Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 19, Issue 2, 2025, Pages 167-175
This paper is available online at http://www.frdint.com/
Published online August 22, 2025

GAPS AND OVERLAPPINGS

VOLKER W. THÜREY

Hegelstr. 101, 28201 Bremen
Germany
e-mail: volker@thuerey.de

Abstract

In the first part, we investigate the tiling of the plane by convex polygons, and we introduce many terms. We will not calculate them. At the end of this paper, we provide an example, where we cover the plane with convex 8-gons. The polygons overlap. In the second part, we take special curves and further convex polygons. We define two new constants.

1. Introduction

It is well-known that we can tile the plane $\mathbb{R}^2$ with n-gons, where n is a natural number larger than 2, (see [1], p. 11). If we restrict our efforts to convex polygons, in most cases it is impossible to cover the plane completely without overlappings. It is known that we can tile the plane with squares and regular 6-gons. We can also tile the plane with convex 5-gons, see the 'Cairo Tiling' in [2]. For natural numbers larger than 6 we believe that it is impossible to tile the plane completely with convex n-

Keywords and phrases: polygon, plane, tiling.

2020 Mathematics Subject Classification: 51P99.

Received June 23, 2025; Accepted July 31, 2025

gons. Either we have to leave gaps or some polygons overlap to cover $\mathbb{R}^2$ completely.

For additional information, see [3].

2. Convex Polygons and Curves

We believe that it is useful to repeat the definition of a *simple polygon*.

A simple polygon with n vertices consists of n different points of the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, called *vertices*, and the straight lines between (x_i, y_i) and (x_{i+1}, y_{i+1}) for $1 \leq i \leq n-1$, called *edges*. Also the straight line between (x_n, y_n) and (x_1, y_1) belongs to the polygon. We demand that it is homeomorphic to a circle, and that there are no three consecutive collinear points (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) for $1 \leq i \leq n-2$. Also the three points (x_n, y_n), (x_1, y_1), (x_2, y_2) and (x_{n-1}, y_{n-1}), (x_n, y_n), (x_1, y_1) are not collinear.

We call this just described simple polygon an n*-gon*.

Theorem 2.1. *There is a covering of the plane with* k*-gons for every natural number* k *larger than* 2.

Proof. Well-known. See, for instance, ([1] page 11).

Note that we work from now on exclusively with convex curves. With 'Polygon' we always mean a convex simple polygon. With 'k-gon' we always mean a convex k-gon.

Let r be any real number larger than or equal to 1, and let k be a natural number larger than 2.

We define a set of polygons.

Definition 2.2. We define k-Polygons as the class of k-gons.

Remark 2.3. In the next definition, we define some constants. They are actually percentages, but we prefer numbers from 0 to 1.

The area of $\mathbb{R}^2$ is regarded as 1.

We try to tile $\mathbb{R}^2$ with simple polygons. For $k > 6$, either we fix tiles without any overlapping and we do not cover $\mathbb{R}^2$ completely, or we cover $\mathbb{R}^2$ completely, where there may be overlappings.

Definition 2.4. Let $k \text{ gap}(r)$ be the supremum of the covered part of $\mathbb{R}^2$. The polygons do not overlap. We use elements from the class k-Polygons, where the quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$.

Let $k \text{ overlap}(r)$ be the infimum of the part of $\mathbb{R}^2$ which is covered by polygons from the class k-Polygons at least twice, where $\mathbb{R}^2$ is covered completely, and the quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$.

Conjecture 2.5. $k \text{ gap}(r) = 1$ and $k \text{ overlap}(r) = 0$ holds for all k for suitable numbers r.

For polygons with 8 vertices, see Proposition 3.3.

Let n be a natural number larger than 2.

Definition 2.6. We define a *cyclic polygon* as a simple polygon such that all vertices are on a circle.

We define an *elliptical polygon* as a simple polygon such that all vertices are on an ellipse.

We define a *convex Cassini polygon* as a simple polygon such that all vertices are on a convex Cassini curve.

We call a *regular* n*-gon* a regular polygon which has precisely n vertices.

We call a *cyclic* n*-gon* a cyclic polygon which has precisely n vertices.

We call an *elliptical* n*-gon* an elliptical polygon which has precisely n vertices.

