\right)d\varphi$$

$$+ \frac{1}{h}\left(x\,\frac{l_\eta}{f} - l_\varphi\right)dz$$

Hypothesis 1. From now on we shall study the equations of motion in the static case $(\partial_\eta = 0)$ and assuming azimuthal symmetry $(\partial_\varphi = 0)$.

The Christoffel symbols belong to 3 groups. The first group, named G1, contains the elements which are null, the second group, named G2, contains the elements which are independent on x at first order (up to $O(x^2)$ terms) and the third one, G3, the elements of which depend on x at order 1.

The groups G2 and G3 correspond, respectively, to the columns 1 and 2 of Table 1.

Note. x is not a parameter. It is a function which is directly linked to the angular momentum J of the source: $x \sim J$. By misuse x will be often used as if it were a parameter.

Table 1 applies also to the Christoffel symbols in the form $\widetilde{\Gamma}^a_{.bc} = h^\gamma_c \widetilde{\Gamma}^a_{.b\gamma}$ because if $\widetilde{\Gamma}^a_{.b\gamma}$ is odd (even) in x then $\widetilde{\Gamma}^a_{.bc}$ is also odd (even). For instance $\tilde{\Gamma}^{a}_{.bc=0} = h^{\gamma}_{c=0}\tilde{\Gamma}^{a}_{.b\gamma} = \frac{1}{f}\tilde{\Gamma}^{a}_{.b\gamma=0}$ and $\tilde{\Gamma}^{a}_{.bc=2} = h^{\gamma}_{c=2}\tilde{\Gamma}^{a}_{.b\gamma}$ $= \frac{1}{h}\tilde{\Gamma}^{a}_{.b\gamma=2} - \frac{x}{fh}\tilde{\Gamma}^{a}_{.b\gamma=0}$.

Table 1. Non zero Christoffel symbols (static case and azimuthal symmetry)

$\tilde{\Gamma}^{a}_{.b\alpha}$ independent on x at first order	$\tilde{\Gamma}^{a}_{.b\alpha}$ dependent on x at lowest order
$\tilde{\Gamma}^{0}_{.10} = \frac{f_\rho}{g}$	$\tilde{\Gamma}^{0}_{.12} = \frac{1}{2g}\left(x_\rho + \frac{xf_\rho}{f}\right)$
$\tilde{\Gamma}^{0}_{.30} = \frac{f_z}{l}$	$\tilde{\Gamma}^{0}_{.21} = \frac{1}{2h}\left(\frac{xf_\rho}{f} - x_\rho\right)$
$\tilde{\Gamma}^{1}_{.22} = -\frac{h_\rho}{g} - \frac{x}{2gh}\left(\frac{xf_\rho}{f} - x_\rho\right)$	$\tilde{\Gamma}^{0}_{.23} = \frac{1}{2h}\left(\frac{xf_z}{f} - x_z\right)$
$\tilde{\Gamma}^{1}_{.31} = \frac{g_z}{l}$	$\tilde{\Gamma}^{0}_{.32} = \frac{1}{2l}\left(x_z + \frac{xf_z}{f}\right)$
$\tilde{\Gamma}^{1}_{.33} = -\frac{l_\rho}{g}$	$\tilde{\Gamma}^{1}_{.20} = \frac{f}{2gh}\left(x_\rho - \frac{xf_\rho}{f}\right)$
$\tilde{\Gamma}^{2}_{.32} = \frac{h_z}{l} + \frac{x}{2hl}\left(\frac{xf_z}{f} - x_z\right)$	$\tilde{\Gamma}^{2}_{.30} = \frac{f}{2hl}\left(\frac{xf_z}{f} - x_z\right)$

2. The Equations of Motion

The equations of motion are obtained from the gauge invariant Lagrangian (2.10) of [1]:

$$-\frac{L}{\beta^2 N} = \Omega^{ab} \wedge * (\omega_a \wedge \omega_b) + \eta\Omega^{ab} \wedge * \Omega_{ab} + \lambda dV + \mu \Sigma^a \wedge * \Sigma_a \quad (2.1)$$

with the constraint $\Sigma^a = d\omega^a + \omega^a_{.b} \wedge \omega^b$, and where $dV = \omega_a \wedge * \omega^a$ and λ is the cosmological constant. β is a free gauge coupling parameter and here $N = 4$.

In [1], the parameters η, λ, μ have been set free, but they should be linked by:

$$\eta = 1/(8\beta^2) \qquad \lambda = 2\beta^2 \qquad \mu = -1 \tag{2.2}$$

In the following $\mu = -1$.

The first term of (2.1) is the Einstein-Hilbert Lagrangian $L_{EH} = RdV$. The second term is $\eta\Omega^{ab} \wedge * \,\Omega_{ab} = \eta R^{ab}_{\ ..cd} R_{ab..}^{\ \ ..cd}$.

The terms involving the Christoffel symbols only are separated from those involving the contorsion. Please note that, the variable used is the contorsion tensor, not the torsion. The curvature tensor becomes:

$$R^{a}_{.b\gamma\delta} = \tilde{R}^{a}_{.b\gamma\delta} + \tilde{D}_\gamma \bar{S}^{a}_{.b\delta} - \tilde{D}_\delta \bar{S}^{a}_{.b\gamma} + \bar{S}^{a}_{.e\gamma}\bar{S}^{e}_{.b\delta} - \bar{S}^{a}_{.e\delta}\bar{S}^{e}_{.b\gamma}$$

where $\tilde{D}$ is the covariant derivative involving the Christoffel symbols only.

