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FOUNDATIONAL TOPOLOGICAL PROPERTIES FOR TOPOLOGICAL PROPERTIES SATISFYING T_0

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Abstract

The regular property was introduced by Vietoris in 1921 and used, together with T_1, to define the T_3 property; a space is T_3 iff it is (regular and T_1). Further investigation of regular and T_3 revealed that the addition of T_1 to regular to obtain T_3 can be relaxed to T_0. In similar fashion, in 1925, Urysohn introduced the completely regular property, which was used along with T_1 to define the $T_{3\frac{1}{2}}$ property. As in the case of regular and T_3, the addition of T_1 to completely regular to get $T_{3\frac{1}{2}}$ can be relaxed to T_0. Because of their origins, through the years,

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without given criteria, regular was thought of as the foundational property for T_3 and completely regular was thought of as the foundational property for $T_{3\frac{1}{2}}$. As given above, the addition of T_0 to the first property to give the second property raises the question of whether or not such a practice would uniquely define a foundational property for a topological property satisfying the T_0 property. In this paper, the answer is shown to be "no" and a unique, meaningful assignment consistent with earlier beliefs about foundational property for topological property satisfying the T_0 property is given.

1. Introduction and Preliminaries

Within the development of modern topology, in order to make topology a more meaningful, useful tool in the continued study and advancement of mathematics, properties that in some manner separate distinct elements or elements and closed sets or distinct closed sets were introduced and investigated. Tietze [16] was the first to use the term "separation axiom" to collectively label such properties. In classical topology, the regular separation axiom was introduced and used, along with T_1, which could be replaced by T_0, to define the T_3 separation axiom; a space is T_3 iff it is (regular and T_0). Also, in classical topology the separation axiom completely regular was introduced and used, along with T_1, which can be relaxed by T_0, to define $T_{3\frac{1}{2}}$. For that reason, in classical topology, regular was considered the foundational property for T_3 and completely regular was considered the foundational property for $T_{3\frac{1}{2}}$.

In 1923, Tietze [16] introduced the normal property, which together with T_1, gave the separation axiom T_4, but, unlike T_3 and $T_{3\frac{1}{2}}$, the requirement of T_1 added to normal to give T_4 can not be relaxed to T_0 and normal was not considered a foundational property for T_4. Thus the following questions arise; (1) does T_4 have a foundational topological property and, if so, what is it?, (2) if T_4 has a foundational property what criteria was used in determining its foundational property?, (3) could that criteria be applied to all other topological properties satisfying the T_0 property to obtain a foundational topological property?, and (4) if the criteria is applied to T_3 and $T_{3\frac{1}{2}}$ does it give regular as the foundational property for T_3 and completely regular as the foundation property for $T_{3\frac{1}{2}}$, tying the new together with the old? Below these questions are addressed.

2. Progress but still no Resolution

In the continued investigation and expansion of topology, in 1943, the R_0 separation axiom was introduced [14].

Definition 2.1. A space (X, T) is R_0 iff for each closed set C and each $x \notin C$, $C \cap Cl(\{x\}) = \phi$.

In 1961 [1], the R_0 separation axiom continued to be investigated and the R_1 separation axiom was introduced and investigated.

Definition 2.2. A space (X, T) is R_1 iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

In the 1961 paper [1], it was shown that a space is T_i iff it is $(R_{i-1}$ and $T_{i-1})$; $i = 1, 2,$ respectively. In that paper [1], the fact that R_1 implies R_0 was used to show a space is T_2 iff it is $(R_1$ and $T_0)$.

In 1988 [2], weakly Urysohn was introduced and investigated.

Definition 2.3. A space (X, T) is weakly Urysohn iff for x and y in X such that $Cl(\{x\}) \neq Cl(\{y\})$, there are open sets U and V such that $x \in U$, $y \in V$, and $Cl(U) \cap Cl(V) = \phi$.

In the 1988 paper [2], it was shown that a space is Urysohn iff it is ((weakly

Urysohn) and T_0).