We call a *convex Cassini* n*-gon* a convex Cassini polygon which has precisely n vertices.

We define a set of shapes.

Definition 2.7. $\text{Shapes} := \{\text{circle, ellipse, convex Cassini curve}\}$.

Definition 2.8. Let Curves be the set of curves of a shape from the set shapes. Let $\text{gap}_{\text{XXX}}(r)$ be the supremum of the covered part of $\mathbb{R}^2$, where we use curves from the set Curves of shape of a XXX, where XXX is an element of shapes. The curves do not overlap. The quotient of the arc lengths of two curves is in the interval $\left[\frac{1}{r}, r\right]$. Let $\text{overlap}_{\text{XXX}}(r)$ be the infimum of the part of $\mathbb{R}^2$ which is covered at least twice where we use curves from the set Curves of shape XXX, where XXX is an element of shapes and $\mathbb{R}^2$ is covered completely. The quotient of the arc lengths of two curves is in the interval $\left[\frac{1}{r}, r\right]$.

Definition 2.9. We define $k\ \text{reg}(r)$ as the supremum of the covered part of the plane, where we use regular k-gons. The polygons do not overlap. The quotient of two edges of the used polygons is in $\left[\frac{1}{r}, r\right]$.

We define k overlap reg(r) as the infimum of the part of the plane which is covered at least twice, where we use regular k-gons. $\mathbb{R}^2$ is covered completely. The quotient of two edges of the used polygons is in the interval $\left[\frac{1}{r}, r\right]$.

We define k cyclic(r) as the supremum of the covered part of the plane, where we use cyclic k-gons. The polygons do not overlap. The quotient of two edges of one or two used polygons is in the interval $\left[\frac{1}{r}, r\right]$.

We define k overlap cyclic(r) as the infimum of the part of the plane which is covered at least twice. We use cyclic k-gons. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$, and $\mathbb{R}^2$ is covered completely.

We define k elliptical(r) as the supremum of the covered part of the plane, where we use elliptical k-gons. The polygons do not overlap. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$.

We define k overlap elliptical(r) as the infimum of the part of the plane which is covered at least twice. We use elliptical k-gons. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$, and $\mathbb{R}^2$ is covered completely.

We define k Cassini (r) as the supremum of the covered part of the plane, where we use convex Casssini k-gons. The polygons do not overlap. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$.

We define k overlap Cassini (r) as the infimum of the part of the plane which is covered at least twice. We use convex Cassini k-gons. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{r}, r\right]$, and $\mathbb{R}^2$ is covered completely.

Remark 2.10. Note that $r = 1$ means that all edges of the polygons or the arc lengths of all curves, respectively, are equal.

Remark 2.11. The used polygons or curves, respectively, can not be arbitrarily small since $\frac{1}{r}$ is a positive number.

We suggest the name 'The first Thuerey constant' for 5 reg (1), and for 5 overlap reg (1), we suggest 'The second Thuerey constant'. Both are real numbers between 0 and 1. They are interesting new constants.

3. Propositions

Proposition 3.1. *The following equations hold for all* r.

$$3\ reg\,(r) = 4\ reg\,(r) = 6\ reg\,(r) = 3\ gap\,(r) = 4\ gap\,(r) = 6\ gap\,(r) = 1$$

as well as

$$3\ overlap\ reg\,(r) = 4\ overlap\ reg\,(r) = 6\ overlap\ reg\,(r)$$

$$= 3\ overlap\,(r) = 4\ overlap\,(r) = 6\ overlap\,(r) = 0.$$

Proof. Well-known.

Proposition 3.2. *The following equations hold for all* r.

$$5\ gap(r) = 1 \qquad and \qquad 5\ overlap\ reg(r) = 0.$$

Hint. See the 'Cairo Tiling' in [2].

Proposition 3.3. *It holds*

$$8\ overlap\ reg(r) = 0$$

for all r *equal to or larger than* 2.

Proof. At first, we tile the plane with squares of sidelength 1. Into every square we inscribe a regular octagon of sidelength $\sqrt{2} - 1$. These octagons cover a part of $\mathbb{R}^2$. We call them 'old' octagons.