We write:

$$R^{a}_{.b\gamma\delta} = \tilde{R}^{a}_{.b\gamma\delta} + K^{a}_{.b\gamma\delta} \tag{2.3}$$

where

$$K^{a}_{.b\gamma\delta} = \tilde{D}_\gamma \bar{S}^{a}_{.b\delta} - \tilde{D}_\delta \bar{S}^{a}_{.b\gamma} + \bar{S}^{a}_{.e\gamma}\bar{S}^{e}_{.b\delta} - \bar{S}^{a}_{.e\delta}\bar{S}^{e}_{.b\gamma} \tag{2.4}$$

which satisfies the same symmetry relations as the curvature tensor:

$$K_{ba\gamma\delta} = -K_{ab\gamma\delta} \qquad K_{ab\gamma\delta} = -K_{ab\delta\gamma} \tag{2.5}$$

The variation of the Lagrangian with respect to the connexion ([1], equation (3.11)) leads to:

$$\eta^{ac}\overline{S}^{b} - \eta^{bc}\overline{S}^{a} + 2\sigma^{abc} = \frac{\mu_g}{4}(D_R^{abc} - 4S_\psi^{abc}) \tag{2.6a}$$

where

$$\overline{S}^{a} = \overline{S}^{da}_{\,..d} \qquad \sigma^{abc} = \overline{S}^{abc} + \overline{S}^{cab} + \overline{S}^{bca} \tag{2.6b}$$

S_ψ^{abc} is the spin tensor of the matter fields.

$$D_R^{abc} = D_d R^{abcd} = \widetilde{D}_d\widetilde{R}^{abcd} + \widetilde{D}_d K^{abcd} + \overline{S}^{a}_{\;.ed}(\widetilde{R}^{ebcd} + K^{ebcd}) + \overline{S}^{b}_{\;.ed}(\widetilde{R}^{aecd} + K^{aecd}) \tag{2.6c}$$

This tensor satisfies the anti-symmetry relation $D_R^{abc} = -D_R^{bac}$ as expected. σ^{abc} is totally anti-symmetric, and $\mu_g = 1/\beta^2 \sim 8\pi G/c^4$.

The equation (2.6c) can be transformed into:

$$\begin{aligned} D_R^{abc} = {} & \widetilde{D}_d\widetilde{R}^{abcd} + h^c_\gamma[-g^{\delta\nu}\widetilde{D}_\delta\widetilde{D}_\nu\overline{S}^{ab\gamma} + \widetilde{R}^{\gamma}_{\,.\varepsilon}\overline{S}^{ab\varepsilon}] \\ & + \overline{S}^{a}_{\;.ed}(2\widetilde{R}^{ebcd} + K^{ebcd}) + \overline{S}^{b}_{\;.ed}(2\widetilde{R}^{aecd} + K^{aecd}) \\ & + h^c_\gamma[g^{\gamma\mu}\widetilde{D}_\mu\widetilde{D}_\delta\overline{S}^{ab\delta} + \widetilde{D}_\delta\overline{S}^{a\,.\,\gamma}_{\;.e.}\overline{S}^{eb\delta} + \overline{S}^{a\,.\,\gamma}_{\;.e.}\widetilde{D}_\delta\overline{S}^{eb\delta} \\ & - \widetilde{D}_\delta\overline{S}^{a\,.\,\delta}_{\;.e.}\overline{S}^{eb\gamma} - \overline{S}^{a\,.\,\delta}_{\;.e.}\widetilde{D}_\delta\overline{S}^{eb\gamma}] \end{aligned} \tag{2.6d}$$

In the second square brackets the first, third and fourth terms can be eliminated by using the gauge condition (A.4), but the Lorenz condition can not be imposed for all ab pairs (see Appendix A). The other terms are quadratic in the contorsion. The second term of (2.6d) looks like the Laplacian of a vector field.

The variation of the Lagrangian with respect to the fields $\{h_a^\alpha\}$ ([1], equation (3.12)) gives the Einstein equations:

$$\tilde{G}^{a}_{.b} + (K^{a}_{.b} - \frac{K}{2}\delta^{a}_{b}) - \lambda\delta^{a}_{b} + \tilde{D}_{c}(\overline{S}_{b}^{\cdot ca} - \overline{S}_{b}^{\cdot ac})$$

$$+ \overline{S}_{bed}(\overline{S}^{eda} - \overline{S}^{ead}) = 2\mu_{g}T^{a}_{.b} \tag{2.7}$$

where $\tilde{G}^{a}_{.b} = \tilde{R}^{a}_{.b} - \frac{\tilde{R}}{2}\delta^{a}_{b}$ is the Einstein tensor obtained with the Christoffel symbol only, and $K_{cd} = K^{e}_{.ced}$.

The left member of (2.7) is rewritten:

$$\tilde{G}^{a}_{.b} + U^{a}_{.b} + Q^{a}_{.b} \tag{2.8}$$

where $U^{a}_{.b}$ represents the terms linear in the contorsion and $Q^{a}_{.b}$ the terms quadratic in the contorsion:

$$U^{a}_{.b} = -\tilde{D}_{b}\overline{S}^{a} - \tilde{D}_{d}\overline{S}^{ad}_{..b} + \tilde{D}_{d}\overline{S}^{d}\delta^{a}_{b} + \mu(\tilde{D}_{c}\overline{S}_{b}^{\cdot ac} - \tilde{D}_{c}\overline{S}_{b}^{\cdot ca}) \tag{2.9}$$

If $a = b$ in $U^{a}_{.b}$ (without sum) the first term inside the brackets is null, and the second and last term cancel ($\mu = -1$). It remains $U^{a}_{.a} = \tilde{D}_{d}\overline{S}^{d} - \tilde{D}_{a}\overline{S}^{a}$ without sum over a. If $a = 0$, we get the divergence term of [2] (3.2).

The quadratic term is:

$$-Q^{a}_{.b} = \overline{S}^{e}\overline{S}^{a}_{.eb} + \overline{S}^{a}_{.ef}\overline{S}^{ef}_{..b} + \overline{S}_{bef}\overline{S}^{efa} + \overline{S}_{bef}\overline{S}^{aef}$$

$$-\frac{1}{2}(\overline{S}^{e}\overline{S}_{e} + \overline{S}_{def}\sigma^{efd})\delta^{a}_{b} \tag{2.10}$$

$\tilde{D}^{abc}_{R} = h^{c}_{\gamma}\tilde{D}_{\delta}\tilde{R}^{ab\gamma\delta}$ is computed for all a, b, c with Hypothesis 1. The calculation details are not given but the result is:

For abc belonging to G1, $\widetilde{D}_R^{abc} = 0$.

For abc belonging to G2, $\widetilde{D}_R^{abc}$ is even with respect to x.

For abc belonging to G3, $\widetilde{D}_R^{abc}$ is odd with respect to x.