In 2011 [3] and 2012 [4], almost perfectly Hausdorff and almost perfectly normal were introduced, respectively, and it was shown that a space is perfectly Hausdorff (perfectly normal) iff it is ((almost perfectly Hausdorff) and T_0) [3] (((almost perfectly normal) and R_0) and T_0) [4]. Thus the study of topology continued to expand, filling in missing gaps with additional properties, and relationships between classical topological properties satisfying the T_0 property and added topological properties were discovered, but there continued to be no well-defined criteria for one topological property to serve as a foundational topological property for another with the T_0 property; and T_4 continued to not have a defined foundational property.

3. Resolution of the Questions

In the same manner as regular was considered the foundational property for T_3 and completely regular was considered the foundational property for $T_{3\frac{1}{2}}$, pseudometrizable served as a foundational property for metrizable. In 1936 [15], T_0-identification spaces were introduced and used to jointly characterize pseudometrizable and metrizable; a space is pseudometrizable iff its T_0-identification space is metrizable. Hence another connection between a topological property satisfying the T_0 property and the property thought to be its foundational property was established, raising the question of whether or not T_0-identification spaces could somehow be used to establish the criteria for a topological property to be the foundational property for a given topological property satisfying the T_0 property in a well-defined, unique manner consistent with earlier beliefs about foundational properties.

Definition 3.1. Let (X, T) be a space, let R be the equivalence relation on X defined by xRy iff $Cl(\{x\}) = Cl(\{y\})$, let X_0 be the set of R equivalence classes of X, let $N : X \to X_0$ be the natural map, and let $Q(X, T)$ be the decomposition topology on X_0 determined by (X, T) and the map N. Then $(X_0, Q(X, T))$ is the T_0-identification space of (X, T).

T_0-identification spaces were cleverly created to generate for a space (X, T) a strongly (X, T) related space $(X_0, Q(X, T))$ with T_0 added [15], potentially making T_0-identification spaces a strong, useful mathematical tool.

In 1975 [13], it was proven that a space is R_1 iff its T_0-identification space is T_2, which along with the result above motivated additional investigations of T_0-identification spaces. Included in those investigations were (1) the fact that the natural map $N : (X, T) \to (X_0, Q(X, T))$ is continuous, onto, open, closed, for each open set O in X, $N^{-1}(N(O)) = O$, and for each closed set C in X, $N^{-1}(N(C)) = C$ [5], revealing T_0-identification spaces as even more powerful and useful; and (2) a space is R_0 iff its T_0-identification space is T_1 [6]. Thus, the question of what other properties behave the same as pseudometrizable and metrizable with respect to T_0-identification spaces arose, leading to the introduction and investigation of weakly Po spaces and properties in 2015 [7].

Definition 3.2. Let P be a topological property for which $P\text{o} = (P \text{ and } T_0)$ exists. Then (X, T) is weakly Po iff $(X_0, Q(X, T))$ has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

In the initial investigation of weakly Po spaces and properties, it was shown that for a topological property P for which weakly Po exists, weakly Po is a unique topological property, weakly Po is simultaneously shared by both a space and its T_0-identification space, and $(\text{weakly } P\text{o})\text{o} = P\text{o}$ [7]. Initially, the search for weakly Po properties and spaces was filled with uncertainties; a topological property Q for which Qo exists was selected, with no certainty that Qo was a weakly Po property, and, then a topological property W was sought for which

W = weakly Qo, with no certainty there is such a W. The process was tedious and, at best, would require much time and effort to search for possible answers. However, in a 2017 paper [8], a major break through occurred. In that paper, it was proven that for a topological property Q for which Qo exists, there exists the topological property QNO such that $W = (Q\text{o or } QNO)$ = weakly $(Q\text{o or } QNO)$o = weakly Qo, and, hence, for every topological property Q for which Qo exists, weakly Qo exists and Qo is a weakly Qo property. Thus the uncertainties of which topological properties satisfying the T_0 property were weakly Po properties and whether there is a topological property W such that W = weakly Qo were replaced by certainty, but just knowing such a topological property W exists gave little insight into the precise, needed topological property W, raising the question of whether known information could somehow be used to more quickly and easily determine W = weakly Qo.

The investigation of that question led to the introduction and investigation of $OXTO$ subsets and the corresponding subspace for each space (X, T) in a 2017 paper [9].

Definition 3.3. Let (X, T) be a space and for each $x \in X$, let C_x be the T_0-identification equivalence class containing x. Then Y is an $OXTO$ subset of X iff Y contains exactly one element from each equivalence class C_x.