Please see Figure 1. There the right square has vertices **A**, **B**, **C** and **D**. Two vertices of the right octagon are **E** and **F**. **G** is a vertex of the left octagon. We add another point **W** on one diagonal of the right square. We connect **E** and **W** and also **F** and **W**. We add a point called **X** on one diagonal of the left square. We connect **E** and **X** and also **G** and **X**. We define two more points **Y** and **Z** on the diagonals of other squares, and in this way, we generate a 'new' 8-gon. Seven of its vertices are **W**, **X**, **Y**, **Z**, **E**, **F** and **G**. The tuples **X**, **A**, **Z** and **Y**, **A**, **W**, **C** are collinear. By this way, we generate infinite many 'new' 8-gons between the 'old' 8-gons.

The 'new' 8-gons cover the area which is not yet covered. With the 'new' 8-gons together with the 'old' 8-gons, $\mathbb{R}^2$ is covered completely, where parts of $\mathbb{R}^2$ are covered twice. The number of 'new' 8-gons is countable. We can choose **W** such that the area of the triangle with vertices **E**, **F**, and **W** is arbitrarily small. Hence, we can choose **W**, **X**,

Y, and **Z** such that the area of $\mathbb{R}^2$ which is covered twice is less than $\frac{1}{2}$. In the next 'new' 8-gon we can choose four vertices of the 8-gon such that the area which is covered twice is less than $\frac{1}{4}$, et cetera.

Therefore, the part of $\mathbb{R}^2$ which is covered twice can be made arbitrarily small. □

In Figure 1, we show two 'old' octagons, and we indicate two 'new' octagons by its vertices.

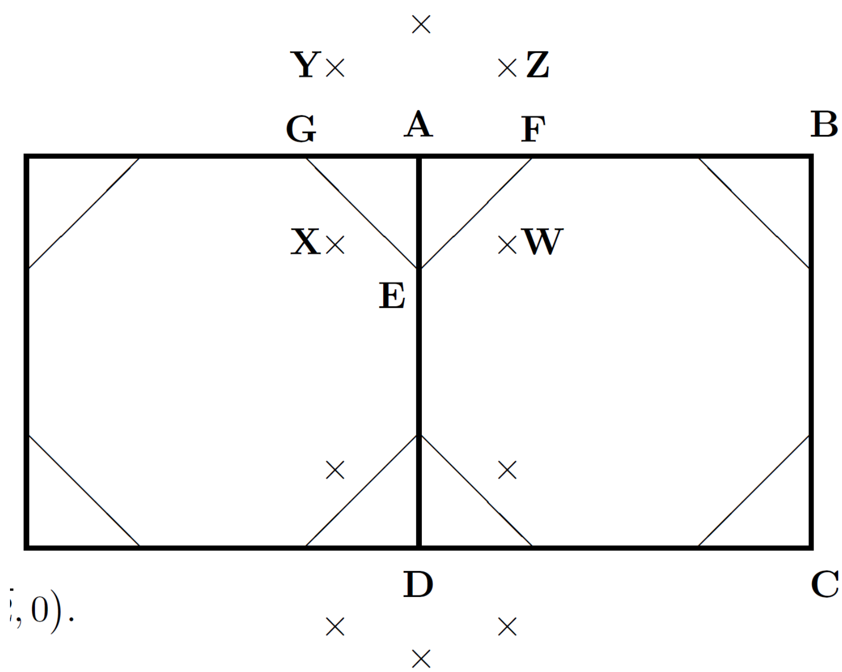

Figure 1.

We see two squares which are partially covered by regular 8-gons. We set $\mathbf{A} = (0,\ 0)$ and $\mathbf{B} = (1,\ 0)$. It holds $\mathbf{E} = (0,\ \frac{1}{2} \cdot \sqrt{2} - 1)$ and $\mathbf{F} = (1 - \frac{1}{2} \cdot \sqrt{2},\ 0)$. **W**, **X**, **Y**, and **Z** are not fixed.

Acknowledgements

We thank Prof. Dr. Wolfgang Fischer for a careful reading of the paper, and as well as Pakize Akin and Selen Yenilmez for support.

References

[1] http://www.willimann.org/A07020-Parkettierungen-Theorie.pdf

[2] http://www.mathematische-Basteleien.de/parkett2.htm

[3] Ehrhard Behrends, Parkettierungen der Ebene, Springer, 2019.