In [1], we obtained, in the case of the Schwarzchild metric and the Lense-Thirring metric: $\widetilde{D}_d \widetilde{R}^{abcd} = 0$. There is no contradiction with the above result if we make the additional assumption that the gravitational field is described by these metrics. The above classification motivates the following working hypothesis [2]:

Hypothesis 2. The contorsion tensor $\overline{S}^{a}_{.\,b\gamma}$ are grouped like the Christoffel symbols $\widetilde{\Gamma}^{a}_{.\,b\gamma}$ or, equivalently, the connexion coefficients are grouped like the Christoffel symbols.

As suggested in [2], we shall assume that the components (abc) belonging to the first column of Table 1 are even in x, and those belonging to the second one are odd in x. For instance, this means that $\overline{S}^{0}_{.10}$ is even in x.

It is easy but long to check that all the equations are consistent with this hypothesis, that is to say that all the non zero terms in a given equation have all the same parity.

The direct consequence is that:

$$\widetilde{D}_K^{abc} = \widetilde{D}_d K^{abcd} \text{ falls in the same groups as } \widetilde{D}_R^{abc} \tag{2.11}$$

The same result holds also for $(\overline{S}\widetilde{R})^{abc} = \overline{S}^{a}_{.\,ed}\widetilde{R}^{ebcd} + \overline{S}^{b}_{.\,ed}\widetilde{R}^{aecd}$.

Hypothesis 2 brings important simplifications to the equation system (2.6), (2.7). First we have: $\overline{S}^0 = \overline{S}^2 = 0$ and if: $(abc) \in \text{G1}$ then

$\sigma^{abc} = 0$. If $(abc) \in \mathrm{G1}$: $\widetilde{D}_R^{abc} = 0$ and $\widetilde{D}_K^{abc} = 0$, then the 12 equations (2.6a) are identically satisfied provided that the source terms $\overline{S}_\psi$ for these indices are null. Therefore it remains 12 equations of type (2.6) and 12 unknown contorsion terms for $(abc) \in \mathrm{G2, G3}$ which are studied in Sections 3 and 4.

Despite Hypotheses 1 and 2 the equations of motion remain very complicated. They depend on two main parameters, the mass of the source and an angular momentum for the function x like for the Kerr metric [5], [6]. There are other parameters like those describing the shape of the source but they will be ignored. In the following, although the function x (and its derivatives) is not a parameter, we shall separate the terms according to their degree: terms independent on x, depending on x at first degree etc. x will be assumed to be small enough to allow developments with respect to it. When the term "lowest order" will be used it means that μ_g times the mass of the source is small and that x is small enough too. In the following the parameter μ_g times the mass of the source is called α.

The expressions of $\widetilde{G}_b^a$ given in Appendix B show that if $b = a$, the explicit dependence on x is even and that $\widetilde{G}_{b=2}^{a=0}$ depends explicitly on x only "linearly". We can get consistent equations if f, g, h, l depend on x by even degree terms, like for the Kerr metric.

In [4], we have shown that isotropic coordinates are very convenient and we have used them in [2] although this is wrong, because, due to the source rotation the problem is no longer isotropic. However, at lowest order (as defined previously), it is possible to assume $l \simeq g$, $h \simeq \rho g$, which simplifies the equations.

In the rest of this text and in order to avoid any confusion when the indices are numbers instead of letters we write: $\widetilde{S}_{abc} = h_c^{\gamma}\overline{S}_{ab\gamma}$.

3. The Contorsion Equations

When the indices of the triplet abc are all different, equations (2.6) are written

$$8\sigma^{abc}/\mu_g = \widetilde{D}_d\widetilde{R}^{abcd} + \widetilde{D}_d K_C^{abcd} + \widetilde{D}_d K_Q^{abcd}$$

$$+ \overline{S}^{a}_{\ .ed}(\widetilde{R}^{ebcd} + \widetilde{K}^{ebcd}) + \overline{S}^{b}_{\ .ed}(\widetilde{R}^{aecd} + \widetilde{K}^{aecd}) - 4S_{\psi}^{abc} \quad (3.1)$$

For $abc \in \mathrm{G}3$, these equations are at lowest order.

$$8\sigma^{120}g^2/\mu_g = \partial_{\rho\rho}\widetilde{S}_{120} + \frac{1}{\rho}\partial_\rho\widetilde{S}_{120} + \partial_{zz}\widetilde{S}_{120} - 4g^2 S_\psi^{120}$$

$$8\sigma^{012}g^2/\mu_g = \partial_{\rho\rho}\widetilde{S}_{012} + \frac{1}{\rho}\partial_\rho\widetilde{S}_{012} + \partial_{zz}\widetilde{S}_{012} + \frac{g}{\rho}L_{02}$$

$$- \frac{2}{\rho^2}(\widetilde{S}_{012} + \widetilde{S}_{021}) + (1 - \frac{g}{l})\partial_z\widetilde{S}_{023} - 4g^2 S_\psi^{012}$$

$$8\sigma^{201}g^2/\mu_g = \partial_{\rho\rho}\widetilde{S}_{201} + \frac{1}{\rho}\partial_\rho\widetilde{S}_{201} + \partial_{zz}\widetilde{S}_{201}$$

The sum of these three equations is:

$$24\sigma^{120}/\mu_g = \left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz}\right)\sigma^{120} - 4\sigma_\psi^{120}$$

$$+ \left(\partial_\rho + \frac{1}{\rho}\right)L_{02} + O(\alpha) + O(x^3) \quad (3.2)$$

where (appendix A) at lowest order:

$$L_{02} = \partial_\rho\widetilde{S}^{021} + \frac{1}{\rho}(\widetilde{S}_{012} + \widetilde{S}_{021}) + \partial_z\widetilde{S}^{023}.$$

The other components are:

$$8\sigma^{230}/g^2\mu_g = \partial_{\rho\rho}\tilde{S}_{230} + \frac{1}{\rho}\partial_\rho\tilde{S}_{230} + \partial_{zz}\tilde{S}_{230} - \frac{1}{\rho^2}\tilde{S}_{230} - 4g^2S_\psi^{230}$$

$$8\sigma^{302}/\mu_g = \frac{1}{g^2}(\partial_{\rho\rho}\tilde{S}_{302} + \frac{1}{\rho}\partial_\rho\tilde{S}_{302} + \partial_{zz}\tilde{S}_{302} - \frac{1}{\rho^2}\tilde{S}_{302} - 4g^2S_\psi^{302})$$

$$8\sigma^{023}/\mu_g = \frac{1}{g^2}(\partial_{\rho\rho}\tilde{S}_{023} + \frac{1}{\rho}\partial_\rho\tilde{S}_{023} + \partial_{zz}\tilde{S}_{023} - \frac{1}{\rho^2}\tilde{S}_{023} - \partial_z(gL_{02}) - 4g^2S_\psi^{023})$$

and the sum of these three equations is:

$$24\sigma^{230}/\mu_g = (\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz})\sigma^{230} - 4\sigma_\psi^{230}$$

$$+ \partial_z L_{02} + O(\alpha) + O(x^3) \tag{3.3}$$

Note that the contribution $\tilde{D}_d\tilde{R}^{abcd}$ in (3.2) and (3.3) is:

$\tilde{D}_d(\tilde{R}^{abcd} + \tilde{R}^{cabd} + \tilde{R}^{bcad})$ which is null, thanks to the Bianchi identities of the first kind.