In the 2017 paper [9], it was shown that for a space (X, T) and each $OXTO$ subset Y of X, (Y, T_Y) is homeomorphic to $(X_0, Q(X, T))$. Thus the T_0-identification space process can be done internally in the initial space avoiding transitions to and from its T_0-identification space, greatly simplifying the search for weakly Qo for a topological property Qo.

Definition 3.4. Let Q be a topological property for which Qo exists. A space (X, T) has property QNO iff (X, T) is "not-T_0" and $(X_0, Q(X, T))$ has property Qo.

Using the results above, for a topological property P for which Po exists, weakly Po can be characterized.

Theorem 3.1. *Let P be a topological property for which Po exists. Then weakly Po is the least element in the set $\mathcal{P} = \{$weakly Qo : Q is a topological property, Qo exists, and Qo implies Po$\}$.*

Proof. Since P is a topological property, Po exists, and Po implies Po, then weakly Po exists and weakly Po $\in \mathcal{P}$. Let W be a topological property for which weakly Wo $\in \mathcal{P}$. Then W is a topological property, Wo exists, and Wo implies Po. Let (X, T) be a space that is weakly *Wo*. Then $(X_0, Q(X, T))$ is Wo, which implies $(X_0, Q(X, T))$ is Po and (X, T) is weakly Po. Thus weakly Po is the least element of $\mathcal{P}$.

As given above, weakly Po proved to be consistent with earlier believed foundational properties; pseudometrizable = weakly (pseudometrizable)o = weakly metrizable, R_0 = weakly (R_0)o = weakly T_1, R_1 = weakly (R_1)o = weakly T_2, regular = weakly (regular)o = weakly T_3, and completely regular = weakly (completely regular)o = weakly $T_{3\frac{1}{2}}$ [6], motivating and justifying the definition below.

Definition 3.5. Let Q be a topological property such that Qo exists. Then weakly Qo is the foundational property for Qo.

Hence, by the results above, foundational properties are uniquely defined for all topological property P for which Po exists, as required.

Since (normal and R_0) = weakly (normal and R_0)o = weakly T_4 [6], then a foundational property for T_4 is finally known; (normal and R_0). Since normal = weakly (normal)o, then normal is the foundational property for (normal and T_0). Also, foundational properties for non-separation axioms are now defined and can be determined. For examples, compact is the foundational property for (compact and

T_0) and connected is the foundational property for (connected and T_0) [6].

By the work above T_0 has a foundational property. What is it?

Within the 2015 paper [7], the search for topological properties that are not weakly Po led to the use of T_0 and "not-T_0". Thus another fundamental role of T_0 in the study of topology was revealed and "not-T_0" proved to be a useful topological property, motivating the addition of "not-P", where P is a topological property for which "not-P" exists, to the study of topology [2]. The addition and use of the many new topological properties provided tools not before studied and used in the study of topology and, in a short time period, has exposed a mathematically fertile, never before imagined territory long overlooked within topology that has already changed and expanded the study of topology.

As an example, in the paper [10], the use of "not-T_0" and "not-P", where "not-P" exists, not only provided needed tools to prove the existence of the never before imagined least of all topological properties L, but, also, provided the needed tools for a quick, easily understood proof of its existence.

Theorem 3.2. *L, the least of all topological properties, is given by $L = (T_0$ or "not-T_0"$) = (P$ or "not-P"$)$, where P is a topological property for which "not-P" exists.*

The discovery of L is precisely what was needed to resolve the question above about T_0; a space is T_0 iff it is $(L$ and $T_0)$ and $L =$ weakly Lo $=$ weakly T_0 [11].

If there is a least topological property, is there a strongest topological property? The addition and use of "not-P", where "not-P" exists quickly and easily gave an answer of "no" [12].

In the study of weakly Po spaces and properties, it was shown that for a topological property P for which Po exists, ((weakly Po) or "not-T_0") is the least topological property, which together with T_0, equals Po [11]. Thus the use of

T_0 as initially given for regular and T_3 and completely regular and $T_{3\frac{1}{2}}$ will not uniquely determine the foundational property for a topological property satisfying the T_0 property, as stated above, and fortunately weakly Po and weakly Po properties were available to fill that need.

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