The equations (3.2) and (3.3) are not decoupled because of the term L_{02}. If the source terms $S_\psi^{\{abc\}\in G3} = 0$, then at lowest order: $\tilde{S}^{\{abc\}\in G3} = 0$.

Equations (2.6) for $a = c$ give:

$$12\bar{S}^b/\mu_g = \tilde{D}_d\tilde{R}^{bd} + \tilde{D}_dK_C^{bd} + \tilde{D}_dK_Q^{bd} + \bar{S}^a_{\ .ed}\tilde{R}^{eb.d}_{..a}$$

$$+ \bar{S}^b_{\ .ed}\tilde{R}^{ed} + \bar{S}^a_{\ .ed}K^{eb.d}_{..a} + \bar{S}^b_{\ .ed}K^{ed} - 4S_{\psi..a}^{\ \ ab} \tag{3.4}$$

The first term on the right is $\tilde{D}_d\tilde{R}^{bd} = \tilde{D}_d\left(\tilde{G}^{bd} + \frac{\tilde{R}}{2}\eta^{bd}\right) = \tilde{D}_d\tilde{R}\,\eta^{bd}/2$.

The calculations show that the quadratic terms in x come from terms of the type: $\widetilde{S}^{\{abc\}\in \mathrm{G2}}$, $\widetilde{S}^{\{abc\}\in \mathrm{G3}} \otimes \widetilde{S}^{\{abc\}\in \mathrm{G3}}$ and $\widetilde{S}^{\{abc\}\in \mathrm{G3}} \otimes x$.

If we take as an example a spinor field, then $S_{\psi}^{aba} = 0$ and the source terms for $\overline{S}^{b}$ come from the other terms. All the terms are even in x.

It was shown in [1], for the Schwarzchild and Lense-Thirring metrics which are solutions of the equations without torsion, that: $\widetilde{D}_d \widetilde{R}^{abcd} = 0$. Therefore, if $S_{\psi}^{abc} = 0$, then $\widetilde{S}^{\{abc\}\in \mathrm{G3}} = 0$ is a solution of equations (3.1), (3.2), (3.3). As a consequence, $\widetilde{S}^{\{abc\}\in \mathrm{G2}} = 0$ at order x^2 included, and, in the limit of weak fields, if $S_{\psi}^{abc} = 0$, the contorsion is null.

4. The Einstein Equations

As said at the end of Section 2, $\widetilde{G}_{b=2}^{a=0}$ provides an equation for x. With (2.9) we have:

$$U^{a=0}_{.\,b=2} = -\widetilde{D}_2 \overline{S}^0 + \widetilde{D}_d \sigma^{0d2} \tag{4.1}$$

where: $\widetilde{D}_2 \overline{S}^0 = \partial_2 \overline{S}^0 + \widetilde{\Gamma}^0_{.\,e2} \overline{S}^e$.

From Section 2: $\overline{S}^0 = \overline{S}^2 = 0$ and: $\overline{S}^1, \overline{S}^3 \sim O(x^2)$, therefore:

$$U^{a=0}_{.\,b=2} = \widetilde{D}_d \sigma^{0d2} + O(x^3) \qquad\qquad U^{\alpha=0}_{.\,\beta=2} = \frac{h}{f} \widetilde{D}_d \sigma^{0d2} + O(x^3)$$

where, without any hypothesis about the value of α:

$$\widetilde{D}_d \sigma^{0d2} = -\frac{1}{g}\left(\partial_\rho + \frac{l_\rho}{l}\right)\sigma^{021} - \frac{1}{l}\left(\partial_z + \frac{g_z}{g}\right)\sigma^{023} \tag{4.2}$$

Neglecting the contorsion quadratic terms (2.10) and keeping only the terms of order 0 in α and order 1 in x the Einstein equation for $G^{\alpha=0}_{.\beta=2}$ is:

$$\frac{1}{2g^2}\left\{x_{\rho\rho} - \frac{1}{\rho}x_\rho + x_{zz}\right\} + \frac{h}{f}\tilde{D}_d\sigma^{0d2} = \mu_g T^{\alpha=0}_{.\beta=2} \tag{4.3}$$

The term in the brackets looks like the Euclidean Laplacian except for the sign of the second term. Setting: $x = \rho\partial_\rho Y$ we get:

$$\left\{x_{\rho\rho} - \frac{1}{\rho}x_\rho + x_{zz}\right\} = \rho\partial_\rho\Delta_E Y$$

where Δ_E means the Euclidean Laplacian.

The Energy-Momentum tensor component $T^{\alpha=0}_{.\beta=2}$ is associated to the local matter momentum p_φ or equivalently: $T^{\alpha=0}_{.\beta=2} = -hp^{c=2}$.

Let us assume that the source is a classical non relativistic matter body uniformly rotating around the Oz axis and that: $p^{c=2} = d\rho\omega$ where d is the matter density and ω is the angular rotation speed. In weak field situation (parameter α small) $f,\ g,\ \overline{h} \simeq 1$, we then have: $\partial_\rho\Delta_E Y = 2\mu_g d\rho\omega$. Far from the source we obtain: $Y = -\mu_g\ L/r$ where L is the angular momentum of the source and r is the Euclidean distance from the origin of the coordinates. Then:

$$x = \mu_g\ L\rho^2/r^3 \tag{4.4}$$

This term looks like the cross term of the Kerr metric but the comparison requires to be careful with the meaning of the coordinates. It is valid only in the limit $\alpha \to 0$ and x small.

Now we look at the other source term in equation (4.3), and consider the equation:

$$\frac{1}{2g^2}\rho\partial_\rho\Delta_E Y = -\frac{h}{f}\tilde{D}_d\sigma^{0d2}$$

From (4.2), neglecting the term of order α we have:

$$\tilde{D}_d\sigma^{0d2} = -\frac{1}{g}\partial_\rho\sigma^{021} - \frac{1}{l}\partial_z\sigma^{023}$$

σ^{abc} is totally antisymmetric then σ^{021} can be replaced by $-\sigma^{120}$. If we can neglect the contribution of σ^{023} (at least far from the source) then:

$$\Delta_E Y = -2\sigma^{120} \quad \text{therefore:} \quad Y \sim \frac{1}{4\pi r}\int 2\sigma^{120}dV$$

Equation (3.2) is an equation of the form: $\tau\sigma = \Delta\sigma - b$ where: $b = 4\sigma_\psi^{120} - \left(\partial_\rho + \frac{1}{\rho}\right)L_{02}$ and where $\tau = 24/\mu_g >> 1$.

When τ is very large the solution of such an equation is approximately a contact term, and the solution is $\tau\sigma \simeq -b$ then:

$$6\int\sigma^{120}dV \simeq -\mu_g\int\sigma_\psi^{120}dV + \frac{\mu_g}{4}\int\left(\partial_\rho + \frac{1}{\rho}\right)L_{02}dV$$

where $dV = (fghl)\rho d\rho dz d\varphi \sim dV_E$ when $\alpha \to 0$ and dV_E is the Euclidean volume element. Integrating by part, the last part of this equation becomes $[\rho L_{02}]$ which is assumed to be negligible far from the source. Then:

$$6\int\sigma^{120}dV \simeq -\mu_g\int\sigma_\psi^{120}dV$$

We assume now that the matter field is a superimposition of spinor fields, then S_ψ^{abc} is cyclic and we have:

$$\int\sigma^{120}dV \simeq -\frac{1}{2}\mu_g\int\tilde{S}_\psi{}^{120}dV$$

$$Y \sim -\frac{1}{4\pi r}\mu_g \int \widetilde{S}_\psi{}^{120} dV$$

Adding the angular momentum (4.4) and spin contributions gives:

$$x \sim -\frac{1}{4\pi r}\frac{\rho^2}{r^2}\mu_g\left(L + \int \widetilde{S}_\psi{}^{120}\, dV\right)$$

which shows that in the limit $\alpha \to 0$ and x small, the angular momentum in the crossed term of the Lense-Thirring and Ker metrics is the sum of the rotation angular momentum and the spin of the source, as expected.

As said at the end of Section 2, the correction to the metric coefficients f, g, h, l are of order x^2 and possibly more.

In the Einstein equations, the terms U_a^a contributes to the metric coefficients at order x^2 since $\widetilde{S}^{cbc}$ is even in x. For instance: $U_0^0 = \widetilde{D}_i \overline{S}^i$ where $i \neq 0$, involves $\overline{S}^1$ and $\overline{S}^3$ which are given by equation (3.4). It results that, because μ_g is small, that U_0^0 is of order $\alpha\mu_g$ or less, and will be neglected.

Conclusion. For weak fields and far from the source, the solution of the equations is given by the Kerr metric whose angular momentum parameter is the sum of the rotation angular momentum plus intrinsic spin, as expected.

5. Extending the Algebra

In [1], we used the algebra formed by the Dirac matrices γ^a and the rotation operators $R^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ to build the gauge theory Lagrangian (2.1). This allowed to introduce the Einstein-Hilbert gravity

Lagrangian naturally. In this section, this algebra is enlarged, introducing new fields.

We define: $\gamma^h = \gamma^{a_1}\gamma^{a_2} \ldots \gamma^{a_h}$ where: $a_1 < a_2 < \ldots < a_h$. If the space-time dimension n is even $Tr(\gamma^h) = 0$ whatever h, and γ^h plus the unit matrix form a basis. We call $\gamma^D = \gamma^0\gamma^1 \ldots \gamma^{n-1}$. In the case $n = 4$, γ^D is usually called γ^5.

In the rest of this note, the space-time dimension n is assumed to be even, and we have: $\gamma^a\gamma^D + \gamma^D\gamma^a = 0$ and: $(\gamma^D)^2 = -(-1)^{n/2} I$.

Now we consider the set: γ^a, R^{ab}, γ^D, R^{aD}, $(a, b < n)$ where:

$$R^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \quad \text{and:} \quad R^{aD} = \frac{1}{4}[\gamma^a, \gamma^D] = \frac{1}{2}\gamma^a\gamma^D$$

The new elements satisfy the following relations:

$$[\gamma^a, R^{bD}] = \eta^{ab}\gamma^D \qquad [\gamma^D, R^{ab}] = 0$$

$$[\gamma^{ab}, R^{cD}] = \eta^{cb}R^{aD} - \eta^{ca}R^{bD}$$

$$[\gamma^D, R^{aD}] = -\eta^{DD}\gamma^a \qquad [R^{aD}, R^{bD}] = R^{ab} \tag{5.1}$$

This is an extension of the algebra (2.1) of [1] where we define: $\eta^{aD} = 0$ if $a < n$ and $\eta^{DD} = -(-1)^{n/2}$.

The trace formulae (2.9) of [1] can be extended to γ^D and R^{aD}. The following properties are used to construct a Lagrangian as in [1]:

$$Tr(\gamma^a\gamma^D) = 0 \qquad Tr(\gamma^D\gamma^D) = N\eta^{DD} \qquad Tr(R^{aD}\gamma^c) = 0$$

$$Tr(R^{ab}R^{cD}) = 0 \qquad Tr(R^{aD}R^{bD}) = \frac{N}{4}(-1)^{n/2}\eta^{ab} \tag{5.2}$$

where N is the dimension of the Dirac matrix representation.

From now on we shall consider only the case $n = 4$. We can proceed like in [1] and define the gauge field:

$$W = \alpha\omega_{ab}R^{ab} + \beta\omega_a\gamma^a + 2\alpha\omega_{a5}R^{a5} + \beta\omega_5\gamma^5$$

As usual, gauge transformations will be of the form: $W' = S^{-1}WS + S^{-1}dS$ where for infinitesimal transformations:

$$S = I + i\varepsilon_a\gamma^a + \varepsilon_{ab}R^{ab} + i\varepsilon_5\gamma^5 + 2\varepsilon_{c5}R^{c5}$$

The gauge field transforms as:

$$W' = W'_A + (\beta\omega_5\gamma^5 + 2\alpha\omega_{c5}R^{c5}) + 2\left[\beta\varepsilon_{c5}\omega^c - i\alpha\varepsilon^c\omega_{c5} + \frac{i}{2}d\varepsilon_5\right]\gamma^5$$

$$+ 4\left[i\beta(\varepsilon_5\omega_c - \varepsilon_c\omega_5) + \alpha(\omega^e_{.5}\varepsilon_{ec} - \omega^e_{.c}\varepsilon_{e5}) + \frac{1}{2}d\varepsilon_{c5}\right]R^{cD}$$

$$+ 2[\beta\omega_5\varepsilon_{f5} - 2i\omega_{f5}\varepsilon_5]\gamma^f + 4\alpha\omega_{e5}\varepsilon_{f5}R^{ef} \qquad (5.3)$$

where W'_A is given by (A.1).

$\omega^5 = h^5_\alpha dx^\alpha$ appears as a pseudo coordinate, and we would like to get rid of it, which means cancelling the factor of γ^5 in (5.3) by using successive infinitesimal gauge transformations. $\omega^5 = 0$ represents 4 constraints $h^5_\alpha = 0$. We require also that there is no contribution from the last bracket whatever the index f, which can be achieved if $\varepsilon_5 = 0$. We then have 5 constraints and five parameters (ε_{c5} and ε_5). There is no degree of freedom left.

The Lagrangian is obtained as usual by defining: $G = dW + W \wedge W$ and $L = Tr(G \wedge * G)$.

In order to lighten the notations we write: $B_c = \omega_{c5} = \omega_{c5a}\omega^a = B_{ca}\omega^a$ (note the place of indices), and:

$$L = L_0 + L_1 + L_2 + L_3 + L_4 \tag{5.4}$$

where L_0 is the Lagrangien (2.1) and the other components are:

$$L_1/dV = \frac{1}{2}F_{x\alpha\beta}F^{x\alpha\beta} + \omega^{xu\alpha}B_{u}^{\cdot\beta}F_{x\alpha\beta} - 4\omega^{xuc}B_{u}^{\cdot f}B_{xg}S^{g}_{\cdot fc}$$

$$+ \omega_{xyc}\omega^{x}_{\cdot ud}(B^{y}_{\cdot e}B^{ue}\eta^{cd} - B^{yd}B^{uc})$$

where $F_{x\alpha\beta} = \partial_\alpha B_{x\beta} - \partial_\beta B_{x\alpha}$

$$L_2/dV = -R_{abef}B^{ae}B^{bf}$$

$$L_3/dV = -4\beta^2 B_{fh}(B^{fh} - B^{hf})$$

$$L_4/dV = -\frac{1}{2}B_{ek}B_{fl}(B^{ek}B^{fl} - B^{el}B^{fk})$$

The Lagrangian (2.1) has been extended to 4 more vector fields: B_a. The only gauge freedom left is the mobile frame rotation invariance. The field B_{ea} can be seen alternatively as a symmetric field with 10 components plus an antisymmetric field with 6 components.

The quadratic terms which are supposed to represent the mass terms show that there are no fixed mass terms, and as in [3], one could consider masses as effective fields. The potential term L_4 is quartic.

We can separate the trace: $D = B^{a}_{\cdot a}$ and write:

$$B_{ea} = D\eta_{ea} + C_{ea} \tag{5.5a}$$

$$F_{eab} = \eta_{eb}d_a D - \eta_{ea}d_b D + d_a C_{eb} - d_b C_{ea} \tag{5.5b}$$

with $F'_{eab} = d_a C_{eb} - d_b C_{ea}$ we obtain:

$$F_{eab}F^{eab} = 6D_a D^a + 4\eta_{eb} D_a F'^{eab} + F'_{eab}F'^{eab}$$

where $D_a = d_a D$. The first term on the right being the Lagrangian of a scalar field.

How are these new fields coupled to matter fields? Here we consider only spinor fields (quark stars?) but other fields like strong electromagnetic fields should also be considered. Coupling terms must respect mobile frame rotation invariance and parity conservation. A first set of minimal coupling invariants (with respect to mobile frame rotation) is $L_C \sim B_{ea}\overline{\psi}\gamma^e\gamma^a\psi$.

With $\gamma^e\gamma^a = \eta^{ea} + \gamma^{ea}$ where $\gamma^{ea} = \frac{1}{2}[\gamma^e, \gamma^a]$ we have two possible coupling terms: $L_{C1} = \mu_1 B_{ea}\overline{\psi}\eta^{ea}\psi$, $L_{C2} = \mu_2 B_{ea}\overline{\psi}\gamma^{ea}\psi$.

The Lagrangian is required to be real. This implies that μ_1 is real and μ_2 is imaginary. We could also have considered:

$$L_{C3} = \mu_3 B_{ea}\overline{\psi}\eta^{ea}\gamma^5\psi \qquad\qquad L_{C4} = \mu_4 B_{ea}\overline{\psi}\gamma^{ea}\gamma^5\psi$$

These coupling terms are real if μ_3 is real and μ_4 is imaginary.

With the introduction of γ^5 arises the problem of parity conservation. Here we define Parity locally as an operation which keep the time direction defined by $\vec{h}_0$ and reverse the direction of the vectors orthogonal to it. The parity operator acting on a spinor is, as in the Minkowski case, $P = \gamma^0$.

If we impose Parity conservation only L_{C1} and L_{C2} remain.

It is possible to introduce other coupling terms. Let us consider the

Lagrangian of a spinor field: $L_\psi = h_a^\alpha \gamma^a (\overline{\psi} i D_\alpha \psi + h.c.)$, where $D_\alpha \psi = \partial_\alpha \psi + \Gamma_{cd\alpha}\gamma^c\gamma^d / 4$; and $h.c.$ means hermitic conjugate.

From it we deduce the Energy-Momentum tensor: $T^a_{\psi\alpha} = \frac{1}{2}(\overline{\psi}\gamma^a i D_\alpha \psi + h.c.)$ and the spin tensor: $S^{ab\alpha}_\psi = \frac{i}{8} h_c^\alpha (\overline{\psi}\gamma^c\gamma^a\gamma^b\psi + \overline{\psi}\gamma^a\gamma^b\gamma^c\psi)$.

Now we can read the Lagrangian, a posteriori, as: $L_\psi = h_a^\alpha T^a_{\psi\alpha}$ and the coupling to the connexion as: $\Gamma_{cd\alpha} S^{cd\alpha}_\psi$. By analogy we could consider couplings like: $F'_{cd\alpha} S^{cd\alpha}_\psi$ where F'_{eab} is given by (5.5). Using the symmetry properties of the spin tensor of a spinor field, one can show that the symmetric part of C_{eb} is not coupled to $S^{cd\alpha}_\psi$. Other minimal couplings are possible. For instance, coupling to the matter current: $d_a D \overline{\psi}\gamma^a\psi$ or coupling to the Energy-Momentum tensor. Which part of the field B_{ea} plays a role depends on the matter coupling.

Appendix A. Gauge Freedom

In [1], we introduced the gauge field: $W = \alpha\omega_{ab}R^{ab} + \beta\omega_a\gamma^a$ where γ^a represents the Dirac matrices and R^{ab} represents the generators of the rotation group (not the Ricci tensor in this appendix).

The gauge field W transforms as: $W' = S^{-1}WS + S^{-1}dS$ where, for infinitesimal transformations: $S = I + i\varepsilon_a\gamma^a + \varepsilon_{ab}R^{ab}$, $\varepsilon_{ab} = -\varepsilon_{ba}$, $|\varepsilon_a|, |\varepsilon_{ab}| << 1$. One has at first order:

$$W' = W + i\alpha\omega_{ab}\varepsilon_e[R^{ab}, \gamma^e] + \alpha\omega_{ab}\varepsilon_{ef}[R^{ab}, R^{ef}] + i\beta\omega_a\varepsilon_e[\gamma^a, \gamma^e]$$
$$+ \beta\omega_a\varepsilon_{ef}[\gamma^a, R^{ef}] + id\varepsilon_e\gamma^e + d\varepsilon_{ef}R^{ef} \quad \text{(A.1)}$$

then, with the algebra ([1], (2.1)):

$$\omega'_a = \omega_a + \varepsilon_{ea}\omega^e - \varepsilon_{ae}\omega^e + \frac{i}{\beta}(d\varepsilon_a + \alpha\omega_{a\,.}^{\;e}\varepsilon_e - \alpha\omega^e_{\;.a}\varepsilon_e)$$

and with $\alpha = \frac{1}{2}$:

$$\omega'_a = \omega_a + \varepsilon_{ea}\omega^e - \varepsilon_{ae}\omega^e + \frac{i}{\beta}D\varepsilon_a \tag{A.2}$$

$$\omega'_{ab} = \omega_{ab} + [d\varepsilon_{ab} - \omega^e_{\;.b}\varepsilon_{ae} - \omega^e_{\;.a}\varepsilon_{eb}] + 2i\beta(\varepsilon_b\omega_a - \varepsilon_a\omega_b) \tag{A.3}$$

In 4 dimensions the total number of parameters is $4 + 6$, but the 1-form ω^a are fixed by the choice (1.8). Requiring, that after an infinitesimal gauge transformation, ω^0 and ω^i keep their form, which means that ω'^i does not depend on $d\eta$ and ω'^0 does not depend on $d\rho$ nor dz leads to the conditions:

$$\varepsilon_0 = \varepsilon_2 = \varepsilon_{01} = \varepsilon_{03} = 0$$

and:
$$2f\varepsilon_{02} + \frac{i}{\beta}(d_\eta\varepsilon_2 + \omega_{2\,.0}^{\;0}\varepsilon_0 + \omega_{2\,.0}^{\;1}\varepsilon_1 + \omega_{2\,.0}^{\;3}\varepsilon_3) = 0$$

Then, after a infinitesimal transformation one wants that the spatial ω^i keep their form, that is: $\omega'^1 \sim d\rho$, $\omega'^2 \sim d\varphi$, $\omega'^3 \sim dz$. This is possible if: $\varepsilon_{12} = \varepsilon_{23} = 0$ and ε_{13} is linked to ε^1 and ε^3 by the two constraints:

$$2l\varepsilon_{13} = -\frac{i}{\beta}(\partial_z\varepsilon_1 + \omega^1_{\;.33}\varepsilon_3) \qquad 2g\varepsilon_{13} = \frac{i}{\beta}(\partial_\rho\varepsilon_3 + \omega^3_{\;.11}\varepsilon_{11})$$

The infinitesimal gauge transformations satisfying the 9 above constraints keeps the form of (1.8) and the classification of Table 1. Therefore it remains only one degree of freedom out of 10. By analogy with the Lorenz condition for electromagnetism (the pair (ab) represents the rotation group indices), we would like to have:

$$\widetilde{D}_\delta \overline{S}^{ab\delta} = 0 \tag{A.4}$$

With Hypothesis 2 in Section 2 and Table 1, one obtains: $\widetilde{D}_\delta \overline{S}^{ab\delta} = 0$ for $(a, b) = (0, 1), (0, 3), (1, 2), (2, 3)$. With $h = \rho\overline{h}$, the two other components are:

$$\begin{aligned} g\widetilde{D}_\delta \overline{S}^{02\delta} = {} & \partial_\rho \widetilde{S}^{021} + \frac{1}{\rho}\widetilde{S}^{021} + \frac{f_\rho}{f}(\widetilde{S}^{021} + \widetilde{S}^{120}) \\ & + \left(\frac{\overline{h}_\rho}{\overline{h}} + \frac{l_\rho}{l}\right)\widetilde{S}^{021} + \left(\frac{\overline{h}_\rho}{\overline{h}} + \frac{1}{\rho}\right)\widetilde{S}^{012} + \frac{g}{l}\partial_z \widetilde{S}^{023} \\ & + \frac{g}{l}\left(\frac{g_z}{g} + \frac{\overline{h}_z}{\overline{h}}\right)\widetilde{S}^{023} + \frac{g}{l}\frac{f_z}{f}(\widetilde{S}^{023} + \widetilde{S}^{320}) \\ & + \frac{g}{l}\frac{h_z}{h}\widetilde{S}^{032} + \frac{1}{2h}\left(x_\rho - x\frac{f_\rho}{f}\right)(\widetilde{S}^{122} - \widetilde{S}^{010}) \\ & + \frac{1}{2h}\frac{g}{l}\left(x_z - x\frac{f_z}{f}\right)(\widetilde{S}^{322} - \widetilde{S}^{030}) \end{aligned} \tag{A.5}$$

$$\begin{aligned} \widetilde{D}_\delta \overline{S}^{13\delta} = {} & \frac{1}{g}\partial_\rho \widetilde{S}^{131} + \frac{1}{g}\left(\frac{f_\rho}{f} + \frac{h_\rho}{h} + \frac{l_\rho}{l}\right)\widetilde{S}^{131} - \frac{1}{l}\partial_z \widetilde{S}^{313} \\ & - \frac{1}{l}\left(\frac{f_z}{f} + \frac{h_z}{h} + \frac{g_z}{g}\right)\widetilde{S}^{313} + \frac{1}{g}\frac{f_\rho}{f}\widetilde{S}^{030} \\ & + \frac{1}{2gh}\left(x_\rho - x\frac{f_\rho}{f}\right)(\widetilde{S}^{032} + \widetilde{S}^{230}) - \frac{1}{g}\frac{h_\rho}{h}\widetilde{S}^{232} - \frac{1}{l}\frac{f_z}{f}\widetilde{S}^{010} \\ & + \frac{1}{2lh}\left(x_z - x\frac{f_z}{f}\right)(\widetilde{S}^{102} + \widetilde{S}^{120}) + \frac{1}{l}\frac{h_z}{h}\widetilde{S}^{212} \end{aligned} \tag{A.6}$$

The remaining degree of freedom can be used to set:

$$\tilde{D}_\delta \overline{S}^{13\delta} = 0 \tag{A.7}$$

As a consequence, the components $L_{02} = \tilde{D}_\delta \overline{S}^{02\delta}$ can not be set to 0 except in particular cases.

The above expressions are internally consistent as far as the parity of the components $\overline{S}^{ab\delta}$ with respect to x is concerned.

Appendix B. Einstein Tensor

Assuming time independence and axial symmetry, the Einstein tensor components used in this note are, without any approximation:

$$\tilde{G}^{a=0}_{b=0} = \frac{1}{g^2}\left[-\frac{h_{\rho\rho}}{h} - \frac{l_{\rho\rho}}{l} + \frac{h_\rho}{h}\left(\frac{g_\rho}{g} - \frac{l_\rho}{l}\right) + \frac{l_\rho g_\rho}{lg}\right]$$

$$+ \frac{1}{l^2}\left[-\frac{h_{zz}}{h} - \frac{g_{zz}}{g} + \frac{h_z}{h}\left(\frac{l_z}{l} - \frac{g_z}{g}\right) + \frac{l_z g_z}{lg}\right]$$

$$+ \frac{3}{4h^2g^2}\left(x_\rho - x\frac{f_\rho}{f}\right)^2 + \frac{3}{4h^2l^2}\left(x_z - x\frac{f_z}{f}\right)^2$$

$$\tilde{G}^{a=1}_{b=1} = -\frac{1}{g^2}\left[\frac{h_\rho f_\rho}{hf} + \frac{h_\rho l_\rho}{hl} + \frac{f_\rho l_\rho}{fl}\right]$$

$$+ \frac{1}{l^2}\left[-\frac{h_{zz}}{h} - \frac{f_{zz}}{f} + \frac{h_z l_z}{hl} + \frac{f_z l_z}{fl} - \frac{f_z h_z}{fh}\right]$$

$$- \frac{1}{4h^2g^2}\left(x_\rho - x\frac{f_\rho}{f}\right)^2 + \frac{1}{4h^2l^2}\left(x_z - x\frac{f_z}{f}\right)^2$$

$$\tilde{G}^{a=3}_{b=3} = -\frac{1}{g^2}\left[\frac{h_{\rho\rho}}{h} + \frac{f_{\rho\rho}}{f} - \frac{f_\rho}{f}\frac{g_\rho}{g} + \frac{f_\rho}{f}\frac{h_\rho}{h} + \frac{h_\rho}{h}\frac{g_\rho}{g}\right]$$

$$- \frac{1}{l^2}\left[\frac{f_z g_z}{fg} + \frac{f_z h_z}{fh} + \frac{h_z g_z}{hg}\right] + \frac{1}{4h^2g^2}\left(x_\rho - x\frac{f_\rho}{f}\right)^2$$

$$- \frac{1}{4h^2l^2}\left(x_z - x\frac{f_z}{f}\right)^2$$

$$\tilde{G}^{a=2}_{b=2} - \tilde{G}^{a=0}_{b=0} = \frac{1}{hg^2}\left[h_{\rho\rho} - h_\rho\frac{g_\rho}{g} + h_\rho\frac{l_\rho}{l}\right]$$

$$+ \frac{1}{hl^2}\left[h_{zz} + h_z\frac{g_z}{g} - h_z\frac{l_z}{l}\right] - \frac{1}{fg^2}\left(f_{\rho\rho} - f_\rho\frac{g_\rho}{g}\right)$$

$$- \frac{1}{fl^2}\left(f_{zz} - f_z\frac{l_z}{l}\right) - \frac{1}{g^2}\frac{f_\rho l_\rho}{fl} - \frac{1}{l^2}\frac{f_z g_z}{fg}$$

$$- \frac{1}{h^2g^2}\left(x_\rho - x\frac{f_\rho}{f}\right)^2 - \frac{1}{h^2l^2}\left(x_z - x\frac{f_z}{f}\right)^2$$

$$\tilde{G}^{a=0}_{b=2} = \frac{1}{2hg^2}\left[\partial_\rho\left(x_\rho - x\frac{f_\rho}{f}\right) + \left(2\frac{f_\rho}{f} - \frac{h_\rho}{h} - \frac{g_\rho}{g} + \frac{l_\rho}{l}\right)\left(x_\rho - x\frac{f_\rho}{f}\right)\right]$$

$$+ \frac{1}{2hl^2}\left[\partial_z\left(x_z - x\frac{f_z}{f}\right) + \left(2\frac{f_z}{f} - \frac{h_z}{h} + \frac{g_z}{g} - \frac{l_z}{l}\right)\left(x_z - x\frac{f_z}{f}\right)\right].$$

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